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REPRESENTATIONS OF EXCEPTIONAL SIMPLE ALTERNATIVE SUPERALGEBRAS OF CHARACTERISTIC 3

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ABSTRACT. We study representations of simple alternative superalgebras $B(1,2)$ and $B(2,4)$. The irreducible bimodules and bimodules with super-involution over these superalgebras are classified, and some analogues of the Kronecker factorization theorem are proved for alternative superalgebras that contain $B(1,2)$ and $B(4,2)$.

1. INTRODUCTION

The simple alternative superalgebras were classified in [6] and [5]. In particular, it was proved in [5] that a simple alternative superalgebra $B = B_0+B_1$, which is not just a Z_2-graded alternative algebra, should necessarily have characteristic 3 and be isomorphic to one of the following superalgebras over a field F of characteristic 3.

1) $B = B(1,2)$, where $B_0 = F \cdot 1$, $B_1 = F \cdot x + F \cdot y$, with 1 being the unit of B and $xy = -yx = 1,\ x^2 = y^2 = 0$.

2) $B = B(4,2)$, where $B_0 = M_2(F)$, $B_1 = F \cdot m_1 + F \cdot m_2$ is the 2-dimensional irreducible Cayley bimodule over B_0; that is, B_0 acts on B_1 by

$$e_{ij} \cdot m_k = \delta_{ik} m_j, \qquad i,j,k \in \{1,2\}, \tag{1}$$
$$m \cdot a = \overline{a} \cdot m, \tag{2}$$

where $a \in B_0$, $m \in B_1$, $a \to \overline{a}$ is the symplectic involution in $B_0 = M_2(F)$. The odd multiplication on B_1 is defined by

$$m_1^2 = -e_{21},\ m_2^2 = e_{12},\ m_1m_2 = e_{11},\ m_2m_1 = -e_{22}.$$

3) **The twisted superalgebra of vector type** $B = B(E, D, \gamma)$. Let E be a commutative and associative algebra over F, D be a nonzero derivation of E such that E is D-simple, and $\gamma \in E$. Denote by $\overline{E}$ an isomorphic copy of the vector space E, with an isomorphism mapping $a \to \overline{a}$. Consider the vector space direct sum $B(E, D, \gamma) = E + \overline{E}$ and define multiplication on it by the rules

$$a \cdot b = ab, \quad a \cdot \overline{b} = \overline{a} \cdot b = \overline{ab}, \quad \overline{a} \cdot \overline{b} = \gamma ab + 2D(a)b + aD(b),$$

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where $a, b \in E$ and ab is the product in E. A Z_2-grading on $B = B(E, D, \gamma)$ is defined by $B_0 = E$ and $B_1 = \overline{E}$. In any characteristic, B is a simple right alternative superalgebra; and when $char\, F = 3$, B is alternative.

In this work, we study birepresentations of $B(1,2)$ and of $B(4,2)$. First, we classify the irreducible superbimodules over these superalgebras. It occurs that, besides a certain two-parametric series of bimodules $V(\lambda, \mu)$ over $B(1,2)$, all the other unital irreducible superbimodules for these superalgebras are regular or opposite to them. As a corollary, we prove that every unital $B(4,2)$-superbimodule is completely reducible. Besides, every alternative superalgebra B that contains $B(4,2)$ as a unital subsuperalgebra admits a graded Kronecker factorization $B = B(4,2)\widetilde{\otimes}U$ for a certain associative commutative superalgebra U.

It was shown in [5] that both $B(1,2)$ and $B(4,2)$ admit J-admissible superinvolutions; that is, superinvolutions with symmetric elements in the nucleus. This was used in [5] for constructing new simple exceptional Jordan superalgebras of characteristic 3 as 3×3 Hermitian matrices over $B(1,2)$ and $B(4,2)$. Motivated by the future study of representations of these Jordan superalgebras, we classify the irreducible bimodules with J-admissible superinvolution over $B(1,2)$ and $B(4,2)$. In the case of $B(4,2)$, the list of irreducible bimodules with superinvolution coincides with that of irreducible bimodules, and for $B(1,2)$ this list contains only regular supermodules and their opposites, while the supermodules $V(\lambda, \mu)$ do not enter in the list. As a corollary, every unital supermodule with J-admissible superinvolution over $B(1,2)$ is completely reducible; and every alternative superalgebra with J-admissible superinvolution that contains $B(1,2)$ as a unital subsuperalgebra admits a Kronecker factorization as above.

