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Black holes and equivariant charge vectors in $\mathcal{N}=2, d=4$ supergravity

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Abstract

We extend previous investigations on the construction of extremal supersymmetric and non-supersymmetric solutions in the H-FGK formalism to *unconventional* solutions with anharmonic terms. We show how the use of *fake* charge vectors equivariant under duality transformations simplifies and clarifies the task of identification of the attractors of the theory.

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Introduction

The intensive search for black-hole solutions of supergravity theories over the last 25 years has been a very rewarding one in respect to the supersymmetric (also known as BPS in the literature, even if this concept is not equivalent, but wider) ones. Even though the existence of extremal non-supersymmetric black holes was discovered long time ago [1, 2] and we know that they are subject to the same attractor mechanism as the supersymmetric ones [3], only a few general families of solutions have been constructed for some classes of theories [4] and we are still far from having a complete understanding of their structure and general properties. The situation w.r.t. non-extremal solutions, which some of us studied recently in [5, 6, 7, 8, 9] is even worse: even if all extremal black-hole solutions may be deformed (*i.e.* heated up) to a non-extremal one, then we do not know the non-extremal deformations of many of them; in general we don't know whether there are obstructions to such a deformation and what they are. We also don't know whether, in each theory, there is only one family of non-extremal black-hole solutions from which all the extremal ones can be obtained by taken the appropriate limits, such as it happens in the few models studied so far [10, 11, 5, 6, 7]. The (stringy) non-extremal black hole landscape is a largely uncharted territory.

It is clear that to answer these questions new tools are needed since the first-order equations associated to unbroken supersymmetry are of no help here and the second-order equations of motion of the FGK effective action [3] are still very hard to solve. Several approaches have been proposed to this end. For instance, it has been shown that in general one can construct first-order flow equations for extremal non-supersymmetric and non-extremal black holes Refs. [12, 13, 14, 15, 16, 17, 18, 19, 20] and many such equations have been constructed. From them one can extract interesting information about the near-horizon and spacelike infinity limits (whence about the entropy and mass of the solutions), but in practice these equations are obtained when the solutions are already known, which somewhat diminishes their usefulness.

The most common approach to the search of stationary black-hole solutions, pioneered in Ref. [21], consists in the dimensional reduction over the time direction. For 4-dimensional theories, this results in a 3-dimensional theory consisting of a non-linear σ-model coupled to gravity (in 3 dimensions the vector fields can be dualized into scalars).[1] When the σ-model corresponds to a homogeneous space one can show that the system is integrable and use the standard techniques to classify and obtain explicit black-hole solutions, see *e.g.* [22]. This approach has been quite a successful one, but for the moment it has not provided complete answers to the above questions.

More recently, a new approach for the 4- and 5-dimensional $\mathcal{N} = 2, d = 4$ supergravity theories coupled to n_V vector supermultiplets has been introduced in Ref. [23][2]. This approach, dubbed the *H-FGK formalism*, consists in using a convenient set of variables in the FGK effective action. These variables arise naturally in the supersymmetric cases [27, 28], but it has been shown that they can be used in more general (but always stationary) cases. The main virtue of the new variables, when compared to the scalar fields present in the FGK effective action, is that they

[1] Further assumptions (staticity plus an ansatz for the 3-dimensional metric) lead to the FGK effective action with its characteristic effective *black-hole potential* [3].

[2] A closely-related approach has been proposed in Ref. [24, 25, 26].

transform linearly under the duality group (embedded in $\mathrm{Sp}(2n_V+2;\mathbb{R})$ in the $d=4$ case and in $\mathrm{SO}(n_V+1)$ in $d=5$ case).

In previous works [7, 9, 8, 29] we have investigated the description of the simplest families of solutions (that we will call *conventional* in Section 3) for which the H-variables are harmonic functions (in the extremal case) or linear combinations of hyperbolic sines and cosines (in the non-extremal case). We have also studied some general features of the formalism, like the invariance of the effective action under local Freudenthal duality rotations [30], but thus far we have not exploited the main feature of the H-variables, namely the linear equivariance under duality transformations of the charges and moduli that characterize a given solution.

Our main goal in this paper is to study this aspect of the formalism and show how to exploit the requirement of linear equivariance in order to find attractors and construct explicit extremal solutions in some already well-studied models: the axidilaton and the $\overline{\mathbb{CP}}^n$ models. We also want to make progress towards answering the questions posed at the beginning of this introduction using these new tools. In the *conventional* cases that we have studied so far, it is known how one can arrive at (extremal) solutions described by harmonic function from (non-extremal) solutions described by hyperbolic sines and cosines: we will apply our new tools to a *non-conventional* (non-supersymmetric) extremal solution of the t^3 model not considered in our previous works Refs. [7, 9]. This solution, which has been known for some time [31, 32, 19, 4], is characterized by H-variables that contain anharmonic terms and its deformation into a non-supersymmetric (finite-temperature) solution has proven elusive [33]. We think that, in order to search for this non-extremal generalization (if it exists), it is necessary to know more about the structure of the extremal solution and we will show how the new tools can help us to this end.

This paper is organized as follows: in Section 1 we briefly review the H-FGK formalism, providing the definitions and relations that we will use in the rest of the article. In Section 2 we explain how equivariant charge vectors enter in black-hole solutions when we express them in the H-variables of this formalism. In Section 3 we explain when the usual harmonic ansatz becomes insufficient to write the general family of solutions associated to some attractor (expressed through an equivariant charge vector). This insufficiency indicates the need of adding anharmonic terms to the H-variables giving rise to what we have called *unconventional* black-hole solutions. Then, in Section 4 we give a general form for the first-order flow equations of any static black-hole solution of these theories that applies, in particular, to the unconventional solutions. In Sections 5 and 6 we review the supersymmetric and non-supersymmetric extremal solutions (which are completely conventional) of two simple models, studying their duality symmetries and their equivariant vectors. In Section 7 we turn to the t^3 model, showing how its extremal, non-supersymmetric solutions are non-conventional. We, then, construct and study this unconventional family of solutions using a basis of equivariant vectors. Our conclusions and comments on further directions of work can be found in Section 8.

1 The H-FGK formalism for $\mathcal{N}=2$, $d=4$ supergravity

As shown in Refs. [26, 23][3] the problem of finding static, single-center, black-hole solutions of any ungauged $\mathcal{N}=2$, $d=4$ supergravity theory coupled to n vector multiplets can be reduced to that of finding solutions to the effective action for the $2(n+1)$ real variables[4] $H^M(\tau)$

$$-I_{\text{H-FGK}}[H] = \int d\tau \left\{ \tfrac{1}{2} g_{MN}\dot{H}^M \dot{H}^N - V \right\}, \tag{1.2}$$

subject to the *Hamiltonian constraint*

$$\tfrac{1}{2} g_{MN}\dot{H}^M \dot{H}^N + V + r_0^2 = 0\,, \tag{1.3}$$

where r_0 is the *non-extremality parameter*. For later reference, we quote the equations of motion that follow from the above action, taking into account that the metric g_{MN} is not invertible [26, 30]

$$g_{MN}\ddot{H}^N + (\partial_N g_{PM} - \tfrac{1}{2}\partial_M g_{NP})\dot{H}^N\dot{H}^P + \partial_M V = 0\,. \tag{1.4}$$

The metric $g_{MN}(H)$ and the potential $V(H)$ of the H-FGK effective action are given in terms of the *Hesse potential* $\mathsf{W}(H)$ by

$$g_{MN}(H) \equiv \partial_M\partial_N \log \mathsf{W} - 2\frac{H_M H_N}{\mathsf{W}^2}\,, \tag{1.5}$$

$$V(H) \equiv \left\{ -\tfrac{1}{4}\partial_M\partial_N \log \mathsf{W} + \frac{H_M H_N}{\mathsf{W}^2} \right\} \mathcal{Q}^M \mathcal{Q}^N\,. \tag{1.6}$$

The Hesse potential contains all the information characterizing the $\mathcal{N}=2, d=4$ supergravity theory under consideration, and defines it (at least in this context) just as the canonically-normalized covariantly-holomorphic symplectic section $(\mathcal{V}^M) = \left(\begin{smallmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{smallmatrix}\right)$ does. The Hesse potential can be derived from $\mathcal{V}^M$ as follows:

1. Introduce an auxiliary complex variable X with the same Kähler weight as $\mathcal{V}^M$, we can define the two Kähler-neutral real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

$$\mathcal{V}^M/X \equiv \mathcal{R}^M + i\mathcal{I}^M\,. \tag{1.7}$$

[3]We will follow the notation and conventions of Ref. [23]. More information about this formalism and the original FGK formalism can be found in *e.g.* Refs. [3, 5, 30].

[4]The indices M, N are $2(n+1)$-dimensional symplectic indices. We use the symplectic metric $(\Omega_{MN}) \equiv \left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right)$ and $\Omega^{MP}\Omega_{NP} = \delta^M{}_N$ to lower and rise the symplectic indices according to the convention

$$H_M = \Omega_{MN}H^N\,, \qquad H^M = H_N\Omega^{NM}\,. \tag{1.1}$$

The components of $\mathcal{R}^M$ can be expressed in terms of those of $\mathcal{I}^M$ (solving the *stabilization equations* a.k.a. *Freudenthal duality equations* [9]). The functions $\mathcal{R}^M(\mathcal{I})$ are characteristic of each theory, but they are always homogeneous of first degree in the $\mathcal{I}^M$.

It can be shown that

$$X = \tfrac{1}{\sqrt{2}} e^{U+i\alpha}\,, \tag{1.8}$$

where e^U is the metric function (or warp factor) and α is a completely arbitrary τ-dependent phase which does not enter in the Lagrangian. Different choices of α give different definitions of the variables H^M which, nevertheless, describe the same physical variables. This freedom gives rise to a local symmetry of the H-FGK action, known as *local Freudenthal duality* [30], that will be discussed later.

2. Given those functions, the *Hesse potential* $\mathsf{W}(\mathcal{I})$ is just

$$\mathsf{W}(\mathcal{I}) \equiv \mathcal{R}_M(\mathcal{I})\mathcal{I}^M\,. \tag{1.9}$$

It is, by construction, homogeneous of second degree in $\mathcal{I}^M$.

It is customary to relabel these variables

$$H^M \equiv \mathcal{I}^M\,, \qquad \tilde{H}^M \equiv \mathcal{R}^M\,, \;\longrightarrow\; \left\{ \begin{array}{rcl} \mathcal{V}^M/X & = & \tilde{H}^M + iH^M \equiv \mathcal{H}^M\,. \\ \mathsf{W}(H) & = & \tilde{H}_M(H)H^M\,. \end{array} \right. \tag{1.10}$$

The relation between the tilded and untilded variables defines the *discrete Freudenthal duality* transformation of the theory [34, 35, 30]: $\tilde{H}^M$ is the Freudenthal dual of H^M. This duality transformation turns out to be an anti-involution, *i.e.*

$$\tilde{\tilde{H}}^M \equiv \tilde{H}^M(\tilde{H}) = -H^M\,, \tag{1.11}$$

and, therefore preserves the Hesse potential

$$\mathsf{W}(\tilde{H}) = \mathsf{W}(H)\,, \tag{1.12}$$

and also the full effective action Eq. (1.2). These discrete duality transformations are associated to the constant shift of the phase of X, $\alpha \to \alpha + \pi$. The continuous, local, transformations $\alpha \to \alpha + f(\tau)$

$$\mathcal{H}'^M = e^{if(\tau)}\,\mathcal{H}^M\,, \tag{1.13}$$

leave invariant the effective action (1.2) and all the physical fields [30]. Since the central charge of these theories $\mathcal{Z}(Z, Z^*, \mathcal{Q})$ is defined in terms of the canonically-normalized covariantly-holomorphic symplectic section $\mathcal{V}^M$ by[5]

$$\mathcal{Z}(\mathcal{Q}) \equiv \mathcal{V}_M \mathcal{Q}^M \,, \tag{1.14}$$

using the definition of the H-variables we find that[6]

$$\mathcal{Z}(\mathcal{Q}) = \frac{e^{-i\alpha}}{\sqrt{2\mathsf{W}}} \mathcal{H}_M \mathcal{Q}^M \,, \tag{1.15}$$

whence under Freudenthal duality

$$\mathcal{Z}'(\mathcal{Q}) = e^{if(\tau)} \mathcal{Z}(\mathcal{Q}) \,. \tag{1.16}$$

The definition of Freudenthal dual can be extended to any symplectic vector of a given theory[7] and, in particular, to the charge vector $\mathcal{Q}^M$. We know that the black-hole potential, which is related to the potential V appearing in the H-FGK action by

$$V_{\rm bh} = -\mathsf{W}\, V \,, \tag{1.17}$$

as a function of the variables H^M, is always extremized by the near-horizon value $B^M = \beta \mathcal{Q}^M$ for any proportionality constant β. Freudenthal symmetry implies that it is also extremized at the same points in terms of the Freudenthal-dual variables $\tilde{B}^M = \beta \mathcal{Q}^M$, which corresponds to $B^M = -\beta \tilde{\mathcal{Q}}^M$ [35, 30]. Freudenthal duality can also be seen as a relation between black holes with identical metrics (and, therefore, entropies) and scalar fields but different charges [34].

2 Explicit solutions and equivariant vectors

The main advantage of the H-FGK formalism is the linear behavior of the variables under transformations of the electric-magnetic duality group G of the theory:

$$H^{M\prime} = S^M{}_N H^N \,, \qquad (S^M{}_N) \in \mathrm{G} \subset \mathrm{Sp}(2n+2; \mathbb{R}) \,. \tag{2.1}$$

This linear behavior can dramatically simplify the construction of explicit solutions to theories with a non-trivial duality group as it implies that any solution must be of the form

$$H^M(\tau) = c^\sigma(\tau)\, U_\sigma^M \,, \tag{2.2}$$

[5]We will often use the lighter notation $\mathcal{Z}(\mathcal{Q})$ or $\mathcal{Z}(B)$ if we replace the charge vector by another equivariant charge vector. Sometimes these equivariant charge vectors are called *fake charges* in which case $\mathcal{Z}(B)$ is referred to as *fake central charge*.

[6]In what follows, W with no arguments will be assumed to be $\mathsf{W}(H)$.

[7]In some theories not all symplectic vectors have a Freudenthal dual. For instance, in the cubic models that we are going to study, only when the Hesse potential, evaluated on a particular vector, is different from zero, is the Freudenthal dual well defined.

where the functions $c^{\sigma}(\tau)$ are duality invariant; the symplectic vectors U_{σ}^{M} are constant vectors that may depend on the physical parameters of the theory (mass M, electric and magnetic charges $\mathcal{Q}^{M}$ and asymptotic values of the scalars Z^{i}_{∞}) and must be *equivariant* w.r.t. the duality group, *i.e.*

$$U_{\sigma}^{M}(M, Z'_{\infty}, Z^{*}_{\infty}{}', \mathcal{Q}') = S^{M}{}_{N} U_{\sigma}^{N}(M, Z_{\infty}, Z^{*}_{\infty}, \mathcal{Q})\,, \tag{2.3}$$

with

$$Z^{i\,\prime} \equiv F^{i}_{S}(Z)\,, \qquad \mathcal{Q}^{M\,\prime} = S^{M}{}_{N}\mathcal{Q}^{N}\,, \tag{2.4}$$

where $F^{i}_{S}(Z)$ is the non-linear realization of the duality transformation $S^{M}{}_{N}$ on the complex scalars.

In some cases, the number of equivariant vectors of the theory can be greater than[8] or equal to the number of variables H^{M}. In that case, one does not win much by using the above ansatz. In other cases, however, the number can be much smaller and we will be left with a small number of invariant functions to be determined.

In the near-horizon limit of extremal black-hole solutions, the value of the variables H^{M} will be dominated by one equivariant vector that we denote by B^{M} and that can be defined, in our conventions, by[9]

$$B^{M} \equiv \lim_{\tau\to-\infty} -\frac{\sqrt{2}H^{M}}{\tau}\,. \tag{2.5}$$

The values of the scalars on the horizon, $Z^{i}_{\rm h}$, are completely determined by this equivariant vector upon use of the general expression of the scalars as functions of the variables H^{M} [23]

$$Z^{i}(H) = \frac{\tilde{H}^{i} + iH^{i}}{\tilde{H}^{0} + iH^{0}}\,, \qquad \Rightarrow \qquad Z^{i}_{\rm h} = Z^{i}(B)\,, \tag{2.6}$$

and also extremize the black-hole potential $V_{\rm bh}(H, \mathcal{Q})$ as a function of the variables H^{M}:

$$\partial_{M} V_{\rm bh}(H, \mathcal{Q})|_{H=B} = 0\,. \tag{2.7}$$

The vectors B^{M}, which in this context can be called attractors, can also be written in the form

$$B^{M} = b^{\sigma} U_{\sigma}^{M}\,, \tag{2.8}$$

where the b^{σ} are duality-invariant constants such that the products bU^{M} have the same dimensions as electric and magnetic charges.

Clearly these vector attractors must contain more information than the values of the scalars on the horizon $Z^{i}_{\rm h}$ (the standard attractors). On the other hand, when the model has a high degree

[8]If it is greater, we can eliminate some from the ansatz, since they will be linearly dependent on the rest.

[9]Observe that this definition is completely general: given the behavior of the 3-dimensional transverse metric in the near-horizon limit as a function of τ and the degree of homogeneity of $e^{-2U} = \mathsf{W}(H)$ as a function of the H-variables, in regular black-hole solutions the functions $H^{M}(\tau)$ are dominated by these constant vectors in the near-horizon limit.

of symmetry the requirement of equivariance imposes strong constraints on the possibilities and it simplifies the task of finding the attractors B^M.

A similar discussion can be made for the values of the variables H^M at spatial infinity, which in the employed coordinate system lies at $\tau = 0$.

The amount of simplification introduced by the above observation that the variables H^M must always be of the form Eq. (2.2) depends on our ability to find a sufficient number of equivariant vectors; the Freudenthal dual of the charge vector $\tilde{\mathcal{Q}}^M$ is, by construction, a prime example of equivariant vector, but there are other systematic ways of finding them. Let us consider, first, equivariant vectors that only depend on the charges. They can be seen as an endomorphism of the $(2n+2)$-dimensional vector space of charges and their equivariance is equivalent to the fact that these endomorphisms commute with the duality transformations (which are also endomorphisms of charge space). Thus, linear (not necessarily symplectic) transformations that commute with G provide a second example of equivariant vectors.

To study non-linear cases, let us expand an equivariant vector and the duality transformations around the identity

$$U^M_\sigma(\mathcal{Q}) \sim \mathcal{Q}^M + \xi^M(\mathcal{Q})\,, \qquad (S\mathcal{Q})^M \sim \mathcal{Q}^M + \alpha^A \eta_A{}^M(\mathcal{Q})\,, \tag{2.9}$$

where $S \in \mathrm{G} \subset \mathrm{Sp}(2n+2;\mathbb{R})$ and, therefore,

$$\eta_A{}^M(\mathcal{Q}) \;=\; (T_A)^M{}_N\, \mathcal{Q}^N\,, \tag{2.10}$$

where $T_A \in \mathrm{Sp}(2n+2;\mathbb{R})$ are the generators of the duality group; the condition of equivariance is equivalent to requiring that the Lie brackets of these two kinds of generators vanish[10]

$$[U, \eta_A] = 0\,, \quad \Rightarrow \quad (T_A)^M{}_N \mathcal{Q}^N \partial_M U^P = (T_A)^P{}_R U^R\,, \quad \text{where} \quad \partial_M U^P \equiv \frac{\partial U^P}{\partial \mathcal{Q}^M}\,. \tag{2.11}$$

On taking the derivative with respect to $\mathcal{Q}^P$ of both sides of this equation we find the integrability condition

$$(T_A)^M{}_N \mathcal{Q}^N \partial_M \mathsf{P} = 0\,, \qquad \mathsf{P} \equiv \partial_M U^M = \Omega^{MN} \partial_M U_N\,. \tag{2.12}$$

which implies that P is an invariant function of the charges. Thus, equivariant vectors are associated to invariants by the above equation. The simplest invariant is just $\mathsf{P} = 0$ and equivariant vectors such that $\partial_{[M} U_{N]} = 0$ are associated to it; clearly there may be more possibilities as locally they must be of the form $U_M = \partial_M h$ for some non-vanishing invariant h (possibly up to additive numerical constants) and one can check that the equivariance condition is automatically satisfied. For instance, if we take $h = \mathsf{W}/2$, then $U_M = \tilde{\mathcal{Q}}_M$.

For equivariant vectors that depend (non-holomorphically) on the moduli Z^i_∞, the equivariance condition takes the form

[10] Obviously, also ξ must be an equivariant vector, whence we can replace ξ by U in what follows for the purpose of writing an equation characterizing equivariant vectors.

$$(T_A)^M{}_N \mathcal{Q}^N \partial_M U^P + k_A{}^i \partial_i U^P + k_A{}^{*\,i^*} \partial_{i^*} U^P = (T_A)^P{}_R U^R\,, \tag{2.13}$$

where $K_A \equiv k_A{}^i(Z)\partial_i + \text{c.c.}$ are the Killing vectors that generate the action of the duality group G on the scalar manifold preserving the holomorphic and Kähler structures. Again, $\mathsf{P} \equiv \partial_M U^M$ must be an invariant and a particularly simple case is $\mathsf{P} = 0$ and $U_M = \partial_M h$ where, now, h is required to be invariant only up to additive functions of the moduli. A recurring example is

$$h \; = \; \log\, \left(\mathcal{Z}(\mathcal{Q})\right)\,, \tag{2.14}$$

where $\mathcal{Z}(\mathcal{Q})$ is the central charge defined in Eq. (1.14). The associated (complex) equivariant vector is

$$U_M = \frac{\partial\, h}{\partial \mathcal{Q}^M} = \frac{\mathcal{V}_M}{\mathcal{Z}(\mathcal{Q})}\,. \tag{2.15}$$

The real and imaginary parts provide two real moduli-dependent equivariant vectors. It should be obvious that one can use, instead of the central charge any fake central charge, but the result may not be a new equivariant vector.

The Lie bracket of two equivariant vectors is also an equivariant vector, so that the equivariant vectors form a Lie algebra that commutes with that of the duality group G.

Finally, in the cases that we are going to study, we will show how one can construct equivariant vectors by using other methods like solution-generating techniques.

3 Conventional and unconventional solutions

As explained in Ref. [23], contracting the equations of motion derived from the H-FGK action Eq. (1.2) with H^M and using the homogeneity properties of the different terms and the Hamiltonian constraint Eq. (1.3) one finds, in the extremal case $r_0 = 0$[11], the equation

$$\mathsf{W}\tilde{H}_M\, \ddot{H}^M + (\dot{H}^M H_M)^2 = 0\,. \tag{3.1}$$

In what we are going to call from now on *conventional* extremal solutions (supersymmetric or not) the variables $H^M(\tau)$ are harmonic functions, *i.e.* they satisfy $\ddot{H}^M = 0$. The above equation implies that they also satisfy the constraint[12]

$$\dot{H}^M H_M = 0\,. \tag{3.2}$$

Conventional extremal solutions have been intensively studied in Ref. [9]. However, how general are these solutions? Can all the extremal black-hole solutions be written in a conventional form? (The answer in the supersymmetric case is yes.) If not, what are the limitations and how can

[11] In this discussion we will only consider the extremal case because in the rest of the paper we are going to restrict ourselves to it.

[12] The converse is not always true: the above constraint can be satisfied for extremal black-hole solutions which are not given by harmonic H^Ms and that we will call *unconventional*.

they be overcome as to obtain the most general extremal black-hole solutions that depend on the maximal number of independent physical parameters?

To investigate these issues, it is convenient to review in detail the construction of conventional extremal black-hole solutions: extremal black-holes are associated to values of the scalar fields $Z^i_{\rm h}$ (attractors) that extremize the black-hole potential [3]. As explained in the previous section, in the H-FGK formulation attractors appear as symplectic vectors B^M that extremize the black-hole potential when written in terms of the H-variables. These attractors B^M are defined up to normalization because the black-hole potential is invariant under rescalings of the H^Ms and also up to global Freudenthal rotations. Furthermore, as functions of the charges and moduli, the attractors B^M are equivariant under duality transformations. A family of extremal black holes closed under duality will be associated to a given equivariant vector expressed as a set of functions of the charge components and moduli $B^M(\mathcal{Q}, Z_\infty, Z^*_\infty)$. We are going to focus on moduli-independent attractors, *i.e.* the so-called *true attractors*.

The attractor B^M determines the near-horizon form of the solution. We can always construct a solution describing the $\mathrm{AdS}^2\times\mathrm{S}_2$ solution that describes the near-horizon geometry by choosing the appropriate normalization of B^M: indeed, one can check that the harmonic functions

$$H^M = -\tfrac{1}{\sqrt{2}} B^M \tau\,, \tag{3.3}$$

always satisfy the equations of motion as long as the condition

$$V_{\rm bh}(B,\mathcal{Q}) = -\tfrac{1}{2}\mathsf{W}(B)\,, \tag{3.4}$$

determining the normalization of B^M is met.

To construct a solution with the same near-horizon behavior and with an asymptotically-flat region we must add to the H^M above a constant vector A^M. The condition Eq. (3.2) and the normalization of the metric at infinity become two constraints for A^M

$$B^M A_M = 0\,, \qquad \mathsf{W}(A) = 1\,, \tag{3.5}$$

that leave $2n$ real constants, which is just the right amount to describe the asymptotic values of the n complex scalars Z^i_∞. Only if we cannot add a vector A^M satisfying these two constraints, then the most general solution associated to the attractor B^M cannot be conventional and we will have to add anharmonic terms to the H^M.

We can reformulate this question as follows: if we add to the H^M in Eq. (3.3) an infinitesimal vector ε^M satisfying $B^M\varepsilon_M = 0$, do we get another solution to the Hamiltonian constraint Eq. (1.3) and equations of motion Eq. (1.4)? To first order in ε^M, the Hamiltonian constraint will be solved by the perturbed solution

$$H'^M = H^M + \varepsilon^M\,, \qquad H^M = -\tfrac{1}{\sqrt{2}} B^M\tau\,, \qquad B^M\varepsilon_M = 0\,, \tag{3.6}$$

if

$$\varepsilon^M \left\{ \tfrac{1}{2}\partial_M g_{NP}\dot{H}^N\dot{H}^P + \partial_M V(H,\mathcal{Q}) \right\} = 0\,. \tag{3.7}$$

Evaluating this equation at the near-horizon solution H^M, using $V_{\rm bh}(H,\mathcal{Q}) = -\mathsf{W}(B)V(H,\mathcal{Q})$, the homogeneity properties of the different terms, the fact that $\partial_M V_{\rm bh}(B,\mathcal{Q}) = 0$ and the condition (3.4), we arrive at

$$\varepsilon^M \left\{\tfrac{1}{4}B^N B^P \partial_M\partial_N\partial_P \log \mathsf{W}(B) - \tfrac{1}{2}\partial_M \log \mathsf{W}(B)\right\} = 0\,, \tag{3.8}$$

which is an equation in the variables B^M (including the partial ∂_M derivatives, which should be understood as partial derivatives with respect to B^M) and is identically satisfied on account of the scale invariance of $\log \mathsf{W}(B)$.

The analogous condition on the equations of motion, Eqs. (1.4), reads

$$\varepsilon^M \left\{\partial_M g_{NP}\ddot{H}^P + \partial_M(\partial_P g_{QN} - \tfrac{1}{2}\partial_N g_{PQ})\dot{H}^P\dot{H}^Q + \partial_M\partial_N V(H,\mathcal{Q})\right\} = 0\,, \tag{3.9}$$

and, after evaluation on the near-horizon solution we get a homogenous equation that, again, can be read as an equation on the variables B^M. Using the same properties we used with the Hamiltonian constraint plus $B^M\varepsilon_M = 0$ we get a non-trivial equation for ε^M

$$\mathfrak{M}_{MN}\varepsilon^N = 0\,, \quad \text{with} \quad \mathfrak{M}_{MN} \equiv \mathsf{W}(B)\partial_M\partial_N \log \mathsf{W}(B) + 2\frac{\tilde{B}_M\tilde{B}_N}{\mathsf{W}(B)} - \partial_M\partial_N V_{\rm bh}(B,\mathcal{Q})\,. \tag{3.10}$$

We are interested in the number of independent solutions to this equation that satisfy the constraint $B^M\varepsilon_M = 0$, *i.e.* in the rank of $\mathfrak{M}_{MN}$. The rank should be at most 1 as this implies a single linear constraint on the components of ε^M, which should be equivalent to $B^M\varepsilon_M = 0$. If the rank of $\mathfrak{M}_{MN}$ happens to be bigger than 1, then there are not enough unconstrained components of ε^M for the family of solutions to have arbitrary values of the moduli and the most general solution based on the chosen attractor, must necessarily contain anharmonic terms.

For cubic models, the need of anharmonic ansätze to construct the most general, generating, non-supersymmetric, extremal, black-hole solution of [31] and [32] was first observed in [19] and later confirmed in [4] and [33]. In the next sections we will see how the obstruction to the fully harmonic ansatz arises in the particular case of the t^3 model. For the non-extremal case of these theories, the situation is still unclear [33].

4 The general first-order flow equations

The central charge of an $\mathcal{N}=2, d=4$ supergravity theory is defined by Eq. (1.14) and, in terms of the H-variables it takes the form of Eq. (1.15) which we copy here for convenience

$$\mathcal{Z}(\mathcal{Q}) = \frac{e^{-i\alpha}}{\sqrt{2\mathsf{W}}}(\tilde{H}_M + iH_M)\mathcal{Q}^M\,. \tag{4.1}$$

Let us consider a generalization of the central charge, denoted by $\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)$, in which we replace the second argument (the charge vector) by the Freudenthal-covariant derivative of H^M introduced in Ref. [30], *i.e.*

$$\mathfrak{D}H^M \equiv \dot{H}^M + A\tilde{H}^M\,, \qquad A \equiv \frac{\dot{H}^N H_N}{\mathsf{W}}\,. \tag{4.2}$$

Since $H_M\mathfrak{D}H^M = 0$ and $\tilde{H}_M\tilde{H}^M = 0$ identically, we immediately find that

$$|\mathcal{Z}(\phi,\sqrt{2}\mathfrak{D}H)| = \pm\frac{\tilde{H}_M\dot{H}^M}{\sqrt{\mathsf{W}}} = \pm\frac{\partial_M\mathsf{W}\,\dot{H}^M}{2\sqrt{\mathsf{W}}} = \pm\frac{d\sqrt{\mathsf{W}}}{d\tau} = \pm\frac{de^{-U}}{d\tau}\,, \tag{4.3}$$

which is the first-order equation for the metric function[13]. Observe that $H_M\mathfrak{D}H^M = 0$ implies that the phase of $\mathcal{Z}(\phi,\sqrt{2}\mathfrak{D}H)$ is equal to the phase of $\pm X$. The sign must be chosen so as to make $\pm\tilde{H}_M\dot{H}^M > 0$ and, since the mass of the solution corresponding to $e^{-2U} = \mathsf{W}(H)$ is given by

$$M = -\tfrac{1}{2}\left.\frac{de^{-2U}}{d\tau}\right|_{\tau=0} = -\tfrac{1}{2}\left.\dot{\mathsf{W}}\right|_{\tau=0} = -\left.\tilde{H}_M\dot{H}^M\right|_{\tau=0}\,, \tag{4.4}$$

we find that for regular solutions (with positive mass) we must choose the lower sign:

$$\frac{de^{-U}}{d\tau} = -|\mathcal{Z}(\phi,\sqrt{2}\mathfrak{D}H)|\,. \tag{4.5}$$

From Eq. (2.8) of Ref. [36] we have that

$$\frac{dZ^i}{d\tau} = -2X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{V}^*_M\dot{H}^M\,. \tag{4.6}$$

We can rewrite $\dot{H}^M$ as

$$\dot{H}^M = \mathfrak{D}H^M - A\tilde{H}^M = \mathfrak{D}H^M - A\left(\frac{\mathcal{V}^M}{2X} + \text{c.c.}\right)\,, \tag{4.7}$$

and plug it into the previous equation to get

$$\begin{aligned}\frac{dZ^i}{d\tau} &= -2X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{Z}^*(\phi,\mathfrak{D}H) = 4Xe^{-i\alpha}\mathcal{G}^{ij^*}\partial_{j^*}|\mathcal{Z}^*(\phi,\mathfrak{D}H)| \\ &= 2e^U\mathcal{G}^{ij^*}\partial_{j^*}|\mathcal{Z}^*(\phi,\sqrt{2}\mathfrak{D}H)|\,,\end{aligned} \tag{4.8}$$

where we have used Eq. (1.8) and the equality of the phases of $-X$ and $|\mathcal{Z}(\phi,\sqrt{2}\mathfrak{D}H)|$. This is the second first-order equation[14].

Some remarks are in order:

[13] This equation reduces to Eq. (5.9) of Ref. [19] in the extremal limit. Observe that the Freudenthal-covariant derivative corresponds to Eq. (5.6) of the same reference.

[14] Again, this equation reduces to Eq. (5.10) of Ref. [19] in the extremal limit.

1. In these derivations we have assumed neither extremality or non-extremality of the solutions nor any explicit form of the variables H^M (harmonic or hyperbolic)[15]. Furthermore, we have not assumed the Freudenthal gauge-fixing condition $\dot{H}^N H_N = 0$. Only the properties of Special Geometry encoded in the H-FGK formalism have been used. Therefore, the first-order Eqs. (4.5) and (4.8) apply to any static black-hole solution of ungauged $\mathcal{N} = 2, d = 4$ supergravity coupled to vector multiplets.

2. These first-order equations reduce to those found in the literature starting from Ref. [3] in the extremal/harmonic (*i.e.* $A = \dot{H}^N H_N = 0$) cases: if $H^M = A^M - \frac{1}{\sqrt{2}} B^M \tau$ for some constant symplectic vectors A^M (which encode the values of the scalars at spatial infinity) and the attractor B^M, then

$$|\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)| = |\mathcal{Z}(\phi, B)| \,, \tag{4.9}$$

which is known as *fake central charge* when $B^M \neq \mathcal{Q}^M$ and coincides with the central charge in the supersymmetric case $B^M = \mathcal{Q}^M$.

3. In the general (non-supersymmetric) case $\mathfrak{D}H$ will be τ-dependent and its near-horizon ($\tau \to -\infty$) and spatial infinity ($\tau \to 0^-$) limits, will not necessarily be equal: in the near-horizon limit $\lim_{\tau\to-\infty} \mathfrak{D}H^M \equiv -\frac{1}{\sqrt{2}} B^M$ and in the spacelike infinity limit $\lim_{\tau\to 0^-} \mathfrak{D}H^M \equiv -\frac{1}{\sqrt{2}} E^M$ and, generically, $B^M \neq E^M$.

$$M = -\lim_{\tau\to 0^-} \frac{de^{-U}}{d\tau} = |\mathcal{Z}(\phi_\infty, E)| \,, \tag{4.10}$$

$$S = \pi \left[\lim_{\tau\to -\infty} \frac{de^{-U}}{d\tau} \right]^2 = \pi |\mathcal{Z}(\phi_{\rm h}, B)|^2 \,, \tag{4.11}$$

where ϕ_∞ and $\phi_{\rm h}$ are the values of the scalars at spatial infinity and on the horizon, respectively. Different fake central charges $\mathcal{Z}(\phi, E)$ and $\mathcal{Z}(\phi, B)$ drive the metric function in the spatial-infinity and near-horizon regions, respectively. This behavior is present in the non-supersymmetric extremal solutions of the cubic models studied in Refs. [31, 37, 38, 19, 4] which have anharmonic H^Ms[16].

4. In Ref. [14] and subsequent literature the first-order flow equations were given in terms of superpotential functions $W(\phi, B)$ which depend only on a constant fake charge vector B^M and which has a structure similar, but not identical, to the central charge. Those first-order equations must be completely equivalent to Eqs. (4.5,4.8), because the same variables, for

[15] Actually, we have written *solutions* but we have not used at any moment the fact that the H^M solve the equations of motion. The first-order equations that we have derived are, therefore, valid for any configuration of the variables H^M, although their use is essentially limited to solutions.

[16] The H^Ms of those solutions do not satisfy the constraint $\dot{H}^M H_M = 0$. A change of Freudenthal gauge can bring the solutions to the $\dot{H}^M H_M = 0$ gauge but cannot make the H^M harmonic [30].

the same solution, cannot obey two different sets of first-order equations. We do not know how to prove this equivalence in general, and it will have to be checked case by case.

5 The axidilaton model

The axidilaton model is defined by the prepotential

$$\mathcal{F} = -i\mathcal{X}^0\mathcal{X}^1 \,, \tag{5.1}$$

and has only one complex scalar that we will denote by λ that is given by

$$\lambda \equiv i\mathcal{X}^1/\mathcal{X}^0 \,. \tag{5.2}$$

In terms of λ and in the $\mathcal{X}^0 = i/2$ gauge, the Kähler potential and metric are

$$\mathcal{K} = -\ln \Im\mathfrak{m}\lambda \,, \qquad \mathcal{G}_{\lambda\lambda^*} = (2\Im\mathfrak{m}\lambda)^{-2} \,, \tag{5.3}$$

and therefore λ, which must take values in the upper half complex plane, parametrizes the coset space $\mathrm{Sl}(2;\mathbb{R})/\mathrm{SO}(2)$.