Now, let us recall some definitions and fix certain notation.

A superalgebra $A = A_0 + A_1$ over a field F is called *alternative* if it satisfies the superidentities

$$(x, y, z) = -(-1)^{d(x)d(y)}(y, x, z) = -(-1)^{d(y)d(z)}(x, z, y),$$

where $(x, y, z) = (xy)z - x(yz)$, $x, y, z \in A_0 \cup A_1$, and $d(r)$ stands for the parity index of a homogeneous element $r:\ d(r) = i$ if $r \in A_i$. In this case, it is easy to see that A_0 is an alternative algebra and A_1 is an alternative bimodule over A_0.

An A-superbimodule $M = M_0 + M_1$ is called an *alternative superbimodule* if the corresponding split extension superalgebra $E = A + M$ is alternative.

For an A-superbimodule M, the *opposite superbimodule* $M^{op} = M_0^{op} + M_1^{op}$ is defined by the conditions $M_0^{op} = M_1$, $M_1^{op} = M_0$, and the following action of A: $a \cdot m = (-1)^{d(a)} am$, $m \cdot a = ma$, for any $a \in A_0 \cup A_1$, $m \in M^{op}$. If M is an alternative A-superbimodule, then one can easily check that so is M^{op}.

A *regular superbimodule*, $Reg\, A$, for a superalgebra A, is defined on the vector superspace A with the action of A coinciding with the multiplication in A.

We will denote, for any homogeneous a and b,

$$\begin{aligned} [a, b] &:= ab - (-1)^{d(a)d(b)}ba, \\ a \circ b &:= ab + (-1)^{d(a)d(b)}ba. \end{aligned}$$

If not stated otherwise, throughout the paper F will denote a field of characteristic 3. All the algebras and superalgebras will be considered over F.

2. Representations of $B(1,2)$

In this section, we classify irreducible superbimodules over the superalgebra $B(1,2)$, defined in the Introduction.

We start with the following general result.

Proposition 2.1. *Let B be a simple commutative non-associative alternative superalgebra, and let V be an irreducible alternative B-superbimodule. Then V is commutative, that is, for any $v \in V_0 \cup V_1,\ a \in B_0 \cup B_1,\ [v,a] = 0$ holds.*

Proof. Let us show first that for any homogeneous $a \in B$ the set $[V,a] := \{[v,a] \mid v \in V\}$ forms a subbimodule of V. Recall two identities that are valid in alternative superalgebras (see [5, 7]):

$$[xy,z] - x[y,z] - (-1)^{d(y)d(z)}[x,z]y - 3(x,y,z) = 0, \tag{3}$$

$$[[x,y],z] - (-1)^{d(y)d(z)}[[x,z],y] - [x,[y,z]] - 6(x,y,z) = 0. \tag{4}$$

Since B is commutative and $char\, B = 3$, we have by (3) for any homogeneous $v \in V,\ b \in B$

$$\begin{aligned} [v,a]b &= (-1)^{d(a)d(b)}[vb,a], \\ b[v,a] &= [bv,a], \end{aligned}$$

which proves that $[V,a]$ is a subbimodule of V. Assume that there exists $z \in B_1$ such that $[V,z] \neq 0$. Then, by irreducibility, $V = [V,z] = [[V,z],z]$. But it follows from (4) that $[[v,z],z] = -[[v,z],z] = 0$; hence $V = [[V,z],z] = 0$, a contradiction. Therefore,

$$[V,B_1] = 0.$$

Now, the set $B_1 + B_1^2$ is an ideal in B. If it were zero, then $B = B_0$ would be a field; so we have $B = B_1 + B_1^2$. Let $x,y \in B_1,\ v \in V$. Then we have by (3)