The canonically-normalized covariantly-holomorphic symplectic section $\mathcal{V}$ is, in the gauge in which the Kähler potential is given by Eq. (5.3),

$$\mathcal{V} = \frac{1}{2(\Im\mathrm{m}\lambda)^{1/2}} \begin{pmatrix} i \\ \lambda \\ -i\lambda \\ 1 \end{pmatrix} \,, \tag{5.4}$$

and the central charge and its holomorphic covariant derivative are

$$\begin{aligned} \mathcal{Z}(\mathcal{Q}) &= \frac{1}{2\sqrt{\Im\mathrm{m}\lambda}} \left[(p^1 - iq_0) - (q_1 + ip^0)\lambda \right] \,, \\ \mathcal{D}_\lambda\mathcal{Z} &= \frac{i}{4(\Im\mathrm{m}\lambda)^{3/2}} \left[(p^1 - iq_0) - (q_1 + ip^0)\lambda^* \right] \,. \end{aligned} \tag{5.5}$$

It is useful to define the fake charge and associated fake central charge

$$\mathcal{P} \equiv \begin{pmatrix} p^0 \\ -p^1 \\ q_0 \\ -q_1 \end{pmatrix} \,, \qquad \mathcal{Z}(\mathcal{P}) \equiv \frac{1}{2\sqrt{\Im\mathrm{m}\lambda}} \left[(-p^1 - iq_0) - (-q_1 + ip^0)\lambda \right] \,, \tag{5.6}$$

in terms of which

$$\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* = |\mathcal{Z}(\mathcal{P})|^2 \,, \tag{5.7}$$

so that the black-hole potential takes the simple form

$$-V_{\rm bh} = |\mathcal{Z}(\mathcal{Q})|^2 + |\mathcal{Z}(\mathcal{P})|^2\,. \tag{5.8}$$

The black-hole solutions of this model have been exhaustively studied in Refs. [39, 40, 11, 5]. Our goal here is to illustrate the general results and methods described in the previous sections using this well-known model. First, let us recall what are the symmetries of this model in its original formulation.

5.1 The global symmetries of the axidilaton model

The full axidilaton model (and not just the axidilaton kinetic term) is invariant under global $\mathrm{Sl}(2;\mathbb{R})$ transformations. Let us start by describing the action of this group on the axidilaton field: parametrize a generic element of $\mathrm{Sl}(2;\mathbb{R})$ as

$$\Lambda \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \qquad \text{with} \quad ad - bd = 1\,, \tag{5.9}$$

then the axidilaton transforms as

$$\lambda' = \frac{a\lambda + b}{c\lambda + d}\,. \tag{5.10}$$

The scalar manifold metric admits 3 holomorphic Killing vectors which can be taken to be

$$K_1 = \lambda\partial_\lambda + \text{c.c.}\,, \qquad K_2 = \tfrac{1}{2}(1-\lambda^2)\partial_\lambda + \text{c.c.}\,, \qquad K_3 = \tfrac{1}{2}(1+\lambda^2)\partial_\lambda + \text{c.c.}\,, \tag{5.11}$$

and satisfy the commutation relations of the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$

$$[K_m, K_n] = \epsilon_{mnq}\eta^{qp}K_p\,, \quad \Rightarrow f_{mn}{}^{p} = -\epsilon_{mnq}\eta^{qp}, \quad (m,n,\ldots = 1,2,3)\,, \tag{5.12}$$

where $\epsilon_{123} = +1$ and $\eta = \mathrm{diag}(++-)$; η is proportional to the Killing metric of $\mathfrak{so}(1,2) \simeq \mathfrak{sl}(2;\mathbb{R}) \simeq \mathfrak{sp}(2;\mathbb{R})$. The infinitesimal $\mathrm{Sl}(2;\mathbb{R})$ transformations of λ can be written using these Killing vectors as

$$\delta_\alpha\lambda = \alpha^m k_m{}^{\lambda} = \tfrac{1}{2}(\alpha^2+\alpha^3) + \alpha^1\lambda - \tfrac{1}{2}(\alpha^2-\alpha^3)\lambda^2\,. \tag{5.13}$$

The infinitesimal linear transformations associated to the above choice of Killing vectors is, in terms of the Pauli matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \mathbb{1}_{2\times 2} + \alpha^m T_m\,, \qquad T_1 = -\tfrac{1}{2}\sigma^3\,,\ T_2 = -\tfrac{1}{2}\sigma^1\,,\ T_3 = \tfrac{i}{2}\sigma^2\,, \tag{5.14}$$

and satisfy the Lie algebra

$$[T_m, T_n] = -\epsilon_{mnq}\eta^{qp}T_p\,. \tag{5.15}$$

The action of the finite $\mathrm{Sl}(2;\mathbb{R})$ transformations on the Kähler potential and on the canonical covariantly-holomorphic symplectic section $\mathcal{V}$ given in Eq. (5.4) is

$$\mathcal{K}'(\lambda) \equiv \mathcal{K}(\lambda'(\lambda)) = \mathcal{K}(\lambda) + 2\mathfrak{Re} f(\lambda)\,, \tag{5.16}$$

$$\mathcal{V}'^{M}(\lambda) \equiv \mathcal{V}^{M}(\lambda'(\lambda)) = e^{-i\mathfrak{Im} f(\lambda)}\, S^{M}{}_{N}\mathcal{V}^{N}\,, \tag{5.17}$$

where the holomorphic function $f(\lambda)$ of the Kähler transformation and the symplectic rotation $S^{M}{}_{N}$ are given by

$$f(\lambda) = \ln(c\lambda + d)\,, \tag{5.18}$$

$$(S^{M}{}_{N}) = \begin{pmatrix} d & & -c & \\ & a & & b \\ -b & & a & \\ & c & & d \end{pmatrix}. \tag{5.19}$$

In this 4-dimensional representation the infinitesimal generators T_m are given by

$$(T_1{}^{M}{}_{N}) = -\tfrac{1}{2}\begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}, \quad (T_2{}^{M}{}_{N}) = -\tfrac{1}{2}\begin{pmatrix} & \sigma^3 \\ \sigma^3 & \end{pmatrix}, \quad (T_3{}^{M}{}_{N}) = \tfrac{1}{2}\begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix}. \tag{5.20}$$

The same transformations act on all the symplectic vectors of the theory and, in particular, on the variables H^M and the charge vectors $\mathcal{Q}^M$. In this formulation of the axidilaton system there seem to be no further symmetries[17].

5.1.1 Equivariant vectors of the axidilaton model

In this model there is no need to solve any equation to find 4 linearly independent equivariant vectors: observe that the symplectic vector of charges is the direct sum of two real $\mathrm{Sl}(2;\mathbb{R})$ doublets a^i and b_i ($i, j = 1, 2$), namely

$$(a^i) \equiv \begin{pmatrix} p^1 \\ q_1 \end{pmatrix}, \qquad (b_i) \equiv (p^0,\, q_0)\,. \tag{5.21}$$

These doublets transform respectively contravariantly and covariantly, that is

[17] We will see, however, that there is an additional $\mathrm{U}(1)$ factor in the symmetry group that only has a non-trivial action on objects with symplectic indices and that coincides with the continuous global Freudenthal duality transformation. The scalars do not transform under this symmetry. On the other hand, only this $\mathrm{U}(1)$ symmetry is also a local symmetry of the H-FGK formalism. We would like to thank Alessio Marrani for clarifying discussions on this point.

$$a'^i = \Lambda^i{}_j\, a^j\,, \qquad b'_i = b_j\, (\Lambda^{-1})^j{}_i\,, \tag{5.22}$$

where $(\Lambda^i{}_j)$ is the matrix given in Eq. (5.9), which furthermore satisfies

$$(\Lambda^{-1})^i{}_j = \Omega^{ki}\, \Lambda^l{}_k\, \Omega_{lj}\,, \qquad (\Omega_{ij}) = (\Omega^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{5.23}$$

because $\mathrm{Sl}(2;\mathbb{R}) \simeq \mathrm{Sp}(2;\mathbb{R})$. We can use the symplectic metric Ω to raise and lower doublet indices such as i and j using the same convention we use for the symplectic indices (see footnote 4), namely $a_i \equiv \Omega_{ij} a^j$ and $b^i = b_j \Omega^{ji}$. The only non-vanishing $\mathrm{Sl}(2;\mathbb{R})$ invariant that can be built out of these two doublets is

$$a^i b_i = p^0 p^1 + q_0 q_1 \equiv \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\,. \tag{5.24}$$

Let us denote by $\mathcal{Q}^M(a,b)$ the standard symplectic charge vector seen as the direct sum of the two doublets a and b. Using the two doublets we can construct three further, up to a global sign, inequivalent charge vectors that under $\mathrm{Sl}(2;\mathbb{R})$ transform in the same way as $\mathcal{Q}^M(a,b)$, *i.e.* equivariantly; the four equivariant charge vectors are

$$\begin{aligned}
\mathcal{Q}^M(a,b) &\equiv \begin{pmatrix} p^0 \\ p^1 \\ q_0 \\ q_1 \end{pmatrix}, \qquad & \mathcal{Q}^M(b,-a) &\equiv \begin{pmatrix} -q_1 \\ -q_0 \\ p^1 \\ p^0 \end{pmatrix}, \\
\mathcal{Q}^M(-a,b) &\equiv \begin{pmatrix} p^0 \\ -p^1 \\ q_0 \\ -q_1 \end{pmatrix}, \qquad & \mathcal{Q}^M(-b,-a) &\equiv \begin{pmatrix} -q_1 \\ q_0 \\ p^1 \\ -p^0 \end{pmatrix}.
\end{aligned} \tag{5.25}$$

These equivariant vectors are generically linearly independent and provide a basis of equivariant vectors; any other equivariant vector, in particular the attractors B^M, can be expanded w.r.t. this base, *e.g.*

$$B^M = b^\sigma U_\sigma^M\,, \quad \text{with} \quad \{U_\sigma\} = \{\mathcal{Q}, \tilde{\mathcal{Q}}, \mathcal{P}, \tilde{\mathcal{P}}\}\,. \tag{5.26}$$

We will plug this general ansatz into the equation $\partial_M V_{\mathrm{bh}}(H,\mathcal{Q})|_{H=B} = 0$ as to find the most general attractor of the theory in Section 5.4, but at this point we already know some general results: The standard charge vector $\mathcal{Q}^M(a,b)$ will be the supersymmetric attractor, as usual, and we are going to see, $\mathcal{Q}^M(b,-a)$ is its Freudenthal dual

$$\mathcal{Q}^M(b,-a) = \tilde{\mathcal{Q}}^M(a,b) = \tilde{\mathcal{Q}}^M\,. \tag{5.27}$$

On the other hand, $\mathcal{Q}^M(-a,b)$ is the non-supersymmetric attractor $\mathcal{P}^M$ and $\mathcal{Q}^M(b,a)$ is its Freudenthal dual

$$\mathcal{Q}^M(-a,b) = \mathcal{P}^M\,, \qquad \mathcal{Q}^M(b,a) = \tilde{\mathcal{Q}}^M(b,-a) = \tilde{\mathcal{P}}^M\,. \tag{5.28}$$

It is easy to see that

$$\mathsf{W}(\tilde{\mathcal{Q}}) = \mathsf{W}(\mathcal{Q}) = -\mathsf{W}(\mathcal{P}) = -\mathsf{W}(\tilde{\mathcal{P}})\,. \tag{5.29}$$

These four vectors are related by $\mathrm{Sp}(4;\mathbb{R})$ transformations that however do not belong to $\mathrm{Sl}(2;\mathbb{R}) \subset \mathrm{Sp}(4;\mathbb{R})$:

$$\tilde{\mathcal{Q}}^M = \mathcal{A}^M{}_N \mathcal{Q}^N\,, \qquad (\mathcal{A}^M{}_N) \equiv \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\,, \tag{5.30}$$

$$\mathcal{P}^M = \mathcal{B}^M{}_N \mathcal{Q}^N\,, \qquad (\mathcal{B}^M{}_N) \equiv \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\,. \tag{5.31}$$

The only non-vanishing symplectic contractions between these four vectors are

$$\tilde{\mathcal{Q}}_M \mathcal{Q}^M = -\tilde{\mathcal{P}}_M \mathcal{P}^M = \mathsf{W}(\mathcal{Q})\,. \tag{5.32}$$

Apart from these moduli-independent equivariant vectors we can construct the generic moduli-dependent ones by taking the real and imaginary parts of Eq. (2.15), in which we can replace $\mathcal{Q}$ by any of the other three equivariant vectors. Observe that when we use the Freudenthal dual charge, we obtain the same complex equivariant vector but multiplied by $-i$.

5.2 H-FGK formalism

The solution of the stabilization equations of this theory is

$$\mathcal{R}_M(\mathcal{I}) = \mathcal{A}_{MN}\mathcal{I}^N\,, \qquad (\mathcal{A}_{MN}) \equiv \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\,, \tag{5.33}$$

where σ^1 is the standard Pauli matrix. $\mathcal{A} = (\mathcal{A}_{MN})$ is a symplectic matrix:

$$\mathcal{A}\Omega\mathcal{A} = \Omega\,, \tag{5.34}$$

which is not surprising since it is just $-\mathcal{M}_{MN}(\mathcal{F})$. It follows that $(\mathcal{A}^M{}_N) = (\Omega^{PM}\mathcal{A}_{PN}) = -\Omega\mathcal{A}$ is also a symplectic matrix.

By definition, the original and tilded, *i.e.* Freudenthal dual, H-variables are related by[18]

$$\tilde{H}_M(H) = \mathcal{A}_{MN}H^N\,, \qquad \tilde{H}^M(H) = \mathcal{A}^M{}_N H^N\,. \tag{5.36}$$

[18]Explicitly, we have

$$(\tilde{H}^M) = \begin{pmatrix} -\sigma^{1\,\Lambda\Sigma}H_\Sigma \\ \sigma^1{}_{\Lambda\Sigma}H^\Sigma \end{pmatrix} = \begin{pmatrix} -H_1 \\ -H_0 \\ H^1 \\ H^0 \end{pmatrix}\,. \tag{5.35}$$

This vector should be compared with $\mathcal{Q}^M(b,-a)$ in Eq. (5.25).

Therefore in this simple model the Freudenthal duality transformation is linear and is, furthermore, a symplectic transformation. It is clearly a transformation that does not belong to the global symmetries that act on the axidilaton (i.e. $\mathrm{Sl}(2;\mathbb{R})$ whose embedding into $\mathrm{Sp}(4;\mathbb{R})$ is given in Eq. (5.19)), but it is a symmetry transformation that acts on objects with symplectic indices such as the vector fields and as such must be considered a part of the duality group of the model[19].

As expected in Freudenthal duality

$$\mathcal{A}^M{}_P\,\mathcal{A}^P{}_N = -\delta^M{}_N\,. \tag{5.37}$$

We can extend the Freudenthal duality transformation to all symplectic vectors. The properties

$$\tilde{X}_M Y^M = \tilde{Y}_M X^M = -Y_M\tilde{X}^M\,, \quad \Rightarrow \quad \tilde{X}_M\tilde{Y}^M = X_M Y^M\,, \tag{5.38}$$

which hold in this particular model for any two symplectic vectors X^M and Y^M because Freudenthal duality is a symplectic transformation, will be used very often.

The Hesse potential is given by the $\mathrm{Sl}(2;\mathbb{R})$ invariant discussed in earlier sections

$$\mathsf{W}(H) \equiv \tilde{H}_M(H)H^M = \mathcal{A}_{MN}H^M H^N = 2(H^0H^1 + H_0H_1)\,, \tag{5.39}$$

and in accordance with the general formalism it determines the model completely: the effective action can be constructed entirely from it and the metric function e^{-2U} and the axidilaton λ are related to the Hesse potential by

$$e^{-2U} = \mathsf{W}(H)\,, \qquad \lambda \equiv iZ = i\frac{\tilde{H}^1 + iH^1}{\tilde{H}^0 + iH^0} = \frac{H^1 + iH_0}{H_1 - iH^0}\,. \tag{5.40}$$

The metric $g_{MN}(H)$ of this system can be written in the form

$$g_{MN} = 2\,\mathfrak{N}_{MNPQ}\,\frac{H^PH^Q}{\mathsf{W}^2}\,, \tag{5.41}$$

where we have defined the constant matrix

$$\mathfrak{N}_{MNPQ} \equiv \mathcal{A}_{MN}\mathcal{A}_{PQ} - 2\mathcal{A}_{MP}\mathcal{A}_{NQ} - \Omega_{MP}\Omega_{NQ}\,. \tag{5.42}$$

Using this notation, the derivatives of the metric take the form

$$\partial_M g_{PQ} = -4\frac{\tilde{H}_M}{\mathsf{W}}g_{PQ} + 4\mathfrak{N}_{PQ(MR)}\frac{H^R}{\mathsf{W}^2}\,, \tag{5.43}$$

and the Christoffel symbols of the first kind are given by[20]

[19] See footnote 17.

[20] We remind the reader that the metric $g_{MN}(H)$ is not invertible, so we cannot use the standard Christoffel symbols $\Gamma_{PQ}{}^M \equiv g^{NM}[PQ,M]$.

$$
\begin{aligned}
[PQ, M] &= 2\frac{\tilde{H}_M g_{PQ} - \tilde{H}_P g_{QM} - \tilde{H}_Q g_{PM}}{\mathsf{W}} \\
&\quad -[6\mathcal{A}_{PQ}\mathcal{A}_{MR} - 4\mathcal{A}_{M(P}\mathcal{A}_{Q)R} + 4\Omega_{M(P}\Omega_{Q)R}]\frac{H^R}{\mathsf{W}^2}\,.
\end{aligned} \tag{5.44}
$$

It is easy to check that $\tilde{H}^M[PQ, M] = 0$, as required by Freudenthal duality invariance.

The potential V can be written in the convenient form

$$
\mathsf{W}^2 V(H, \mathcal{Q}) = -\tfrac{1}{2}\mathsf{W}(\mathcal{Q})\mathsf{W} + (H^M\tilde{\mathcal{Q}}_M)^2 + (H^M\mathcal{Q}_M)^2\,, \tag{5.45}
$$

and its derivative reads

$$
\partial_M V = -4\frac{\tilde{H}_M}{\mathsf{W}}\left[V + \tfrac{1}{4}\frac{\mathsf{W}(\mathcal{Q})}{\mathsf{W}}\right] + 2(\mathcal{Q}_M\mathcal{Q}_N + \tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N)\frac{H^N}{\mathsf{W}^2}\,; \tag{5.46}
$$

using the properties Eq. (5.38) it is easy to see that $\tilde{H}^M\partial_M V = 0$, which is the last requirement for having local Freudenthal duality [30].