$$[xy,v] = x[y,v] + (-1)^{d(v)}[x,v]y = 0.$$

Thus, $[B,V] = 0$, proving the proposition. □

Corollary 2.1. *Every unital alternative superbimodule V over the superalgebra $B = B(1,2)$ satisfies the condition*

$$[[[V,B],B],B] = 0.$$

Proof. It was proved above that, for any $v \in V,\ z \in B_1$, the equality $[[v,z],z] = 0$ holds. Linearizing it, we have $[[v,x],y] = -[[v,y],x]$. In particular, $[[V,B],B] = [[V,x],y] = [[V,y],x]$. Therefore,

$$[[[V,B],B],x] = [[[V,y],x],x] = 0,$$

and similarly $[[[V,B],B],y] = 0$, proving the corollary. □

Denote by $V(\lambda,\mu)$, for $\lambda,\mu \in F$, the commutative superbimodule over $B(1,2) = F \cdot 1 + F \cdot x + F \cdot y$, with the basis

$$v_0,\ v_1 y,\ v_0 y^2 \quad \text{for } V_0, \quad v_1,\ v_0 y,\ v_1 y^2 \quad \text{for } V_1,$$

and the action of x and y defined as follows. Let v stand for any of the elements v_0, v_1 and $v_i^s = v_{1-i}$. Then

$$\begin{gathered} vy^j \cdot y = vy^{j+1}, \quad j = 0,1; \quad vy^2 \cdot y = \mu v^s; \\ vy^j \cdot x = \lambda v^s y^j + j v y^{j-1}, \quad j = 0,1,2. \end{gathered}$$

Proposition 2.2. *The superbimodule $V(\lambda,\mu)$ is alternative for any λ,μ and irreducible if $\lambda \neq 0$ or $\mu \neq 0$.*

Proof. It is easy to see that in any commutative superalgebra the equality

$$(a,b,c) = -(-1)^{d(a)d(b)+d(a)d(c)+d(b)d(c)}(c,b,a)$$

holds. This implies easily that every right alternative commutative superbimodule over a commutative superalgebra is also left alternative. Hence, it suffices to prove that $V(\lambda,\mu)$ is right alternative. For this we need to check the following identities:

$$(u,x,y) - (u,y,x) = 0, \tag{5}$$

$$(x,u,y) + (-1)^{d(u)}(x,y,u) = 0, \tag{6}$$

$$(y,u,x) + (-1)^{d(u)}(y,x,u) = 0, \tag{7}$$

$$(x,u,x) + (-1)^{d(u)}(x,x,u) = 0, \tag{8}$$

$$(y,u,y) + (-1)^{d(u)}(y,y,u) = 0, \tag{9}$$

where u is any element of the base. Let us start with (5). For $u = vy^j,\ j = 0,1,$ we have

$$\begin{aligned} (vy^j,x,y) &= \lambda v^s y^{j+1} + (j-1)vy^j, \\ (vy^j,y,x) &= \lambda v^s y^{j+1} + (j+1)vy^j + vy^j, \end{aligned}$$

which gives (5) since $char\, F = 3$. Similarly,

$$\begin{aligned} (vy^2,x,y) &= \lambda\mu v + 2vy^2 - vy^2, \\ (vy^2,y,x) &= \mu v^s \cdot x + vy^2 = \mu\lambda v + vy^2, \end{aligned}$$

which proves (5).

Furthermore, by commutativity,

$$\begin{aligned} (x,u,y) &= (-1)^{d(u)}(ux\cdot y + uy \cdot x), \\ (x,y,u) &= u\cdot xy + uy\cdot x; \end{aligned}$$

hence $(x,u,y) + (-1)^{d(u)}(x,y,u) = (-1)^{d(u)}(ux\cdot y + uy\cdot x + u\cdot xy + uy\cdot x) = (-1)^{d(u)}(ux\cdot y - uy\cdot x - u\cdot xy + u\cdot yx) = (-1)^{d(u)}((u,x,y) - (u,y,x)) = 0$ by (5). Similarly, we have (7). Finally, we have

$$\begin{aligned} (x,u,x) &= (-1)^{d(u)}(ux\cdot x + ux\cdot x), \\ (x,x,u) &= ux\cdot x, \end{aligned}$$

which proves (8) and, similarly, (9). Hence, the module $V(\lambda,\mu)$ is alternative. One can easily check that if $\lambda \neq 0$ or $\mu \neq 0$, then this module is irreducible. $\square$

Observe that the opposite bimodule $(V(\lambda,\mu))^{op}$ is isomorphic to $V(\lambda,\mu)$ under the isomorphism $vy^j \mapsto v^s y^j$. It is also easy to see that the modules $V(\lambda,\mu)$ and $V(\lambda',\mu')$ are isomorphic if and only if $(\lambda,\mu) = \pm(\lambda',\mu')$.

Theorem 2.1. *Every irreducible unital alternative superbimodule V over $B(1,2)$, in the case where the ground field F (of characteristic 3) is algebraically closed, is isomorphic to one of the bimodules: $Reg\, B(1,2)$, $(Reg\, B(1,2))^{op}$, $V(\lambda,\mu)$.*[1]

[1] V. N. Zhelyabin informed the authors that a classification of irreducible alternative superbimodules over $B(1,2)$ was also obtained by M. Trushina.

Proof. According to Proposition 2.1, we can assume that V is commutative; so we may restrict ourselves to considering only the right actions $\rho(x)$ and $\rho(y)$ of x and y on V. Let us prove first that the elements $\rho(x)^3$ and $\rho(y)^3$ lie in the centralizer of V as a right $B(1,2)$-module.

We will use in this proof non-graded (ordinary) commutators, which we will denote by

$$[a,b]_0 := ab - ba,$$

in order to distinguish them from the graded commutators, defined in the Introduction. By super-rightalternativity, we have for any $v \in V$

$$(vx)y - (vy)x = v(xy - yx) = 2v = -v,$$

which gives

$$[\rho(x), \rho(y)]_0 = -id_V. \tag{10}$$

Now

$$\begin{aligned} [\rho(x)^3, \rho(y)]_0 &= \rho(x)^2[\rho(x),\rho(y)]_0 + [\rho(x),\rho(y)]_0\rho(x)^2 + \rho(x)[\rho(x),\rho(y)]_0\rho(x) \\ &= -3\rho(x)^2 = 0. \end{aligned}$$

Thus $\rho(x)^3$ lies in the centralizer of V, and similarly so does $\rho(y)^3$.

Consider the two possible cases separately.

1°. $\rho(x)^3 = \rho(y)^3 = 0$.

Let us prove that in this case V is isomorphic to $Reg\, B(1,2)$ or to its opposite bimodule. Observe first that $\rho(x)^2 \neq 0$. In fact, we have by (10)

$$[\rho(x)^2, \rho(y)]_0 = \rho(x)[\rho(x),\rho(y)]_0 + [\rho(x),\rho(y)]_0\rho(x) = -2\rho(x);$$

so $\rho(x)^2 = 0$ would imply $\rho(x) = 0$, which is impossible. Assume that $\rho(x)^2|_{V_i} \neq 0$ for some $i \in \{0,1\}$, that is, there exists $v \in V_i$ such that $u = (vx)x \neq 0,\ u \in V_i$. Then we have

$$ux = ((vx)x)x = 0.$$

Observe that, by (10), $(ux)y - (uy)x = -u \neq 0$; hence $uy \neq 0$. Furthermore,

$$\begin{aligned} (uy)x &= (ux)y + u = u, \\ ((uy)y)y &= 0, \\ ((uy)y)x &= uy + ((uy)x)y = uy + uy = -uy. \end{aligned}$$

Therefore, the elements $u,\ uy,\ (uy)y$ span a $B(1,2)$-submodule of V, which, by irreducibility, coincides with V. It it easy to check that if $i = 0$, then $V \cong (Reg\ B(1,2))^{op}$, and if $i = 1$, then $V \cong Reg\ B(1,2)$.