Observe that, in this model, a Freudenthal duality transformation of the charge vectors *only* (that is: not of the variables H^M), not only preserves $\mathsf{W}(\mathcal{Q})$ but also the complete potential and black-hole potential, *i.e.*

$$
\mathsf{W}(\tilde{\mathcal{Q}}) = \mathsf{W}(\mathcal{Q}) \quad\Rightarrow\quad V(H, \tilde{\mathcal{Q}}) = V(H, \mathcal{Q})\,, \quad \text{and} \quad V_{\rm bh}(H, \tilde{\mathcal{Q}}) = V_{\rm bh}(H, \mathcal{Q})\,. \tag{5.47}
$$

On the other hand, using the definition of the fake charge Eq. (5.6) one can show that for any values of H^M

$$
-V_{\rm bh}(\mathcal{Q}) = -\tfrac{1}{2}\mathsf{W}(\mathcal{Q}) + 2|\mathcal{Z}(\mathcal{Q})|^2 = -\tfrac{1}{2}\mathsf{W}(\mathcal{P}) + 2|\mathcal{Z}(\mathcal{P})|^2 = -V_{\rm bh}(\mathcal{P})\,, \tag{5.48}
$$

$$
|\mathcal{Z}(\mathcal{P})|^2 = |\mathcal{Z}(\mathcal{Q})|^2 - \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\,. \tag{5.49}
$$

(5.50)

The first identity means that, if $\mathcal{Q}$ is an attractor, so will $\mathcal{P}$. The fact that it is an identity for arbitrary values of H^M means that replacing $\mathcal{Q}$ by $\mathcal{P}$ in an extremal solution gives another extremal solution with the attractor $\mathcal{P}$. The second identity is a consequence of the first and implies that

$$
\begin{aligned}
\mathsf{W}(\mathcal{Q}) &< 0\,, \;\Rightarrow\; |\mathcal{Z}(\mathcal{P})| > |\mathcal{Z}(\mathcal{Q})|\,, \\
\mathsf{W}(\mathcal{Q}) &> 0\,, \;\Rightarrow\; |\mathcal{Z}(\mathcal{Q})| > |\mathcal{Z}(\mathcal{P})|\,,
\end{aligned} \tag{5.51}
$$

for all values of H^M. The second case should correspond to the supersymmetric attractor in which the evaporation process stops when the mass equals the largest central charge, which in this case is the true one.

Finally, observe that this black-hole potential satisfies the curious interchange property

$$V_{\rm bh}(H,\mathcal{Q}) = \frac{\mathsf{W}(H)}{\mathsf{W}(\mathcal{Q})}\, V_{\rm bh}(\mathcal{Q},H)\,. \tag{5.52}$$

5.3 The symmetries in the H-FGK formalism

In Section 5.1 we discussed the global symmetries of the axidilaton model (more precisely, of its scalar manifold metric) when it is described in terms of the standard fields and have studied the embedding of these symmetries into $\mathrm{Sp}(4;\mathbb{R})$. It is in this form that we expect these symmetries to be present in the H-FGK formalism. On the other hand, there may be additional non-obvious symmetries such as Freudenthal duality (which is in general non-linear) in the H-FGK formalism.

Let us consider first the kinetic term: if we consider only linear transformations of the H^M

$$\delta H^M = T^M{}_N H^N\,, \tag{5.53}$$

it is evident that they will leave the kinetic term invariant if they are symplectic transformations, *i.e.*

$$\Omega_{P(M}T^P{}_{N)} = 0\,, \tag{5.54}$$

and are furthermore symmetries of the Hesse potential

$$\delta\mathsf{W} = 2\tilde{H}_M \delta H^M = 2\tilde{H}_M\, T^M{}_N\, H^N = 0 \quad\longrightarrow\quad [\Omega\mathcal{A}, T] = \beta \mathbb{1}_{4\times 4}\,, \tag{5.55}$$

where β is a real constant that can vanish. It is not difficult to see that for infinitesimal symplectic transformations, β must indeed vanish, and the only independent generators that solve the above equation are the three $\mathrm{Sl}(2;\mathbb{R})$ generators T_i given in Eq. (5.20) plus

$$T_4 = \tfrac{1}{2}\mathcal{A}\Omega\,, \tag{5.56}$$

which generates the Freudenthal transformations and commutes with the generators of $\mathrm{Sl}(2;\mathbb{R})$[21].

It can be checked that these symmetries leave invariant the metric g_{MN}. Actually, the metric is invariant under the constant rescalings of the H^M

$$T_5 \equiv \tfrac{1}{4}\mathbb{1}_{4\times 4}\,, \tag{5.57}$$

which are not symplectic transformations and leave the Hesse potential invariant only up to a multiplicative constant, in the same way as the Kähler potential is invariant under isometries of the Kähler metric only up to Kähler transformations.

We can study now the invariance of the potential using the expression for $\partial_M V$ given in Eq. (5.46). The first term cancels for $i = 1,2,3,4$ (we do not need to check $i = 5$: the potential is homogeneous of degree -2 and $\delta_5 V = -2V \neq 0$ in general) and the rest gives

[21] it is not difficult to see that the Hesse potential of the axidilaton model is not determined by $\mathrm{Sl}(2;\mathbb{R})$ invariance alone: one must require invariance under Freudenthal duality.

$$\delta_i V = -2H^N T_i{}^M{}_N(\mathcal{Q}_M\mathcal{Q}_N + \tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N)\frac{H^N}{\mathsf{W}^2}\,, \tag{5.58}$$

which vanishes only for the Freudenthal transformation $i = 4$ unless we also perform the same transformation on the charge vector: this means that $\mathrm{Sl}(2;\mathbb{R})$ is only a *pseudo-symmetry* of the system, since the constants that enter the action are rotated. The charges appear as integration constants of the solution of the equations of motion for the electrostatic and magnetostatic potentials in Ref. [3] and $\mathrm{Sl}(2;\mathbb{R})$ is probably a (standard) symmetry of the effective theory before that.

There are no conserved quantities associated to pseudo-symmetries, whence there is only one conserved current: the one associated to the Freudenthal duality. This current vanishes, however, identically, which is a generic feature of the formalism.

5.4 Critical points

The critical points of this model are equivariant vectors B^M satisfying the equations

$$\partial_M V_{\rm bh}|_{H=B} = -2\frac{\tilde{B}_M}{\mathsf{W}(B)}\left[V_{\rm bh}(B,\mathcal{Q}) - \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\right] - 2(\mathcal{Q}_M\mathcal{Q}_N + \tilde{\mathcal{Q}}_M\tilde{\mathcal{Q}}_N)\frac{B^N}{\mathsf{W}(B)} = 0\,. \tag{5.59}$$

Using the basis of equivariant vectors $\{U_\sigma\} = \{\mathcal{Q}, \tilde{\mathcal{Q}}, \mathcal{P}, \tilde{\mathcal{P}}\}$ constructed in Section 5.1.1, we can write any such solution as

$$B^M = a\mathcal{Q}^M + \tilde{a}\tilde{\mathcal{Q}}^M + b\mathcal{P}^M + \tilde{b}\tilde{\mathcal{P}}^M\,. \tag{5.60}$$

The only non-vanishing symplectic products of the four basis vectors are

$$\tilde{\mathcal{Q}}_M\mathcal{Q}^M = \mathsf{W}(\mathcal{Q})\,, \qquad \tilde{\mathcal{P}}_M\mathcal{P}^M = -\mathsf{W}(\mathcal{Q})\,, \tag{5.61}$$

and a very simple calculation gives

$$\begin{aligned}\partial_M V_{\rm bh}|_{H=B} \;\; = \;\; & \frac{-2}{(a^2+\tilde{a}^2-b^2-\tilde{b}^2)}\Big\{\tilde{a}(b^2+\tilde{b}^2)\mathcal{Q}_M - a(b^2+\tilde{b}^2)\tilde{\mathcal{Q}}_M \\ & +\tilde{b}(a^2+\tilde{a}^2)\mathcal{P}_M - b(a^2+\tilde{a}^2)\tilde{\mathcal{P}}_M\Big\} = 0\,,\end{aligned} \tag{5.62}$$

which only admits two non-trivial solutions: $b = \tilde{b} = 0$ and $a = \tilde{a} = 0$. The first solution, up to global normalization (which is undetermined in this formalism because the black-hole potential is scale-invariant), corresponds to a global Freudenthal rotation with arbitrary angle of the standard supersymmetric attractor $B^M = \mathcal{Q}^M$ and the second corresponds to a global Freudenthal rotation of the standard non-supersymmetric attractor $B^M = \mathcal{P}^M$ [5].

We obtain the following relations

$$V_{\rm bh}(\mathcal{P},\mathcal{P}) = -V_{\rm bh}(\mathcal{Q},\mathcal{P}) = V_{\rm bh}(\mathcal{P},\mathcal{Q}) = -V_{\rm bh}(\mathcal{Q},\mathcal{Q}) = \tfrac{1}{2}\mathsf{W}(\mathcal{Q})\,, \tag{5.63}$$

that are necessary to have the corresponding near-horizon solutions, see Eq. (3.4).

5.5 Conventional extremal solutions

As a first simple illustration of the methods proposed in the first section of this paper, we are going to review the construction of the extremal solutions[22] performed in Ref. [9].

From the results of that paper we know that all of them (including the extremal non-supersymmetric ones) are going to be conventional, but it is important for us to understand why. Thus, we start from the near-horizon solutions given by Eq. (3.3) where B^M takes the values of the attractors found in the previous section, normalized so that (see Eq. (3.4))

$$V_{\rm bh}(B,\mathcal{Q}) = V_{\rm bh}(B,B) = -\tfrac{1}{2}\mathsf{W}(B)\,. \tag{5.64}$$

The attractors that satisfy these conditions are global Freudenthal rotations of the standard supersymmetric attractor $\mathcal{Q}^M$ and of the non-supersymmetric one $\mathcal{P}^M$, *i.e.*

$$\begin{aligned} \text{either} \qquad B^M &= \cos\theta\,\mathcal{Q}^M + \sin\theta\,\tilde{\mathcal{Q}}^M\,, \\ \text{or} \qquad B^M &= \cos\theta\,\mathcal{P}^M + \sin\theta\,\tilde{\mathcal{P}}^M\,. \end{aligned} \tag{5.65}$$

The results of Section (3) guarantee that Eq. (3.3) provides a near-horizon solution for these choices of B^M. Now, to see if we can extend these solutions to asymptotically flat solutions by adding an infinitesimal constant vector to these H^M as in Eq. (3.6), we have to compute the rank of $\mathfrak{M}_{MN}$ in Eq. (3.10) to find how many independent solutions ε^M exist.

It is enough to consider a charge configuration whose orbit covers the complete charge space (see Appendix A) and, therefore, we set $p^0 = p^1 = 0$, getting, for the supersymmetric (+) and non-supersymmetric (−) cases, the matrix

$$(\mathfrak{M}_{MN}) = \tfrac{1}{2}\begin{pmatrix} \frac{1}{q_1^2} & \pm\frac{1}{q_0q_1} & 0 & 0 \\ \pm\frac{1}{q_0q_1} & \frac{1}{q_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.66}$$

This matrix has rank 1 and, furthermore, the three independent solutions to Eq. (3.10) satisfy the constraint $B^M\varepsilon_M = 0$. This means that there is no obstruction to the addition of arbitrary (up to normalization $\mathsf{W}(A) = 1$ and the condition $B^M A_M = 0$) constants A^M to the near-horizon harmonic functions, which now take the form

[22] The axidilaton model is a particular case ($n = 1$) of the $\overline{\mathbb{CP}}^n$ model. We will construct the most general non-extremal solutions of that model (and, hence, of the axidilaton model) later.

$$H^M = A^M - \tfrac{1}{\sqrt{2}} B^M \tau \,. \tag{5.67}$$

The two independent components of A^M describe the two real moduli of this theory $\Re\mathfrak{e}(\lambda_\infty)$, $\Im\mathfrak{m}(\lambda_\infty)$ and A^M is given by [9]

$$A^M = \sqrt{2}\,\Im\mathfrak{m}\left(\frac{\mathcal{Z}^*(\phi_\infty, B)}{|\mathcal{Z}(\phi_\infty, B)|}\mathcal{V}^M_\infty\right) \,. \tag{5.68}$$

To show that the equations of motion are satisfied for finite constants A^M (which is only needed in the non-supersymmetric case) we can proceed as follows: from the linearity of the H^M it is possible to show that these configurations satisfy first-order flow equations [36]. These, in turn can be shown to imply the standard second-order equations of motion if and only if the identity

$$V_{\rm bh}(H, \mathcal{Q}) = V_{\rm bh}(H, B) \,, \tag{5.69}$$

is satisfied for arbitrary values of H. This is evident for $B^M = \mathcal{Q}^M$ (the supersymmetric attractor) and has been shown for $B^M = \mathcal{P}^M$ (the non-supersymmetric attractor) in Eq. (5.8) and the invariance of the black-hole potential under Freudenthal transformations of the charges extends this result to the other two (physically indistinguishable) attractors and proves that these configurations are classical solutions of the model.

5.6 Unconventional solutions

We do not expect more extremal black-hole solutions to the axidilaton model since the solutions constructed in the previous section already have the maximal number of independent physical parameters (charges $\mathcal{Q}^M$ and moduli λ_∞) which are constrained only by the requirement that the horizon has a non-vanishing area, *i.e.* $\mathsf{W}(B) > 0$.

On the other hand, we can rewrite these solutions in an unconventional form (*i.e.* so that $\dot{H}^M H_M \neq 0$) by using local Freudenthal duality transformations, but in this case doing so merely complicates the form of the solution in the H-FGK formalism.

6 The $\overline{\mathbb{CP}}^n$ model

The prepotential of the $\overline{\mathbb{CP}}^n$ model is given by[23]

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma \,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-) \,. \tag{6.1}$$

The $\overline{\mathbb{CP}}^n$ model contains n scalar fields given by

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0 \,, \tag{6.2}$$

[23]The black-hole solutions of this model have been studied in [5].

but it is convenient to add $Z^0 \equiv 1$ and we define

$$(Z^\Lambda) \equiv \left(\mathcal{X}^\Lambda/\mathcal{X}^0\right) = (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (\eta_{\Lambda\Sigma}Z^\Sigma) = (1, Z_i) = (1, -Z^i)\,. \tag{6.3}$$

The Kähler potential, the Kähler metric, the inverse Kähler metric and the covariantly holomorphic symplectic section read

$$\begin{aligned}
\mathcal{K} &= -\log\left(Z^{*\Lambda}Z_\Lambda\right)\,,\\
\mathcal{G}_{ij^*} &= -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z^*_i Z_{j^*}\right)\,,\\
\mathcal{G}^{ij^*} &= -e^{-\mathcal{K}}\left(\eta^{ij^*} + Z^i Z^{*j^*}\right)\,,\\
\mathcal{V} &= e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2}Z_\Lambda \end{pmatrix}\,.
\end{aligned} \tag{6.4}$$

It is also convenient to define the following complex charge combinations

$$\Gamma_\Lambda \equiv q_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}p^\Sigma\,, \tag{6.5}$$

in terms of which the central charge, its holomorphic Kähler-covariant derivative and the black-hole potential are

$$\begin{aligned}
\mathcal{Z} &= e^{\mathcal{K}/2}Z^\Lambda\Gamma_\Lambda \equiv \mathcal{Z}(\Gamma)\,,\\
\mathcal{D}_i\mathcal{Z} &= e^{3\mathcal{K}/2}Z^*_i Z^\Lambda\Gamma_\Lambda - e^{\mathcal{K}/2}\Gamma_i\,,\\
-V_{\rm bh} &= 2e^{\mathcal{K}}|Z^\Lambda\Gamma_\Lambda|^2 - \Gamma^{*\,\Lambda}\Gamma_\Lambda\,.
\end{aligned} \tag{6.6}$$

We can extend this complex notation to any symplectic vector:

$$\text{if}\quad (A^M) = \begin{pmatrix} a^\Lambda \\ b_\Lambda \end{pmatrix} \quad \text{then} \quad \left\{\begin{array}{lcl} \mathcal{A}_\Lambda & \equiv & b_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma}a^\Sigma\,,\\ \\ \mathcal{A}^\Lambda & \equiv & \eta^{\Lambda\Sigma}\mathcal{A}_\Sigma \;=\; \eta^{\Lambda\Sigma}b_\Sigma + \frac{i}{2}a^\Lambda\,, \end{array}\right. \tag{6.7}$$

and the symplectic product of two vectors becomes

$$A_M B^M = -2\Im\mathfrak{m}(\mathcal{A}_\Lambda\mathcal{B}^{*\,\Lambda})\,, \tag{6.8}$$

where of course $\mathcal{A}_\Lambda\mathcal{B}^{*\,\Lambda} = \mathcal{A}^\Lambda\mathcal{B}^*{}_\Lambda$. We will use both notations, based on convenience.

6.1 The global symmetries of the $\overline{\mathbb{CP}}^n$ model

The n complex scalars of the $\overline{\mathbb{CP}}^n$ model parametrize the symmetric coset space $\mathrm{SU}(1,n)/\mathrm{SU}(n)$, and the full theory is invariant under global SU$(1,n)$ transformations[24]. If $\Lambda^\Lambda{}_\Sigma$ is a generic element in the fundamental representation of $\mathrm{SU}(1,n)$, *i.e.* if it satisfies

$$\Lambda^{*\Gamma}{}_\Lambda\,\eta_{\Gamma\Delta}\,\Lambda^\Delta{}_\Sigma = \eta_{\Lambda\Sigma}\,, \qquad (\text{or} \quad \Lambda^\dagger\eta\Lambda = \eta)\,, \qquad \det\Lambda = 1\,, \tag{6.9}$$

then its action on the scalars is given by

$$Z'^{\,\Lambda} = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,, \qquad Z'_{\,\Lambda} = \frac{\Lambda_\Lambda{}^\Sigma Z_\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,, \tag{6.10}$$

where we have raised and lowered the indices of the $\mathrm{SU}(1,n)$ matrix with the metric η. In the fundamental representation the $n(n+2)$ infinitesimal generators of $\mathfrak{su}(1,n)$

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\,T_m{}^\Lambda{}_\Sigma\,, \tag{6.11}$$

are matrices such that $T_{m\,\Lambda\Sigma} = \eta_{\Lambda\Gamma}\,T_m{}^\Gamma{}_\Sigma$ is anti-Hermitean. Substituting the infinitesimal linear transformations in the non-linear transformation rules of the scalars, Eq. (6.10), we find that they take the form

$$Z'^{\,\Lambda} = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z)\,, \tag{6.12}$$

where $k_m{}^\Lambda(Z)$, the holomorphic part of the Killing vectors K_m, is given by[25]

$$k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\,Z^\Sigma - T_m{}^0{}_\Omega\,Z^\Omega Z^\Lambda\,. \tag{6.13}$$

The commutation relations of the generators T_m and the Lie brackets of the Killing vectors are related as usual:

$$[T_m, T_n] = f_{mn}{}^p\,T_p\,, \qquad [K_m, K_n] = -f_{mn}{}^p\,K_p\,. \tag{6.14}$$

The action of the finite $\mathrm{SU}(1,n)$ transformations on the Kähler potential and on the canonical covariantly-holomorphic symplectic section $\mathcal{V}$ are given by the obvious generalization of Eqs. (5.16) and (5.17) where now

$$f(Z) \;=\; \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right)\,, \tag{6.15}$$

$$(S^M{}_N) \;=\; \begin{pmatrix} \mathfrak{Re}\Lambda^\Lambda{}_\Sigma & -2\mathfrak{Im}\Lambda^{\Lambda\Sigma} \\ \tfrac{1}{2}\mathfrak{Im}\Lambda_{\Lambda\Sigma} & \mathfrak{Re}\Lambda_\Lambda{}^\Sigma \end{pmatrix}\,, \tag{6.16}$$

[24]Actually, the coset space can also be described as $\mathrm{U}(1,n)/\mathrm{U}(n)$, which would imply that the global symmetry group of the model is $\mathrm{U}(1,n)$. As in the axidilaton model (the $n=1$ case), the extra $\mathrm{U}(1)$, that does not act on the scalars, is the Freudenthal duality group (see footnote 17). We thank Alessio Marrani for clarifying discussions on this point.

[25]The $\Lambda = 0$ component vanishes, as it should, but it is useful to keep it.

where once again we have raised and lowered the indices of $\Lambda^{\Lambda}{}_{\Sigma}$ with η. The condition $\Lambda^{\dagger}\eta\Lambda = \eta$ implies for the real and imaginary parts of Λ

$$\mathfrak{Re}\Lambda_{\Delta\Lambda}\,\mathfrak{Im}\Lambda^{\Delta}{}_{\Sigma} = \mathfrak{Im}\Lambda_{\Delta\Lambda}\,\mathfrak{Re}\Lambda^{\Delta}{}_{\Sigma}\,, \qquad \mathfrak{Re}\Lambda_{\Delta\Lambda}\,\mathfrak{Re}\Lambda^{\Delta}{}_{\Sigma} + \mathfrak{Im}\Lambda_{\Delta\Lambda}\,\mathfrak{Im}\Lambda^{\Delta}{}_{\Sigma} = \eta_{\Lambda\Sigma}\,, \tag{6.17}$$

and implies that the matrix $(S^{M}{}_{N})$ constructed above satisfies $S^{T}\Omega S = \Omega$ and therefore belongs to $\mathrm{Sp}(2n+2;\mathbb{R})$. The infinitesimal generators in this representation, *i.e.* $(T_{m}{}^{M}{}_{N})$, can be constructed in the same way, leading to

$$(T_{m}{}^{M}{}_{N}) = \begin{pmatrix} \mathfrak{Re}T_{m}{}^{\Lambda}{}_{\Sigma} & -2\mathfrak{Im}T_{m}{}^{\Lambda\Sigma} \\ \\ \frac{1}{2}\mathfrak{Im}T_{m\,\Lambda\Sigma} & \mathfrak{Re}T_{m\,\Lambda}{}^{\Sigma} \end{pmatrix}. \tag{6.18}$$

6.1.1 Equivariant vectors

The search for equivariant vectors is simplified by using the complex combinations defined above: we look for vectors $\mathcal{B}^{\Lambda}$ behaving as Γ^{Λ} under duality transformations, *i.e.* such that its complex conjugate transforms in the fundamental representation of $\mathrm{SU}(1,n)$

$$\Gamma^{*\,\prime\,\Lambda} = \Lambda^{\Lambda}{}_{\Sigma}\,\Gamma^{*\,\Sigma}\,, \quad \Rightarrow \quad \mathcal{B}^{*\,\prime\,\Lambda} = \Lambda^{\Lambda}{}_{\Sigma}\,\mathcal{B}^{*\Sigma}\,. \tag{6.19}$$

Observe that $\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$ and $\mathcal{B}^{*\,\Lambda}\mathcal{B}_{\Lambda}$ are duality invariants.

The simplest equivariant vectors are, up to a complex constant, just equal to the charge vector Γ^{Λ}. This constant is relevant because, as we will see, the complex form of the Freudenthal dual of the charge vector

$$\tilde{\mathcal{Q}}^{M} = \begin{pmatrix} -2\,\eta^{\Sigma\Lambda}q_{\Lambda} \\ \\ \frac{1}{2}\,\eta_{\Lambda\Sigma}p^{\Lambda} \end{pmatrix}, \tag{6.20}$$

is just $\tilde{\Gamma}^{\Lambda} = -i\Gamma^{\Lambda}$, whence the phase of the constant corresponds to a global Freudenthal duality rotation. This immediately implies that the $\mathrm{SU}(1,n)$ invariants $\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$ and $\mathcal{B}^{*\,\Lambda}\mathcal{B}_{\Lambda}$ are also invariant under Freudenthal $\mathrm{U}(1)$ duality. There may be other equivariant vectors which are functions of the charges only, but we will not need them.

We can use the moduli Z^{Λ}_{∞} in order to construct more equivariant vectors. Again, up to normalization, the only one we will need is the generic vector given in Eq. (2.15). Multiplying it by the invariant $\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$ as to give it the right dimensions for later convenience, we have the equivariant vector

$$\Sigma^{\Lambda} \equiv \frac{Z^{*\,\Lambda}_{\infty}}{Z^{*\,\Sigma}_{\infty}\Gamma^{*}_{\Sigma}}\,\Gamma^{*\,\Sigma}\Gamma_{\Sigma}\,. \tag{6.21}$$

We will see that in order to find the most general solutions of this model, it is enough to consider complex linear combinations of the two equivariant vectors constructed thus far:

$$\mathcal{B}^\Lambda = \alpha\Gamma^\Lambda + \beta\Sigma^\Lambda\,, \tag{6.22}$$

where α and β are complex duality invariants (including pure numbers).

Using this information we can see that in this model (for generic n), in distinction to the axidilaton model, we cannot define a fake charge $\mathcal{B}^\Lambda$ and its associated fake central charge $\mathcal{Z}(\mathcal{B})$ such that

$$\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* = |\mathcal{Z}(\mathcal{B})|^2 = e^{\mathcal{K}}|Z^\Lambda\Gamma_\Lambda|^2 - \Gamma^{*\Lambda}\Gamma_\Lambda\,, \tag{6.23}$$

or such that

$$V_{\rm bh}(\mathcal{Q}) = V_{\rm bh}(\mathcal{B})\,, \tag{6.24}$$

for arbitrary values of the scalars. This fact has important implications for the construction of extremal non-supersymmetric solutions as the first-order equations do not imply the second order ones, which therefore have to be solved explicitly. In this paper we are going to construct directly the general non-extremal solutions from which all the extremal ones can be obtained in the appropriate limits.

6.2 H-FGK formalism

The stabilization equations of this model are solved by a linear relation between $\mathcal{R}_M$ and $\mathcal{I}^M$, as in the axidilaton case:

$$\mathcal{R}_M(\mathcal{I}) = \mathcal{A}_{MN}\mathcal{I}^N\,, \qquad (\mathcal{A}_{MN}) = \begin{pmatrix} \tfrac{1}{2}\eta_{\Lambda\Sigma} & 0 \\ 0 & 2\eta^{\Lambda\Sigma} \end{pmatrix}\,, \tag{6.25}$$

which implies that the Freudenthal dual can be expressed as

$$\tilde{H}^M = \mathcal{A}^M{}_N H^N\,, \qquad (\mathcal{A}^M{}_N) = (\Omega^{PM}\mathcal{A}_{PN}) = \begin{pmatrix} 0 & -2\eta^{\Lambda\Sigma} \\ \tfrac{1}{2}\eta_{\Lambda\Sigma} & 0 \end{pmatrix}\,. \tag{6.26}$$

As in the axidilaton case, $\mathcal{A}_{MN}$ is a symplectic matrix, but, in contradistinction to that case, $\mathcal{A}^M{}_N$ is not. In terms of the complex H-variables[26]

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma\,, \tag{6.27}$$

discrete Freudenthal duality is equivalent to multiplication by a factor of $-i$.

The Hesse potential reads

$$\mathsf{W}(H) = \mathcal{A}_{MN}H^M H^N = \tfrac{1}{2}\eta_{\Lambda\Sigma}H^\Lambda H^\Sigma + 2\eta^{\Lambda\Sigma}H_\Lambda H_\Sigma = 2\mathcal{H}^{*\Lambda}\mathcal{H}_\Lambda\,, \tag{6.28}$$

[26]Observe that, in his notation, $\mathcal{H}^\Lambda \equiv \eta^{\Lambda\Sigma}\mathcal{H}_\Sigma$ but $H^\Lambda \neq \eta^{\Lambda\Sigma}H_\Sigma$.

and the metric function e^{-2U} and the scalars Z^i can be easily obtained from it as

$$e^{-2U} = \mathsf{W}(H)\,, \qquad Z^i = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0} = \frac{H_i + \frac{i}{2}H^i}{-H_0 + \frac{i}{2}H^0} = \frac{\mathcal{H}_i^*}{\mathcal{H}_0^*}\,. \tag{6.29}$$

The metric $g_{MN}(H)$ and the potential $V(H)$ have the same structure as in the axidilaton case when we write them in terms of the matrix $\mathcal{A}_{MN}$ (which, evidently, is different). Then, the expressions from Eq. (5.41) to Eq. (5.46) are also valid here upon use of the new matrix $\mathcal{A}_{MN}$.

The central charge of the model, Eq. (6.6), takes in the H-FGK formalism the form

$$\mathcal{Z}(H, \mathcal{Q}) = -\frac{(H_0 + \frac{i}{2}H^0)}{\left|H_0 + \frac{i}{2}H^0\right|}\,\frac{(\tilde{H}_M + iH_M)\mathcal{Q}^M}{\sqrt{2\mathsf{W}(H)}}. \tag{6.30}$$

It is easy to check that, like in the axidilaton case, this black-hole potential satisfies

$$V_{\rm bh}(H, \mathcal{Q}) = \frac{\mathsf{W}(\mathcal{Q})}{\mathsf{W}(H)}\, V_{\rm bh}(\mathcal{Q}, H)\,. \tag{6.31}$$

6.3 Critical points

Using the complex notation we can write the equation for the critical points $\mathcal{B}_\Lambda$ of the black-hole potential of this model in the form

$$\tfrac{i}{2}\mathsf{W}(\mathcal{B})\, \partial^*_\Lambda V_{\rm bh}|_{\mathcal{H}=\mathcal{B}} = \frac{\mathcal{B}^\Sigma \Gamma^*_\Sigma}{\mathsf{W}(B)}\left[\mathcal{B}^{*\,\Delta}\Gamma_\Delta \mathcal{B}_\Lambda - \mathcal{B}^{*\,\Delta}\mathcal{B}_\Delta \Gamma_\Lambda\right] = 0\,, \tag{6.32}$$

and can be solved by

$$\mathcal{B}^\Sigma \Gamma^*_\Sigma = 0\,, \quad \text{or} \quad \mathcal{B}^{*\,\Delta}\Gamma_\Delta \mathcal{B}_\Lambda - \mathcal{B}^{*\,\Delta}\mathcal{B}_\Delta \Gamma_\Lambda = 0\,. \tag{6.33}$$

Inserting the general ansatz (6.22) into the first condition we find that it is satisfied for

$$\alpha = -\beta\,, \quad \Rightarrow \quad \mathcal{B}^\Lambda = \alpha(\Gamma^\Lambda - \Sigma^\Lambda)\,, \tag{6.34}$$

which, up to normalization (which is not fixed in this approach), leaves us with one arbitrary global phase associated to Freudenthal duality: this is the moduli-dependent attractor found in Ref. [5].

Inserting our ansatz (6.22) into the second condition we get the equation

$$\beta(\alpha^* + \beta^*)\Gamma^{*\,\Delta}\Gamma_\Delta \Sigma_\Lambda - \left[2\Re\mathfrak{e}(\alpha\beta^*) + \frac{|\beta|^2 \Gamma^{*\,\Sigma}\Gamma_\Sigma}{|\mathcal{Z}_\infty(\Gamma)|^2}\right]\Gamma^{*\,\Delta}\Gamma_\Delta \Gamma_\Lambda = 0\,. \tag{6.35}$$

The coefficients of the two equivariant vectors must vanish separately, which can only happen for $\beta = 0$, whence $\mathcal{B}^\Lambda = \alpha\Gamma^\Lambda$: up to normalization and the Freudenthal duality phase, this is the supersymmetric attractor.

6.4 Conventional non-extremal solutions

In this section we are going to show how the knowledge of the equivariant vectors of the model simplifies the construction of solutions in the H-FGK formalism. We are going to see that the most general solution can be written as

$$\mathcal{H}^\Lambda(\tau) = a(\tau)\Gamma^\Lambda + b(\tau)\Sigma^\Lambda\,, \tag{6.36}$$

where $a(\tau)$ and $b(\tau)$ are two complex, duality-invariant functions of τ to be determined. Already, at this stage, we see that this ansatz reduces dramatically the number of real functions to be found, from $2n+2$ to just 4, and all of this without any loss of generality.

First of all, we are going to impose the usual Freudenthal gauge-fixing condition $\dot{H}^M H_M = 0$ [30] which in complex notation takes the form

$$\Im\mathfrak{m}(\dot{\mathcal{H}}^{*\,\Lambda}\mathcal{H}_\Lambda) = 0\,. \tag{6.37}$$

As shown in Ref. [30], assuming this condition, the contraction of the equations of motion with H^M leads to the equation

$$\tilde{H}_M\left(\ddot{H}^M - r_0^2 H^M\right) = 0\,, \tag{6.38}$$

which can always be solved by

$$\ddot{H}^M = r_0^2 H^M\,, \quad \Rightarrow \quad \ddot{\mathcal{H}}^\Lambda = r_0^2\mathcal{H}^\Lambda\,. \tag{6.39}$$

This is not necessarily the only solution of Eq. (6.38), but as we are going to see it allows us to solve the rest of the equations without imposing unnecessary constraints on the physical parameters of the solution. This equation combined with the equivariant ansatz leads to

$$\mathcal{H}^\Lambda(\tau) = \left[c_1 e^{r_0\tau} + c_3 e^{-r_0\tau}\right]\Gamma^\Lambda + \left[c_2 e^{r_0\tau} + c_4 e^{-r_0\tau}\right]\Sigma^\Lambda\,, \tag{6.40}$$

so it only remains to determine the four complex invariants c_i $(i = 1, \cdots, 4)$ in terms of the charges Γ_Λ, the moduli Z^Λ_∞ and the mass M (or alternatively of the non-extremality parameter r_0).

These four constants can be constrained even further by requiring that the ansatz gives the right asymptotic behavior for the physical fields in Eq. (6.29): requiring that $Z^\Lambda_\infty = \mathcal{H}^{*\,\Lambda}_\infty/\mathcal{H}^{*\,0}_\infty$ we get[27]

$$c_1 + c_3 = 0\,. \tag{6.41}$$

Asymptotic flatness requires that $\mathcal{H}^{*\,\Lambda}_\infty\mathcal{H}_{\Lambda,\infty} = \frac{1}{2}$ which, upon use of the above condition, gives

$$|c_2 + c_4|^2 - \frac{|\mathcal{Z}_\infty(\Gamma)|^2}{2(\Gamma^{*\,\Lambda}\Gamma_\Lambda)^2} = 0\,, \tag{6.42}$$

[27] In the (H-)FGK coordinate system, spatial infinity corresponds to the limit $\tau \to 0^-$.

where $\mathcal{Z}_\infty(\Gamma)$ is the central charge at spatial infinity. The gauge-fixing condition (6.37) gives (again, upon use of Eq. (6.41))

$$\Im\mathfrak{m}\,[c_3^*(c_2+c_4)] + \Im\mathfrak{m}\,[c_2^* c_4]\,\frac{\Gamma^{*\,\Lambda}\Gamma_\Lambda}{|\mathcal{Z}_\infty(\Gamma)|^2} = 0\,. \tag{6.43}$$

Finally, we can still make global Freudenthal duality rotations, which are not fixed by Eq. (6.37): this freedom cannot be used to solve Eq. (6.43) but can be used to simplify it by fixing the phase of one of the constants to a convenient value.

Using the gauge-fixing condition (6.37), the Hamiltonian constraint takes the form

$$\left[\dot{\mathcal{H}}^{*\,\Lambda}\dot{\mathcal{H}}_\Lambda - \tfrac{1}{2}\Gamma^{*\,\Lambda}\Gamma_\Lambda\right]\mathcal{H}^{*\,\Sigma}\mathcal{H}_\Sigma - 2(\dot{\mathcal{H}}^{*\,\Lambda}\mathcal{H}_\Lambda)^2 + \left|\mathcal{H}^{*\,\Lambda}\Gamma_\Lambda\right|^2 - r_0^2(\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda)^2 = 0\,, \tag{6.44}$$

and using the gauge-fixing condition plus Eq. (6.39) and the Hamiltonian constraint above, the equations of motion take the form

$$\mathcal{H}^*_\Lambda\left[2(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)^2 - \left|\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma\right|^2\right] + \Gamma^*_\Lambda(\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta) - 2\dot{\mathcal{H}}^*_\Lambda(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta) = 0\,. \tag{6.45}$$

The coefficients of the two equivariant vectors Γ_Λ and Σ_Λ must vanish independently, which implies that we must solve the following equations

$$a^*\left[2(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)^2 - \left|\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma\right|^2\right] + (\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta) - 2\dot{a}^*(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta) = 0\,, \tag{6.46}$$

$$b^*\left[2(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)^2 - \left|\mathcal{H}^{*\,\Sigma}\Gamma_\Sigma\right|^2\right] - 2\dot{b}^*(\dot{\mathcal{H}}^{*\,\Sigma}\mathcal{H}_\Sigma)(\mathcal{H}^{*\,\Delta}\mathcal{H}_\Delta) = 0\,. \tag{6.47}$$

The coefficients of b^* and $\dot{b}^*$ in the last equation are real (on account of the gauge-fixing condition) and this implies that the phases of c_2 and c_4 must be the same up to π (the global sign) so that $\Im\mathfrak{m}(c_2^* c_4) = 0$. Then, Eq. (6.43) states that the phase of c_3 must be the same as that of c_2 and c_4, again up to π. We know that in the near-horizon limit (*i.e.* $\tau \to -\infty$) of the extremal non-supersymmetric case the phases of c_3 and c_4 must differ by π and, since this difference is constant, this must always be the case. Furthermore, in the extremal non-supersymmetric case $\mathcal{Z}_\infty(\Gamma) = 0$ and Eq. (6.42) implies that c_2 and c_4 must also have opposite global signs. Therefore we find

$$\arg(c_3) = \arg(c_2) = \arg(c_4) + \pi \equiv \theta\,, \tag{6.48}$$

and, by making use of the global Freudenthal duality freedom

$$|c_2| - |c_4| = -\frac{|\mathcal{Z}_\infty(\Gamma)|}{\sqrt{2}\Gamma^{*\,\Lambda}\Gamma_\Lambda}\,. \tag{6.49}$$

To simplify the calculations further, we introduce the constant A

$$|c_2| + |c_4| = -\frac{|\mathcal{Z}_\infty(\Gamma)|}{\sqrt{2}\Gamma^{*\Lambda}\Gamma_\Lambda} A\,, \tag{6.50}$$

which allows us to rewrite Eq. (6.40) as

$$\mathcal{H}^\Lambda(\tau) = e^{i\theta}\left\{-2|c_3|\sinh r_0\tau\Gamma^\Lambda + \frac{|\mathcal{Z}_\infty(\Gamma)|}{\sqrt{2}\Gamma^{*\Lambda}\Gamma_\Lambda}\left[(1+A)e^{-r_0\tau} + (1-A)e^{r_0\tau}\right]\Sigma^\Lambda\right\}\,. \tag{6.51}$$

It is now straightforward to solve the equations of motion for the three constants θ, A and $|c_3|$, for which it is convenient to express the final result using the mass M (defined in Eq. (4.4))

$$M = r_0\left[A + 2\sqrt{2}|c_3||\mathcal{Z}_\infty(\Gamma)|\right]\,. \tag{6.52}$$

The final result is

$$|c_3| = \frac{|\mathcal{Z}_\infty(\Gamma)|}{2\sqrt{2}Mr_0}\,, \tag{6.53}$$

$$A = \frac{M^2 - |\mathcal{Z}_\infty(\Gamma)|^2}{Mr_0}\,, \tag{6.54}$$

$$e^{i\theta} = \pm\frac{\mathcal{Z}_\infty(\Gamma)}{|\mathcal{Z}_\infty(\Gamma)|}\,, \tag{6.55}$$

$$M^2 r_0^2 = \left[M^2 - |\hat{\mathcal{Z}}_\infty|^2\right]\left[M^2 - |\mathcal{Z}_\infty(\Gamma)|^2\right]\,, \tag{6.56}$$

which is precisely the result obtained in Ref. [5].