2°. $\rho(x)^3 \neq 0$.

We claim that in this case V is isomorphic to a module of the type $V(\lambda,\mu)$. Let $A = alg_F\langle\rho(x),\rho(y)\rangle$ be a subalgebra of $End_F V$ generated by $\rho(x), \rho(y)$. Since V is irreducible, the center $Z = Z(A)$ is a graded division algebra; besides, $Z_1 \ni \rho(x)^3 \neq 0$. It is easy to see that in this case $Z = Z_0 + Z_0 s$ for any fixed $0 \neq s \in Z_1$; in particular, $\rho(x)^3 = \alpha s,\ \rho(y)^3 = \mu s$ for some $\alpha, \mu \in Z_0$. Let $E = alg_F\langle\alpha,\mu,s^2\rangle$. Then $E \subseteq Z_0$ and A is spanned over E by the elements $\rho(x)^i\rho(y)^j,\ s\rho(x)^i\rho(y)^j,\ 0 \leq i,j \leq 2$. In particular, V is finite dimensional over Z_0. Since V is a commutative supermodule, by [1, Proposition 4.2], it is irreducible as an ordinary (non-graded) A-module. This implies, by the density theorem, that $A = End_{Z_0} V$. Let us show that $Z_0 = E$. Consider some $z \in Z_0$, $z = \alpha_0 + \alpha_1\rho(y) + \alpha_2\rho(y)^2$, where α_i depend only on $\rho(x)$ and s. We have $0 = [z, \rho(x)]_0 = \alpha_1 + 2\alpha_2\rho(y)$. Multiplying this by $\rho(y)$ and subtracting from z, we get $z = \alpha_0 - \alpha_2\rho(y)^2$. Commuting z with $\rho(x)$ again, we get $\alpha_2\rho(y) = 0$ and so $z = \alpha_0 = \beta_0 + \beta_1\rho(x) + \beta_2\rho(x)^2$, where $\beta_0, \beta_2 \in E$, $\beta_1 \in Es$. Commuting now z with $\rho(y)$ and arguing as before, we obtain finally that $z = \beta_0 \in E$.

Thus, the field Z_0 is a finitely generated algebra over F. Since F is algebraically closed, this implies that $Z_0 = F$. We can now choose $s \in Z_1$ such that $s^2 = 1$. Let $0 \neq \lambda \in F$ be a root of the polynomial $X^3 - \alpha$ and $v \in V$ such that $s\rho(x)(v) = v^s \cdot x = \lambda v$. We can assume, without loss of generality, that $v = v_0 \in V_0$. Denote $v_1 := v^s$, $\rho(y)^j(v_i) := v_i y^j$ for $0 \leq j \leq 2$. Then we have

$$
\begin{aligned}
&v_0 \cdot x = \lambda v_1,\ v_1 \cdot x = \lambda v_0;\\
&v_i y^j \cdot y = v_i y^{j+1},\ j < 2;\quad v_i y^2 \cdot y = v_i \rho(y)^3 = \mu v_{1-i};\\
&v_i y \cdot x = v_i[\rho(y), \rho(x)]_0 + (v_i \cdot x) \cdot y = v_i + \lambda v_{1-i} y;\\
&v_i y^2 \cdot x = v_i y[\rho(y), \rho(x)]_0 + (v_i y \cdot x) \cdot y = v_i y + \lambda v_{1-i} y^2 + v_i y = \lambda v_{1-i} y^2 + 2 v_i y.
\end{aligned}
$$