We do not expect any other Freudenthal-inequivalent solutions to this model since the solutions we just found have the maximal number of independent physical parameters.

7 The t^3 model

The t^3-model is characterized by the prepotential

$$\mathcal{F}(\mathcal{X}) = -\tfrac{5}{6}\frac{(\mathcal{X}^1)^3}{\mathcal{X}^0}\,. \tag{7.1}$$

In terms of the coordinate $t = \mathcal{X}^1/\mathcal{X}^0$, the Kähler potential and the scalar-manifold metric are given by

$$\mathcal{K} = -3\ln\Im\mathfrak{m}\,t - \ln\tfrac{20}{3}\,, \qquad \mathcal{G}_{tt^*} = \tfrac{3}{4}\left(\Im\mathfrak{m}\,t\right)^{-2}\,; \tag{7.2}$$

the covariantly holomorphic symplectic section reads

$$\mathcal{V}(t,t^*) = e^{\mathcal{K}/2}\begin{pmatrix} 1 \\ t \\ \frac{5}{6}t^3 \\ -\frac{5}{2}t^2 \end{pmatrix}, \tag{7.3}$$

and the central charge, its covariant derivative, the black-hole potential and its partial derivative read

$$\mathcal{Z} \equiv e^{\frac{1}{2}\mathcal{K}}\hat{\mathcal{Z}}, \tag{7.4}$$

$$\mathcal{D}_t\mathcal{Z} \equiv \frac{i}{2}\frac{e^{\frac{1}{2}\mathcal{K}}}{\Im\mathfrak{m}\, t}\hat{\mathcal{W}}, \tag{7.5}$$

$$-V_{\rm bh} = e^{\mathcal{K}}\left[|\hat{\mathcal{Z}}|^2 + \tfrac{1}{3}|\hat{\mathcal{W}}|^2\right], \tag{7.6}$$

$$-\partial_t V_{\rm bh} = \tfrac{i}{20}(\Im\mathfrak{m}\, t)^{-4}\left[(\hat{\mathcal{W}}^*)^2 + 3\hat{\mathcal{W}}\hat{\mathcal{Z}}^*\right], \tag{7.7}$$

where we have defined

$$\hat{\mathcal{Z}} = \tfrac{5}{6}p^0t^3 - \tfrac{5}{2}p^1t^2 - q_1t - q_0, \tag{7.8}$$

$$\hat{\mathcal{W}} = \tfrac{5}{2}p^0t^2t^* - \tfrac{5}{2}p^1t(t+2t^*) - q^1(2t+t^*) - 3q^0. \tag{7.9}$$

Observe that all these objects are well defined only iff $\Im\mathfrak{m}\, t > 0$.

7.1 The global symmetries of the t^3 model

The t^3 model as a theory of $\mathcal{N} = 2, d = 4$ supergravity is invariant under global $\mathrm{Sl}(2;\mathbb{R})$ transformations, just like the axidilaton model, since their Kähler metrics are identical up to a numerical factor. The action of $\mathrm{Sl}(2;\mathbb{R})$ on t is identical to its action on λ, which was discussed in Section 5.1. The transformations of the Kähler potential and covariantly-holomorphic symplectic section Eqs. (5.16,5.17) are determined by the holomorphic function $f(t)$ and the $\mathrm{Sp}(4;\mathbb{R})$ matrix $S^M{}_N$ given by

$$
\begin{aligned}
f(t) &= 3\ln(ct+d)\,, \qquad (7.10)\\
(S^M{}_N) &= \begin{pmatrix} d^3 & 3d^2c & \frac{6}{5}c^3 & -\frac{6}{5}dc^2 \\ bd^2 & (ad+2bc)d & \frac{6}{5}ac^2 & -\frac{2}{5}(2ad+bc)c \\ \frac{5}{6}b^3 & \frac{5}{2}ab^2 & a^3 & -a^2b \\ -\frac{5}{2}b^2d & -\frac{5}{2}(2ad+bc)b & -3a^2c & (ad+2bc)a \end{pmatrix}\,. \qquad (7.11)
\end{aligned}
$$

In this case the 4-dimensional representation of the generators T_m are given by

$$
(T_1{}^M{}_N) = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -3 & \\ & & & -1 \end{pmatrix}\,, \qquad (T_2{}^M{}_N) = \begin{pmatrix} & -3 & & \\ -1 & & & 4/5 \\ & & & 1 \\ & 5 & 3 & \end{pmatrix}\,,
$$
$$
(T_3{}^M{}_N) = \begin{pmatrix} & -3 & & \\ 1 & & & 4/5 \\ & & & -1 \\ & -5 & 3 & \end{pmatrix}\,. \qquad (7.12)
$$

As in the axidilaton model, the same transformations act on all the symplectic vectors of the theory and, in particular on H^M and $\mathcal{Q}^M$. There are no more symmetries in this formulation of the model.

7.1.1 Equivariant vectors of the t^3 model

It is not difficult to see that, from the point of view of $\mathrm{Sl}(2;\mathbb{R})$, the symplectic vectors such as the charge vector $\mathcal{Q}^M$ transform as a quadruplet, i.e. a fully symmetric 3-index covariant tensor $\mathcal{Q}_{ijk} = \mathcal{Q}_{(ijk)}$ (in the notation used in Section 5.1). The relation between the components of this tensor and those of the charge vector is

$$
\mathcal{Q}_{111} = p^0\,, \quad \mathcal{Q}_{112} = -p^1\,, \quad \mathcal{Q}_{122} = -\tfrac{2}{5}q_1\,, \quad \mathcal{Q}_{222} = -\tfrac{6}{5}q_0\,. \qquad (7.13)
$$

It is useful to observe that the contraction of two quadruplets is related to the symplectic product by

$$
A_{ijk}B^{ijk} = -\tfrac{6}{5}A^M B_M\,. \qquad (7.14)
$$

By definition, any new $\mathrm{Sl}(2;\mathbb{R})$ quadruplet that we construct out of t_∞ and $\mathcal{Q}_{ijk}$ can be transformed according to the above rules into an equivariant symplectic vector of the t^3-model. The $\mathrm{Sl}(2;\mathbb{R})$ index notation makes this construction easy, but, as we are going to see, insufficient.

In order to construct $\mathrm{Sl}(2;\mathbb{R})$ invariants and other quadruplets it is useful to define the matrix

$$m^i{}_j \equiv \mathcal{Q}^{ikl}\mathcal{Q}_{jkl}\,, \tag{7.15}$$

whose components take the values

$$m^1{}_1 = -m^2{}_2 = -\tfrac{2}{5}(p^1 q_1 + 3p^0 q_0)\,, \quad m^1{}_2 = \tfrac{12}{5}p^1 q_0 - \tfrac{8}{25}(q_1)^2\,, \quad m^2{}_1 = \tfrac{4}{5}p^0 q_1 + 2(p^1)^2\,. \tag{7.16}$$

The square of this matrix is

$$m^i{}_k\, m^k{}_j = -\tfrac{36}{25}\, J_4(\mathcal{Q})\, \delta^i{}_j\,, \tag{7.17}$$

where, since $\delta^i{}_j$ is an invariant tensor, the coefficient $J_4(\mathcal{Q})$ must be an invariant of order four in the charges; this quartic invariant is explicitly given by

$$J_4(\mathcal{Q}) \equiv \tfrac{8}{45}p^0(q_1)^3 + \tfrac{1}{3}(p^1 q_1)^2 - (p^0 q_0)^2 - 2p^0 q_0 p^1 q_1 - \tfrac{10}{3}(p^1)^3 q_0\,. \tag{7.18}$$

This is the only independent invariant that can be constructed from the charge alone. We can construct invariants taking traces of powers of m and taking also the determinant: the traces of odd powers vanish and those of even powers are proportional to $J_4(\mathcal{Q})$. Furthermore, the determinant is also proportional to $J_4(\mathcal{Q})$, *i.e.*

$$\det(m) = \tfrac{36}{25}J_4(\mathcal{Q})\,. \tag{7.19}$$

The simplest quadruplet that can be built out of the original one $\mathcal{Q}_{ijk}$ is

$$\mathcal{Q}_{(ij|l}\, m^l{}_{|k)}\,. \tag{7.20}$$

This tensor is necessarily proportional to the Freudenthal dual of $\mathcal{Q}_{ijk}$ since

$$\mathcal{Q}_{(ij|l}m^l{}_{|k)} = \tfrac{1}{4}\frac{\partial \mathrm{Tr}\, m^2}{\partial \mathcal{Q}^{ijk}} = -\tfrac{18}{25}\frac{\partial J_4(\mathcal{Q})}{\partial \mathcal{Q}^{ijk}}\,. \tag{7.21}$$

Using higher powers of m does not give anything new as

$$\mathcal{Q}_{(i|lm}m^l{}_{|j}m^m{}_{k)} = \mathcal{Q}_{(ij|l}m^l{}_m m^m{}_{|k)} = -\tfrac{36}{25}\, J_4(\mathcal{Q})\, \mathcal{Q}_{ijk}\,. \tag{7.22}$$

We must use, therefore, contractions of $\mathcal{Q}_{ijk}$ such that the free indices are not those of $m^i{}_j$. At cubic order in $\mathcal{Q}_{ijk}$ there is only one possibility, which vanishes identically

$$\mathcal{Q}_{(i|lm}\mathcal{Q}_{|j|n}{}^l\mathcal{Q}_{|k)}{}^{mn} = 0\,, \tag{7.23}$$

due to the antisymmetry of the symplectic metric Ω_{ij}. At order five in $\mathcal{Q}_{ijk}$ we can consider

$$\mathcal{Q}_{i,i_1,i_2}\mathcal{Q}_{j,j_1,j_2}\mathcal{Q}_{k,k_1,k_2}\mathcal{Q}^{i_1,j_1,k_1}\mathcal{Q}^{i_2,j_2,k_2} \;=\; -\tfrac{36}{25}\, J_4(\mathcal{Q})\, \mathcal{Q}_{ijk}\,, \tag{7.24}$$

$$\mathcal{Q}_{(i|mn}\mathcal{Q}_{|j|pq}\mathcal{Q}_{|k)}{}^{mp}m^{nq} \;=\; 0\,. \tag{7.25}$$

Up to at least order 9 there are no quadruplets other than $\mathcal{Q}_{ijk}$ and its Freudenthal dual that can be constructed by these tensor methods.

To find more, we have to solve Eq. (2.11). Since this is a very complicated task, we are going to restrict ourselves to a generating charge configuration with $p^0 = q_1 = 0$, *i.e.*

$$(\mathcal{Q}^M) = \begin{pmatrix} 0 \\ p^1 \\ q_0 \\ 0 \end{pmatrix} . \tag{7.26}$$

This subspace is preserved by the $\mathrm{Sl}(2;\mathbb{R})$ transformations with $b = c = 0$ and $d = 1/a$ (or equivalently by the infinitesimal transformations generated by T_1), to which by analogy we shall refer to as the *small group*. It is not difficult to see that by acting on this charge vector with the transformations with appropriate charge-dependent parameters $b \neq 0\,,\ c \neq 0$ (or, equivalently, by the infinitesimal transformations generated by T_2 and T_3) we can generate the complete generic charge vector with four unrestricted charge components.

It should be clear that if we construct vectors in the subspace $p^0 = q_1 = 0$ that are equivariant under the small group, then by acting on these vectors with the same transformations that generate the complete charge vector, we will obtain vectors that are equivariant under the full duality group, *i.e.* $\mathrm{Sl}(2;\mathbb{R})$, and which reduce to the former when we set $p^0 = q_1 = 0$. Since duality transformations preserve linear independence, a base for the small-group-equivariant vectors will be transformed into a base of the duality-group-equivariant vectors; seeing this reasoning we shall refer to a small-group-equivariant vector as an *equivariant-generating vector*.

The equation that these equivariant-generating vectors have to solve is the restriction of Eq. (2.11) to just T_1 and allow for no dependence on p^0 nor q_1, *i.e.*

$$p^1\frac{\partial U^P}{\partial p^1} - 3q_0\frac{\partial U^P}{\partial q_0} = \beta^{(P)}U^{(P)}\,, \qquad (\beta^P) = \begin{pmatrix} 3 \\ 1 \\ -3 \\ -1 \end{pmatrix} , \tag{7.27}$$

which is solved by

$$U^P = \sum_i a_i^{(P)}(p^1)^{\alpha_i^{(P)}}(q_0)^{\frac{\alpha_i^{(P)}-\beta^{(P)}}{3}}\,, \tag{7.28}$$

for arbitrary constants a_i^P, α_i^P (the parenthesis enclosing the indices P indicate that they are not summed over and the index i runs over an arbitrary number of terms). For simplicity, we can choose them to depend only on p^1 ($\alpha^P = \beta^P$) or only on q_0 ($\alpha_i^P = 0$) and take them to have only one term:

$$U^P = a^{(P)}(p^1)^{\beta^{(P)}}\,, \qquad U^P = a^{(P)}(q_0)^{-\beta^{(P)}/3}\,. \tag{7.29}$$

To avoid charges with fractional components, we choose the first option and get a basis of equivariant-generating vectors

$$U_\sigma{}^P \sim \delta_\sigma{}^{(P)}(p^1)^{\beta(P)}\,. \tag{7.30}$$

We have found it convenient to normalize these vectors and give them names $\{R, S, U, V\}$

$$R \equiv \begin{pmatrix} \frac{10}{3}(p^1)^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,,\quad S \equiv \begin{pmatrix} 0 \\ 0 \\ (\frac{10}{3}(p^1)^3)^{-1} \\ 0 \end{pmatrix}\,,\quad U \equiv \begin{pmatrix} 0 \\ p^1 \\ 0 \\ 0 \end{pmatrix}\,,\quad V \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/p^1 \end{pmatrix}\,. \tag{7.31}$$

The only non-vanishing symplectic contractions of these four vectors are

$$R_M S^M = -1\,, \qquad U_M V^M = -1\,, \tag{7.32}$$

and they satisfy the completeness relation

$$R^M S_N - S^M R_N + U^M V_N - V^M U_N = \delta^M{}_N\,. \tag{7.33}$$

We can decompose any equivariant-generating vector, such as $\mathcal{Q}^M$ w.r.t. this basis and the expression will have the same form after acting with the duality group. For $\mathcal{Q}^M$ we find

$$R_M \mathcal{Q}^M = -\tfrac{10}{3}(p_1)^3 q_0 = J_4(\mathcal{Q})|_{p^0=q_1=0}\,, \qquad V_M \mathcal{Q}^M = 1\,, \tag{7.34}$$

from which we find that in general

$$\mathcal{Q}^M = U^M - J_4(\mathcal{Q}) S^M\,. \tag{7.35}$$

The Freudenthal dual charge vector is (using the results of the next section) given by

$$\tilde{\mathcal{Q}}^M = \frac{1}{\mathsf{W}(\mathcal{Q})} R^M + \tfrac{3}{4}\mathsf{W}(\mathcal{Q}) V^M\,, \qquad \mathsf{W}(\mathcal{Q}) = 2\sqrt{J_4(\mathcal{Q})}\,. \tag{7.36}$$

As for the moduli-dependent equivariant vectors, we can use the generic construction in Eq. (2.15) replacing $\mathcal{Q}$ with different equivariant vectors.

7.2 H-FGK formalism

The stabilization equations can be solved in a completely general way [41] and the result is summarized by the Hesse potential which, in terms of the quartic invariant

$$J_4(H) \equiv \tfrac{8}{45} H^0 (H_1)^3 + \tfrac{1}{3}(H^1 H_1)^2 - (H^0 H_0)^2 - 2H^0 H_0 H^1 H_1 - \tfrac{10}{3}(H^1)^3 H_0\,, \tag{7.37}$$

can be expressed as

$$\mathsf{W}(H) = 2\sqrt{J_4(H)}\,. \tag{7.38}$$

It is convenient to introduce the fully symmetric rank-4 $\mathbb{K}$-tensor [42, 43], implicitly defined by[28]

$$\mathbb{K}_{MNPQ}H^M H^N H^P H^Q \equiv J_4(H)\,. \tag{7.39}$$

Using this tensor, we can write

$$\tilde{H}_M = \frac{\partial_M J_4}{\mathsf{W}} = 4\frac{\mathbb{K}_{MNPQ}H^N H^P H^Q}{\mathsf{W}}\,, \tag{7.40}$$

$$\mathcal{M}_{MN}(\mathcal{F}) = -\frac{\partial_M\partial_N J_4}{\mathsf{W}} + 2\frac{\partial_M J_4 \partial_N J_4}{\mathsf{W}^3} = -12\frac{\mathbb{K}_{MNPQ}H^P H^Q}{\mathsf{W}} + 2\frac{\tilde{H}_M\tilde{H}_N}{\mathsf{W}}\,, \tag{7.41}$$

$$g_{MN} = 24\frac{\mathbb{K}_{MNPQ}H^P H^Q}{\mathsf{W}^2} - 8\frac{\tilde{H}_M\tilde{H}_N}{\mathsf{W}^2} - 2\frac{H_M H_N}{\mathsf{W}^2}\,, \tag{7.42}$$

and one can check (*e.g.* using a symbolic manipulation program) the following properties:

$$J_4(\tilde{H}) = J_4(H)\,, \tag{7.43}$$

$$\mathbb{K}_{MNPQ}\tilde{H}^N\tilde{H}^P\tilde{H}^Q = -\tfrac{1}{4}\mathsf{W}H_M\,, \tag{7.44}$$

$$\mathbb{K}_{MNPQ}\tilde{H}^P\tilde{H}^Q = \mathbb{K}_{MNPQ}H^P H^Q + \tfrac{1}{6}(H_M H_N - \tilde{H}_M\tilde{H}_N)\,, \tag{7.45}$$

$$\mathbb{K}_{MNPQ}H^P\tilde{H}^Q = -\tfrac{1}{6}H_{(M}\tilde{H}_{N)}\,. \tag{7.46}$$

These properties (which hold for any symplectic vector with non-vanishing quartic invariant which implies the existence of the Freudenthal dual) imply the invariance under Freudenthal duality of W, $\mathcal{M}_{MN}(\mathcal{F})$ and the potential $V(H)$; the latter can be rewritten in the manifestly Freudenthal-duality-invariant form

$$V(H) = -3\mathsf{W}^{-2}\left\{\mathbb{K}_{MNPQ}\left(H^P H^Q + \tilde{H}^P\tilde{H}^Q\right) - \tfrac{1}{2}\left(H_M H_N + \tilde{H}_M\tilde{H}_N\right)\right\}\mathcal{Q}^M\mathcal{Q}^N\,. \tag{7.47}$$

It is, however, not possible to express it in a form manifestly invariant under the Freudenthal duality transformation of the charge vector $\mathcal{Q}^M \to \tilde{\mathcal{Q}}^M$.

The physical fields are given in terms of the H-variables by the usual expressions

$$e^{-2U} = 2\mathsf{W} = 2\sqrt{J_4(H)}\,, \tag{7.48}$$

$$t = \frac{\tilde{H}^1 + iH^1}{\tilde{H}^0 + iH^0} = -\frac{3H^0H_0 + H^1H_1}{5(H^1)^2 + 2H^0H_1} + i\frac{3\mathsf{W}}{2\left[5(H^1)^2 + 2H^0H_1\right]}\,. \tag{7.49}$$

[28]In most of what follows, the exact form of the $\mathbb{K}$-tensor will be irrelevant. The formulae and results obtained will, therefore, be valid for any $\mathcal{N}=2, d=4$ theory with Hesse potential of the same generic form.

7.2.1 Very small vectors

The vectors R^M and S^M turn out to be very small charge vectors of this model [4, 44], owing to the following properties:

$$\mathbb{K}_{MNPQ}R^P R^Q = -\tfrac{1}{6}R_M R_N \,, \qquad \mathbb{K}_{MNPQ}S^P S^Q = -\tfrac{1}{6}S_M S_N \,, \tag{7.50}$$

that leads to (in obvious shorthand notation)

$$\mathbb{K}_M R^3 = \mathbb{K}_M S^3 = 0 \,, \qquad J_4(R) = J_4(S) = 0 \,. \tag{7.51}$$

On the other hand, the vectors U^M and V^M are both small vectors

$$J_4(U) = J_4(V) = 0 \,. \tag{7.52}$$

7.3 Critical points

The complexity of this model forces us to use a symbolic manipulation program and, further, impose the restriction $p^0 = q_1 = 0$ on the charges to search for the critical points of the black-hole potential. Apart from the standard supersymmetric attractor $B^M = \mathcal{Q}^M$ we find only one physically acceptable attractor given by

$$(B^M) = \begin{pmatrix} 0 \\ p^1 \\ -q_0 \\ 0 \end{pmatrix} \,. \tag{7.53}$$

It is an equivariant vector and we can write it in the form

$$B^M = U^M + J_4(\mathcal{Q})S^M = \mathcal{Q}^M + 2J_4(\mathcal{Q})S^M \,. \tag{7.54}$$

The quartic invariant for this vector can be computed readily using Eqs. (7.50–7.52), and

$$S_M \mathcal{Q}^M = 0 \,, \qquad \tilde{\mathcal{Q}}_M S^M = -1/\mathsf{W}(\mathcal{Q}) \,, \tag{7.55}$$

and, by Eq. (7.36), it reads

$$\begin{aligned} J_4(B) &= \mathbb{K}B^4 = \mathbb{K}[\mathcal{Q} + 2J_4(\mathcal{Q})S]^4 = \mathbb{K}\mathcal{Q}^4 + 8J_4(\mathcal{Q})\mathbb{K}\mathcal{Q}^3 S \\ &= J_4(\mathcal{Q}) + 2J_4(\mathcal{Q})\mathsf{W}(\mathcal{Q})\tilde{\mathcal{Q}}_M S^M \\ &= -J_4(\mathcal{Q}) \,. \end{aligned} \tag{7.56}$$

7.4 Conventional extremal solutions

The supersymmetric solutions of this model are constructed as usual, and we will focus on the extremal non-supersymmetric ones which are associated to the attractor $B^M = U^M + J_4(\mathcal{Q})S^M$. For the near-horizon solutions, the H^M take the standard form Eq. (3.3) since Eq. (3.4) is satisfied. Now we must investigate whether we can add constant terms A^M to these harmonic functions satisfying only the normalization condition $\mathsf{W}(A) = 1$ and the constraint $B^M A_M = 0$, which is equivalent, at the infinitesimal level, to investigating the space of solutions to Eq. (3.10). For simplicity, we work with a generating charge configuration with $p^0 = q_1 = 0$. We find for the non-supersymmetric attractor

$$(\mathfrak{M}_{MN}) = \frac{1}{2}\begin{pmatrix} \frac{21}{20}\frac{q_0}{(p^1)^3} & 0 & 0 & -\frac{3}{20}\frac{1}{(p^1)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{3}{20}\frac{1}{(p^1)^2} & 0 & 0 & \frac{1}{4}\frac{1}{p^1 q_0} \end{pmatrix}, \tag{7.57}$$

whose rank is 2. The solutions to Eq. (3.10) have the form $(\varepsilon^M) = \left(\begin{smallmatrix} 0 \\ \varepsilon^1 \\ \varepsilon_0 \\ 0 \end{smallmatrix}\right)$ and satisfy $B^M \varepsilon_M = 0$ but we still have to impose the normalization condition $\mathsf{W}(A) = 1$ on the two non-vanishing components, which leaves us with only one independent solution that can only describe one independent real moduli; this modulus turns out to be $\Im\mathfrak{m}(t_\infty)$. It can be shown that the solution takes the form [33]

$$\left(H^M\right) = \begin{pmatrix} 0 \\ s^1\left\{\sqrt{\frac{3}{10\Im\mathfrak{m}\, t_\infty}} - \frac{1}{\sqrt{2}}|p^1|\tau\right\} \\ -s_0\left\{\sqrt{\frac{5(\Im\mathfrak{m}\, t_\infty)^3}{24}} - \frac{1}{\sqrt{2}}|q_0|\tau\right\} \\ 0 \end{pmatrix}, \tag{7.58}$$

where we have defined

$$s^M \equiv \mathrm{sgn}(\mathcal{Q}^M)\,, \tag{7.59}$$

and where we have to require $s^1 = s_0$ for the solution to be regular.

Having $\Re\mathfrak{e}\, t_\infty = 0$ poses a very important problem because even though the charge vector with $p^0 = q_1$ can generate via $\mathrm{Sl}(2;\mathbb{R})$ duality transformations a complete charge vector with four independent charges, it cannot at the same time generate an independent $\Re\mathfrak{e}\, t_\infty \neq 0$. In other words, this solution is not a *generating solution*; its orbit under $\mathrm{Sl}(2;\mathbb{R})$ rotations will not fully cover the space of parameters. A necessary and sufficient condition for a solution to be generating is that all the $\mathrm{Sl}(2;\mathbb{R})$ invariants of the theory are independent when evaluated on the charges and moduli of that solution [45, 46]. As we show in detail in Appendix A.2, the solution (7.58) does not satisfy this condition.

In order to have a generating solution for the class of extremal non-supersymmetric black-hole solutions associated to the attractor $B^M = U^M + J_4(\mathcal{Q})S^M$, we need to add $\mathfrak{Re}\, t_\infty \neq 0$ to the solution and it should be clear that this cannot be done if we make a conventional, *i.e.* harmonic, ansatz: the H^M must contain anharmonic terms.

For future use, it is useful to have symplectic-covariant expressions for the constraints on A^M imposed by the equations of motion for a harmonic ansatz:

$$A_M U^M = 0\,, \qquad A_M S^M = 0\,. \tag{7.60}$$

$A_M B^M = 0$ only imposes the weaker condition $A_M(U^M + J_4(\mathcal{Q})S^M) = 0$. The above constraints imply that A^M has to take the form

$$A^M = aU^M + bS^M\,, \tag{7.61}$$

for some invariant coefficients a and b, and it cannot contain terms proportional to the vectors R^M and V^M.

7.5 Unconventional extremal solutions

The missing free parameter must be added to the above solution by adding anharmonic terms to the harmonic ansatz: let us don the harmonic functions of the undeformed solution with hats, so that

$$\hat{H}^M = A^M - \tfrac{1}{\sqrt{2}} B^M \tau\,, \tag{7.62}$$

where B^M is given by the attractor (7.54) and A^M satisfies the constraints Eqs. (7.60) but is otherwise arbitrary (up to asymptotic flatness normalization). Observe that this implies that

$$\hat{H}_M U^M = \hat{H}_M S^M = 0\,, \quad \Rightarrow \quad \hat{H} = a(\tau)U^M + b(\tau)S^M\,, \tag{7.63}$$

where $a(\tau)$ and $b(\tau)$ are duality-invariant harmonic functions of τ. Terms proportional to R^M and V^M are excluded if the coefficients are harmonic functions; a term proportional to V^M can always be eliminated by a local Freudenthal duality transformation, whence we expect that it is enough to add a (necessarily anharmonic) term proportional to R^M. It turns out that such a solution [33][29] has the form[30]

$$H^M = \hat{H}^M - \frac{\chi R^M}{R_N H^N}\,, \tag{7.64}$$

[29]This solution can be obtained by truncation from the STU-model solution in Ref. [32] and is also a particular case of the general extremal non-supersymmetric solutions of cubic models of Ref. [4]. It has also been obtained by using integrability methods in the action that one obtains in the approach of Ref. [21] (see also [20]): its derivation can be found in Section 9.4 (page 76) of Ref. [47]. The solution belongs to the orbit $\mathcal{O}^3_{22}$ in the classification of Ref. [48] (see Table 2 of that reference).

[30]This definition is not recursive because $R_N H^N = R_N \hat{H}^N$.

where χ is another independent parameter, like A^M. The values of χ and A^M are determined by requiring that the physical fields have the right asymptotic behavior at spatial infinity ($e^{-2U} \to 1\,,\ \ t \to t_\infty$ when $\tau \to 0^-$) as follows: first of all, observe that as a consequence of Eq. (7.63) the property

$$H_M U^M = 0\,, \tag{7.65}$$

is satisfied everywhere and in particular at spatial infinity where

$$H^M \stackrel{\tau\to 0^-}{\longrightarrow} H^M_\infty = A^M - \frac{\chi R^M}{R_N A^N}\,. \tag{7.66}$$

Then, using the definition of $H^M = \mathcal{I}^M$, Eq. (1.7), in Eq. (7.65) plus Eq. (1.8) at spatial infinity we find

$$0 = H_{M\,\infty} U^M = \Im\mathfrak{m}\left(\frac{\mathcal{V}_{M\,\infty}}{X_\infty}\right) U^M = \Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(U)}{X_\infty}\right) = \sqrt{2}\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(U)}{e^{i\alpha_\infty}}\right)\,. \tag{7.67}$$

This implies that

$$e^{i\alpha_\infty} = \pm\frac{\mathcal{Z}_\infty(U)}{|\mathcal{Z}_\infty(U)|}\,, \tag{7.68}$$

which can be used again in the definition of $H^M = \mathcal{I}^M$ to give

$$H^M_\infty = \pm\sqrt{2}\Im\mathfrak{m}\left(\frac{\mathcal{V}^M_\infty}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,. \tag{7.69}$$

To determine the overall sign we will demand that the functions $H^M(\tau)$ never vanish for $\tau \in [-\infty, 0)$, a condition that is usually related to the positivity of the mass. Contracting the above result with S^M and using Eq. (7.63) we get

$$\frac{\chi}{R_N A^N} = \pm\sqrt{2}\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(S)}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,, \tag{7.70}$$

which, after substitution in Eq. (7.66) gives the value of the constants A^M, satisfying Eqs. (7.60), as an equivariant symplectic vector, function of the physical parameters of the solution

$$A^M = \pm\sqrt{2}(\delta^M{}_N - R^M S_N)\Im\mathfrak{m}\left(\frac{\mathcal{V}^M_\infty}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,. \tag{7.71}$$

With this information we can compute $R_N A^N$ to find, from Eq. (7.70) the value of the invariant parameter χ as a function of the physical parameters of the solution[31]

[31] In terms of the invariants $i_1, \cdots, i_5$ of the theory given in Eqs. (A.1)-(A.5)

$$\chi = \tfrac{1}{4}(-J_4(\mathcal{Q}))^{-1/6}\left\{\left(i_1 + i_2 - \frac{(i_1 - i_2/3)^3}{J_4(\mathcal{Q})} - \frac{4\,i_3}{\sqrt{-J_4(\mathcal{Q})}}\right)^{1/3} - \left(i_1 + i_2 - \frac{(i_1 - i_2/3)^3}{J_4(\mathcal{Q})} + \frac{4\,i_3}{\sqrt{-J_4(\mathcal{Q})}}\right)^{1/3}\right\}\,. \tag{7.72}$$

$$\chi = -2\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(R)}{\mathcal{Z}_\infty(U)}\right)\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(S)}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|^2\,. \tag{7.73}$$

For $p^0 = q_1 = 0$, the solution takes the explicit (but not manifestly equivariant) form

$$\left(H^M\right) = \begin{pmatrix} -\frac{1}{2}\frac{\Re\mathfrak{e}\, t_\infty}{\Im\mathfrak{m}\, t_\infty}\dfrac{1}{H_0}\,, \\ \\ s^1\left\{\sqrt{\frac{3}{10\Im\mathfrak{m}\, t_\infty}} - \frac{1}{\sqrt{2}}|p^1|\tau\right\} \\ \\ -s_0\left(\frac{|t_\infty|}{\Im\mathfrak{m}\, t_\infty}\right)^2\left\{\sqrt{\frac{5\Im\mathfrak{m}\, t_\infty}{24}} - \frac{1}{\sqrt{2}}|q_0|\tau\right\} \\ \\ 0 \end{pmatrix}\,. \tag{7.74}$$

The mass of this solution can be computed using the general formula Eq. (4.4). From the definition of $\tilde{H}_M$ we have

$$\tilde{H}_M(0) = \pm\sqrt{2}\Re\mathfrak{e}\left(\frac{\mathcal{V}_{\infty\, M}}{\mathcal{Z}_\infty(U)}\right)|\mathcal{Z}_\infty(U)|\,, \tag{7.75}$$

and

$$\dot{H}^M(0) = -\tfrac{1}{\sqrt{2}}\left[B^M - \frac{\chi J_4(\mathcal{Q})}{(RA)^2}R^M\right]\,, \tag{7.76}$$

from which we get the covariant expression

$$M = \pm|\mathcal{Z}_\infty(U)|\left\{1 - \tfrac{1}{3}J_4(\mathcal{Q})\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(V)}{\mathcal{Z}_\infty(U)}\right)\left[\Im\mathfrak{m}\left(\frac{\mathcal{Z}_\infty(R)}{\mathcal{Z}_\infty(U)}\right)\right]^{-1}\right\}\,. \tag{7.77}$$

This last expression reduces for $p^0 = q_1 = 0$ (selecting the upper sign in Eq. (7.69)) to

$$M = e^{\mathcal{K}_\infty/2}\left(|q_0| + \tfrac{5}{2}|t_\infty|^2|p^1|\right)\,. \tag{7.78}$$

Observe that the value of the mass differs from the absolute value of the associated fake central charge B^M:

$$M \neq |\mathcal{Z}(\phi_\infty, B)|\,. \tag{7.79}$$

The above result should be compared to the mass of the supersymmetric black hole which is given by the standard formula $M = |\mathcal{Z}_\infty(\mathcal{Q})|$ and reduces for $p^0 = q_1 = 0$ to[32] the following expression,

[32]We have used that $p^1 q_0 > 0$ for the non-supersymmetric case and $p^1 q_0 < 0$ for the supersymmetric one.

$$M = e^{\mathcal{K}_\infty/2}\sqrt{\left[|q_0| - \tfrac{5}{2}(\Re\mathfrak{e}t_\infty)^2|p^1|\right]^2 + \tfrac{25}{4}(\Im\mathfrak{m}t_\infty)^4|p^1|^2 + 5(\Im\mathfrak{m}t_\infty)^2|q_0p^1|}\,, \tag{7.80}$$

which can be rewritten in the equivalent form

$$M = e^{\mathcal{K}_\infty/2}\sqrt{\left[|q_0| + \tfrac{5}{2}|t_\infty|^2|p^1|\right]^2 - 10(\Re\mathfrak{e}t_\infty)^2|q_0p^1|}\,, \tag{7.81}$$

which shows that the mass of the supersymmetric black hole is always smaller than the mass of the non-supersymmetric one with charges of equal absolute value.

The entropy is given by the square of the fake central charge at the horizon

$$S = \pi|\mathcal{Z}(\phi_h, B)|^2 = \pi \mathsf{W}(B)/2 = \pi\sqrt{-J_4(\mathcal{Q})}\,. \tag{7.82}$$

As discussed in Section 4, an interesting characteristic of the unconventional solutions is that, in distinction to what happens for the conventional ones, the flow of the black-hole metric function e^{-U} from infinity to the horizon is not governed by a simple fake central charge $\mathcal{Z}(\phi, B)$ since the near-horizon limit of the metric is related to $\mathcal{Z}(\phi_{\rm h}, B)$ but the spacelike infinity limit is not related to $\mathcal{Z}(\phi_\infty, B)$. The first-order flow equations for these black holes can be written in terms of a superpotential $W(\phi, B)$ or, equivalently, in terms of the "fake central charge" $\mathcal{Z}(\phi, \sqrt{2}\mathfrak{D}H)$ defined in Section 4.

It is possible to prove analytically that the general configuration Eq. (7.64) solves the equations of motion by using the duality-invariant properties of the equivariant vectors A^M, B^M and R^M that appear in its definition (that is: not reducing the equations to the $p^0 = q_1$ case) and the properties of the $\mathbb{K}$-tensor of this model, see Eqs. (7.56). As an intermediate step, we derive the following relations, which are valid only for the $H^M s$ of our ansatz:

$$\begin{aligned}
\mathbb{K}_{MN}\hat{H}^2 &= \tfrac{1}{2}(VH)^2R_{(M}V_{N)} + \tfrac{1}{2}(VH)(RH)V_MV_N + \tfrac{1}{18}(VH)^2U_MU_N \\
&\quad -\tfrac{1}{3}(VH)(RH)U_{(M}S_{N)} - \tfrac{1}{6}(RH)^2S_MS_N\,,
\end{aligned} \tag{7.83}$$

$$\begin{aligned}
\mathbb{K}_{MN}\hat{H}\mathcal{Q} &= \tfrac{1}{2}(VH)R_{(M}V_{N)} + \tfrac{1}{4}[J_4(\mathcal{Q})(VH) + (RH)]V_MV_N + \tfrac{1}{18}(VH)U_MU_N \\
&\quad -\tfrac{1}{6}[J_4(\mathcal{Q})(VH) + (RH)]U_{(M}S_{N)} - \tfrac{1}{6}J_4(\mathcal{Q})(RH)S_MS_N\,,
\end{aligned} \tag{7.84}$$

$$\mathbb{K}_{MN}\hat{H}R = -\tfrac{1}{3}(RH)R_{(M}S_{N)} - \tfrac{1}{6}(RH)U_{(M}V_{N)} - \tfrac{1}{6}(VH)R_{(M}U_{N)}\,. \tag{7.85}$$

Using these identities it is easy to show, for instance, that

$$J_4(H) = J_4(\hat{H}) - \chi^2\,, \qquad J_4(\hat{H}) = (VH)^3\,(RH)\,. \tag{7.86}$$

8 Conclusions

In this paper we have shown how the equivariance of the H variables under duality transformations translates into equivariance of the constant symplectic vectors that occur in their explicit expressions. Using the H-FGK formalism we have studied under what conditions the extremal solutions associated to a given attractor can be described, for all values of the charges and moduli, by harmonic Hs alone and when it is necessary to add anharmonic terms to them. We have called these two kinds of solutions conventional, respectively unconventional.

As mentioned in the introduction, it is not known how unconventional extremal solutions (which are necessarily non-supersymmetric, since we know that all the supersymmetric ones are conventional) can be deformed into non-extremal solutions, with non-zero temperature but the same values of the charges and moduli. The H-FGK formalism and the use of equivariant vectors can help us to solve this problem and, as a first step, we have shown how to apply these methods to well-known examples of theories with conventional and unconventional solutions.

In the case of the unconventional extremal solutions of the t^3-model we have shown, first of all, how the criterion found in Section 3 indicates the need for anharmonic terms and which equivariant vectors these terms should depend on. We have then described the solution entirely in terms of these objects and we have computed the general form of the mass and the entropy. The second has a well-known form in terms of the near-horizon limit $\mathcal{Z}(\phi_{\rm h}, B)$ of a *fake central charge*, $\mathcal{Z}(\phi, B)$, constructed from what we have called (in the context of the H-FGK formalism) attractor B^M. The mass instead is not given by the spacelike infinity limit of this fake central charge $M = |\mathcal{Z}(\phi_\infty, B)|$ but rather by the spacelike infinity of a different one $\mathcal{Z}(\phi, E)$ with $E^M \neq B^M$. The first-order flow equations that govern the system (which have been given in Refs. [19, 4]) are written in term of non-standard fake central charge $\mathcal{Z}(\phi, \sqrt{2}\mathcal{D}H)$ whose second argument is τ-dependent and correctly interpolates between B^M (on the horizon) and E^M (at spacelike infinity).

The behavior of the metric function in the unconventional solutions gets modified in the asymptotic region but remains unchanged in the near-horizon region, where it is still governed by the attractor mechanism. This behavior is reminiscent, but opposite, to that of the colored non-Abelian supersymmetric black holes of Refs. [49] in which the near-horizon geometry is modified by the non-Abelian effects while the asymptotic one is unchanged by them.

The formalism and the methods presented in this paper can be applied to the problem of finding the non-extremal generalization of the unconventional solutions studied in this paper. Work in this direction is in progress.

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A Generating new solutions via duality

As mentioned in Section 7.5, a necessary and sufficient condition for a solution to be generating is that all the $\mathrm{Sl}(2;\mathbb{R})$ invariants of the theory are independent when evaluated on the charges and moduli of that solution [50, 51, 45, 46]. In this appendix we are going to study whether or not and why the solution considered in that section is a generating one. We start by stating some general properties which we, then, apply to the (toy) axidilaton model and then to the t^3 model.

There are in general 5 independent invariants that characterize each $\mathcal{N}=2$ symmetric supergravity model. They are [52]:

$$i_1 = |\mathcal{Z}|^2 , \tag{A.1}$$

$$i_2 = \mathcal{G}^{ij^*}\mathcal{Z}_i\mathcal{Z}^*_{j^*} , \tag{A.2}$$

$$i_3 = -\tfrac{1}{3}\mathfrak{Re}\left[\mathcal{Z}\mathcal{N}_3(\mathcal{Z}^*)\right] , \tag{A.3}$$

$$i_4 = \tfrac{1}{3}\mathfrak{Im}\left[\mathcal{Z}\mathcal{N}_3(\mathcal{Z}^*)\right] , \tag{A.4}$$

$$i_5 = \mathcal{G}^{ij^*}\mathcal{C}_{ijk}\mathcal{C}^*_{i^*j^*k^*}\mathcal{G}^{jl^*}\mathcal{G}^{km^*}\mathcal{G}^{j^*l}\mathcal{G}^{k^*m}\mathcal{Z}^*_{l^*}\mathcal{Z}^*_{m^*}\mathcal{Z}_l\mathcal{Z}_m , \tag{A.5}$$

where $\mathcal{Z}$ is the central charge, $\mathcal{G}^{ij^*}$ the inverse Kähler metric,

$$\mathcal{Z}_i \equiv \mathcal{D}_i\mathcal{Z} , \tag{A.6}$$

are the “matter” central charges,

$$\mathcal{C}_{ijk} \equiv \mathcal{D}_i\mathcal{V}_M\mathcal{D}_j\mathcal{D}_k\mathcal{V}^M , \tag{A.7}$$

and

$$\mathcal{N}_3(\mathcal{Z}^*) \equiv \mathcal{C}_{ijk}\mathcal{G}^{il^*}\mathcal{G}^{jm^*}\mathcal{G}^{kn^*}\mathcal{Z}^*_{l^*}\mathcal{Z}^*_{m^*}\mathcal{Z}^*_{n^*} . \tag{A.8}$$

All these invariants are function of the charges and the scalars but their combination

$$J_4(\mathcal{Q}) = (i_1-i_2)^2 + 4i_4 - i_5 , \tag{A.9}$$

depends quartically on the charges only. Sometimes it is advantageous to work with $J_4(\mathcal{Q})$ instead of i_5.

A.1 2-charge generating solutions of the axidilaton model

The minimal number of non-vanishing charges that are necessary for an extremal, supersymmetric[33], black hole of axidilaton theory to be regular is two. Taking into account the form of the Hesse potential Eq. (5.39) and of the axidilaton Eq. (5.40), it is easy to see that there are only two possible non-singular 2-charge configurations, namely $(p^0, p^1, 0, 0)^T$ and $(0, 0, q_0, q_1)^T$.

In this model, the tensor $\mathcal{C}_{ijk}$ vanishes identically, and so does $\mathcal{N}_3(\mathcal{Z}^*)$ and the invariants i_3, i_4, i_5. The model is characterized by the two invariants i_1 and i_2, which are, respectively, the squares of the absolute values of the true and fake central charges at infinity

$$i_1 = |\mathcal{Z}(\lambda_\infty, \mathcal{Q})|^2 \,, \qquad i_2 = |\hat{\mathcal{Z}}(\lambda_\infty, \mathcal{Q})|^2 \,, \tag{A.10}$$

and both are independent for any 2-charge solution (for $\mathfrak{Re}\,\lambda_\infty = 0$ or not) and, in principle, it should be a generating solution. However, depending on our choice of harmonic functions, the regular solutions with two charges may have a vanishing $\mathfrak{Re}\,\lambda_\infty$ and the subgroup of $\mathrm{Sl}(2;\mathbb{R})$ that generates a non-vanishing $\mathfrak{Re}\,\lambda_\infty$, which consists of matrices of the form $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ do not leave invariant the 2-charge configurations. Therefore, the $\mathrm{Sl}(2;\mathbb{R})$ orbit of the regular 2-charge configurations may not cover the full parameter space.

It is interesting to see how the impossibility of generating a solution containing the maximal number of independent parameters arises in practice in this simple case, starting from a configuration characterized by the charges $(0, 0, \hat{q}_0, \hat{q}_1)^T$ and the moduli $\hat{\lambda}_\infty = i\mathfrak{Im}\,\hat{\lambda}_\infty$ (we reserve the unhatted symbols for the final charges and moduli). This solution is determined by two harmonic functions:

$$(\hat{H}^M) = \begin{pmatrix} 0 \\ 0 \\ \frac{s}{\sqrt{2}}\left\{(\mathfrak{Im}\,\hat{\lambda}_\infty)^{1/2} - |\hat{q}_0|\tau\right\} \\ \frac{s}{\sqrt{2}}\left\{(\mathfrak{Im}\,\hat{\lambda}_\infty)^{-1/2} - |\hat{q}_1|\tau\right\} \end{pmatrix} \,, \tag{A.11}$$

where

$$s \equiv \mathrm{sgn}(\hat{q}_0) = \mathrm{sgn}(\hat{q}_1) \,. \tag{A.12}$$

The $\mathrm{Sl}(2;\mathbb{R})$ rotated solution will depend on the original physical parameters $\hat{q}_0, \hat{q}_1, \mathfrak{Im}\,\hat{\lambda}_\infty$ plus the parameters of the $\mathrm{Sl}(2;\mathbb{R})$ transformation a, b, c, d (only 3 of which are independent).

[33] The discussion can also be held for the non-supersymmetric solutions to this model, reaching the same conclusions.

We have to determine $\hat{q}_0, \hat{q}_1, \Im\mathfrak{m}\,\hat{\lambda}_\infty, a, b, c, d$ in terms of the final physical parameters to write the rotated solution in terms of its own physical parameters only.

$\mathrm{Sl}(2;\mathbb{R})$ acts on the charge vector through the matrix Eq. (5.19) so

$$\begin{pmatrix} p^0 \\ p^1 \\ q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} d & & -c & \\ & a & & b \\ -b & & a & \\ & c & & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \hat{q}_0 \\ \hat{q}_1 \end{pmatrix} = \begin{pmatrix} -c\hat{q}_0 \\ b\hat{q}_1 \\ a\hat{q}_0 \\ d\hat{q}_1 \end{pmatrix} . \tag{A.13}$$

From these relations we determine a, b, c, d in terms of the final and original charges:

$$a = q_0/\hat{q}_0 \,, \quad b = p^1/\hat{q}_1 \,, \quad c = -p^0/\hat{q}_0 \,, \quad d = q_1/\hat{q}_1 \,. \tag{A.14}$$

On the other hand, from the transformation rule Eq. (5.10) we get

$$\Re\mathfrak{e}\,\lambda_\infty = \frac{bd + ac(\Im\mathfrak{m}\,\hat{\lambda}_\infty)^2}{d^2 + c^2(\Im\mathfrak{m}\,\hat{\lambda}_\infty)^2} \,, \qquad \Im\mathfrak{m}\,\lambda_\infty = \frac{\Im\mathfrak{m}\,\hat{\lambda}_\infty}{d^2 + c^2(\Im\mathfrak{m}\,\hat{\lambda}_\infty)^2} \,, \tag{A.15}$$

and replacing in these relations the transformation parameters a, b, c, d by the values in Eq. (A.14), we get 2 equations that relate the 3 original to the 6 final physical parameters:

$$p^0 q_0 (\hat{q}_1)^2 (\Im\mathfrak{m}\,\hat{\lambda}_\infty)^2 + \frac{\Re\mathfrak{e}\,\lambda_\infty}{\Im\mathfrak{m}\,\lambda_\infty} (\hat{q}_0\hat{q}_1)^2 \Im\mathfrak{m}\,\hat{\lambda}_\infty - p^1 q_1 (\hat{q}_0)^2 = 0 \,, \tag{A.16}$$

$$\Im\mathfrak{m}\,\lambda_\infty (p^0)^2 (\hat{q}_1)^2 (\Im\mathfrak{m}\,\hat{\lambda}_\infty)^2 - (\hat{q}_0\hat{q}_1)^2 \Im\mathfrak{m}\,\hat{\lambda}_\infty + \Im\mathfrak{m}\,\lambda_\infty (q_1)^2 (\hat{q}_0)^2 = 0 \,. \tag{A.17}$$

The invariance of W implies that

$$\hat{q}_0\hat{q}_1 = p^0 p^1 + q_0 q_1 \,, \tag{A.18}$$

and allows us to eliminate $\hat{q}_1$ from the above two equations. We can solve (A.16) and (A.17) for $\Im\mathfrak{m}\,\hat{\lambda}_\infty$ as a function of the 6 final physical parameters and $\hat{q}_0$ and, for both equations, we find $\Im\mathfrak{m}\,\hat{\lambda}_\infty \hat{q}_0^{-2}$ as a function of those 6 parameters:

$$\Im\mathfrak{m}\,\hat{\lambda}_\infty \hat{q}_0^{-2} = f_1(\mathcal{Q}, \lambda_\infty) \,, \qquad \Im\mathfrak{m}\,\hat{\lambda}_\infty \hat{q}_0^{-2} = f_2(\mathcal{Q}, \lambda_\infty) \,. \tag{A.19}$$

The consistency condition $f_1(\mathcal{Q}, \lambda_\infty) = f_2(\mathcal{Q}, \lambda_\infty)$ determines one of the two final real moduli as a complicated function of the final charges. In other words: the final solution cannot have 6 independent physical parameters, which implies that the original solution is not a generating solution.

On top of this, there seems to be another problem: we cannot solve separately the 3 original physical parameters in terms of the 6 final ones. "Fortunately" only the combination $\Im\mathfrak{m}\,\hat{\lambda}_\infty \hat{q}_0^{-2}$ appears in the rotated solution or, equivalently, in the H^M variables. Using Eqs. (A.13,A.14) and (A.18) we find the these are given by

$$H^M = A^M - \tfrac{1}{\sqrt{2}}\mathcal{Q}^M\tau\,, \qquad \begin{pmatrix} A^0 \\ A^1 \\ A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{2}}p^0(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{1/2} \\ \frac{s}{\sqrt{2}}p^1(p^0p^1+q_0q_1)^{-1}(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{-1/2} \\ \frac{s}{\sqrt{2}}q_0(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{1/2} \\ \frac{s}{\sqrt{2}}q_1(p^0p^1+q_0q_1)^{-1}(\Im\mathfrak{m}\,\hat{\lambda}_\infty\hat{q}_0^{-2})^{-1/2} \end{pmatrix}, \tag{A.20}$$

In the supersymmetric case we know that we can construct a new solution which has, on top of the two non-trivial harmonic functions, two constant ones. If we write all of them in the form

$$\hat{H}^M = \hat{A}^M - \tfrac{1}{\sqrt{2}}\hat{\mathcal{Q}}^M\tau\,, \tag{A.21}$$

then $(\hat{\mathcal{Q}}^M)^T = (0,0,\hat{q}_0,\hat{q}_1)^T$ and, according to the general results of Ref. [9],

$$(\hat{A}^M) = \frac{1}{\sqrt{2\Im\mathfrak{m}\,\hat{\lambda}_\infty}}\Im\mathfrak{m}\left\{\frac{\hat{q}_1\hat{\lambda}^*_\infty - i\hat{q}_0}{|\hat{q}_1\hat{\lambda}^*_\infty - i\hat{q}_0|}\begin{pmatrix} i \\ \hat{\lambda}_\infty \\ -i\hat{\lambda}_\infty \\ 1 \end{pmatrix}\right\}. \tag{A.22}$$

This solution has two independent charges at any generic point in moduli space and should be a generating solution. The difference with the previous case is that, instead of the Eqs. (A.15), we can invert (5.10) and use Eqs. (A.14) and (A.18) to get two independent real equations that do not lead to constraints in the final physical parameters:

$$\hat{\lambda}_\infty\hat{q}_0^{-2} = \frac{1}{(p^0p^1+q_0q_1)}\frac{q_1\lambda_\infty - p^1}{p^0\lambda_\infty + q_0}\,. \tag{A.23}$$

The only combinations of the 4 original physical parameters that appear in the rotated solution are precisely the real and imaginary parts of $\hat{\lambda}_\infty\hat{q}_0^{-2}$ and we obtain a solution with 6 independent physical parameters.

A.2 2-charge solutions of the t^3 model

Again, the minimal number of non-vanishing charges that a regular, extremal, black hole of this model can have is two. A choice of charge vector that leads to regular supersymmetric and non-supersymmetric black holes is $(0,p^1,q_0,0)^T$. In the supersymmetric case, the coefficient of $-\frac{1}{\sqrt{2}}\tau$ in H^M (that we call attractor in the context of this formalism) is given by

$$(B^M) = (\mathcal{Q}^M) = \begin{pmatrix} 0 \\ p^1 \\ q_0 \\ 0 \end{pmatrix}, \tag{A.24}$$

and in the non-supersymmetric one, by

$$(B^M) = \begin{pmatrix} 0 \\ p^1 \\ -q_0 \\ 0 \end{pmatrix} . \tag{A.25}$$

In order to see if these charge configurations lead to generating solutions, we study the values of the invariants. For cubic models with prepotential of the form

$$\mathcal{F} = \tfrac{1}{3!} d_{ijk} \frac{\mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0} , \tag{A.26}$$

one has $\mathcal{C}_{ijk} = e^{\mathcal{K}} d_{ijk}$. The prepotential of the t^3 model is given in Eq. (7.1) and has $d_{111} = -5$ so $\mathcal{C}_{ttt} = \frac{3}{4}(\Im\mathfrak{m} t)^{-3}$. For this model it can be proven that only three invariants are independent and that the other two can be written as a their combination. Specifically, one finds that [53]

$$i_4 = -\sqrt{\tfrac{4}{27} i_2^3 i_1 - i_3^2} , \tag{A.27}$$

$$i_5 = \tfrac{3}{4} i_2^2 , \tag{A.28}$$

and we can take, as independent basis of invariants i_1, i_2 and i_3 (which we can replace by J_4).

Now let us evaluate these invariants for the solutions with charge vector $(0, p^1, q_0, 0)^T$. The result is

$$i_1 = \frac{3}{20(\Im\mathfrak{m}\, t_\infty)^3} \left| -\tfrac{5}{2} p^1 t_\infty^2 - q_0 \right|^2 , \tag{A.29}$$

$$i_2 = \frac{1}{20(\Im\mathfrak{m}\, t_\infty)^3} \left| -\tfrac{5}{2} p^1 t_\infty (t_\infty + 2t_\infty^*) - 3q_0 \right|^2 , \tag{A.30}$$

$$i_3 = -\frac{1}{75(\Im\mathfrak{m}\, t_\infty)^6} \Re\mathfrak{e} \left\{ -\tfrac{i}{8} \left(-\tfrac{5}{2} p^1 t_\infty^2 - q_0 \right) \left[-\tfrac{5}{2} p^1 t_\infty (t_\infty + 2t_\infty^*) - 3q_0 \right]^3 \right\} , \tag{A.31}$$

and it is easy to see that if $\Re\mathfrak{e}\, t_\infty = 0$ (the *axion-free* case) they simplify to

$$i_1 = \frac{3}{20(\Im\mathfrak{m}\, t_\infty)^3} \left[\tfrac{5}{2} p^1 (\Im\mathfrak{m}\, t_\infty)^2 - q_0 \right]^2 , \tag{A.32}$$

$$i_2 = \frac{1}{20(\Im\mathfrak{m}\, t_\infty)^3} \left[\tfrac{5}{2} p^1 (\Im\mathfrak{m}\, t_\infty)^2 + 3q_0 \right]^2 , \tag{A.33}$$

$$i_3 = 0 , \tag{A.34}$$

We see then that in the axion-free case only two invariant are independent and according to the argument in [46] the solutions cannot be seed (generating) solutions.

It is necessary to have $\Re\mathfrak{e}\; t \neq 0$ for the the three invariants $i_1, i_2, i_3 \neq 0$ to be independent from each other and the two-charge solution to be a generating solution.

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