These relations show that V is a homomorphic image of the module $V(\lambda, \mu)$. In order to prove that V is isomorphic to $V(\lambda, \mu)$, it suffices to prove that the elements $v_0, v_1 y, v_0 y^2$ are linearly independent over F. It is easy to see that they are nonzero. Assume that

$$\alpha v_0 + \beta v_1 y + \gamma v_0 y^2 = 0 \tag{11}$$

for some $\alpha, \beta, \gamma \in F$. Applying s to this equality, we get

$$\alpha v_1 + \beta v_0 y + \gamma v_1 y^2 = 0. \tag{12}$$

On the other hand, multiplying (11) by x, we get

$$\alpha\lambda v_1 + \beta(\lambda v_0 y + v_1) + \gamma(\lambda v_1 y^2 + 2 v_0 y) = 0,$$

which, by (12), gives

$$\beta v_1 + 2\gamma v_0 y = 0. \tag{13}$$

Applying s to (13), we get $\beta v_0 + 2\gamma v_1 y = 0$, and multiplying (13) by x, we obtain

$$0 = \beta\lambda v_0 + 2\gamma(\lambda v_1 y + v_0) = \lambda(\beta v_0 + 2\gamma v_1 y) + 2\gamma v_0 = 2\gamma v_0.$$

Thus $\gamma = 0$, which implies easily that $\beta = \alpha = 0$ as well. This finishes the proof of the theorem. □

3. Representations of $B(4, 2)$

We will use in this section certain results about alternative bimodules over composition algebras that were proved in [5]. For the convenience of the reader, we state these results below.

Recall that a bimodule V over a composition algebra C is called a *Cayley bimodule* if it satisfies the relation

$$av = v\overline{a}, \tag{14}$$

where $a \in C$, $v \in V$, and $a \to \overline{a}$ is the canonical involution in C.

Proposition 3.1 ([5, Lemma 11 and its proof])**.** *Let $B = B_0 + B_1$ be a unital alternative superalgebra over a field F which contains an even composition subalgebra C with the same unit. Assume that a subspace V of B is C-invariant and satisfies (14). Then, the following identities hold for any $a, b \in C$, $r \in B$, $u, v \in V$.*

$$(ab)v = b(av),\ v(ab) = (vb)a, \tag{15}$$

$$a(ur) = u(\overline{a}r), \tag{16}$$

$$a(uv) = u(va),\ (uv)a = (au)v, \tag{17}$$

$$(u, v, a) = [uv, a]. \tag{18}$$

Proposition 3.2 ([5, Lemma 12 and its proof])**.** *Let H be a generalized quaternion algebra. Then, any unital alternative H-bimodule V admits the decomposition $V = V_a \oplus V_c$, where V_a is an associative H-bimodule and V_c is a Cayley bimodule over H; moreover, the subbimodule V_c coincides with the subspace (V, H, H).*

In this section we are going to prove the following theorems which describe the alternative superbimodules over the superalgebra $B(4, 2)$.

Theorem 3.1. *Let V be a unital irreducible alternative superbimodule over $B(4, 2)$. Then V is isomorphic to $Reg(B(4, 2))$ or to $Reg(B(4, 2))^{op}$.*

Theorem 3.2. *Every unital alternative superbimodule over $B(4, 2)$ is completely reducible.*

We divide the proof into a sequence of lemmas.

Let $B = B(4, 2) = H + M$, with $H = M_2(F)$, $M = F \cdot m_1 + F \cdot m_2$, the 2-dimensional Cayley H-bimodule defined by (1) and (2), and let V be a unital irreducible alternative superbimodule over B. By Proposition 3.2, $V = V_a \oplus V_c$ where V_a is an associative H-bimodule and V_c is a Cayley H-bimodule.

Lemma 3.1. *Let $V = V_a \oplus V_c$ be a unital alternative superbimodule over $B(4, 2) = H + M$. Then, for any $v \in V_c$, $m \in M$, $a \in H$,*

$$(vm)a = (av)m, \tag{19}$$

$$(mv)a = (am)v, \tag{20}$$

and for any $u \in V_a$, $m \in M$, $a, b \in H$,

$$(um)a = (u\overline{a})m, \tag{21}$$

$$a(mu) = m(\overline{a}u), \tag{22}$$

$$((um)a)b = (um)(ba), \tag{23}$$

$$b(a(mu)) = (ab)(mu), \tag{24}$$

$$(um, a, b) = (um)[b, a], \tag{25}$$

$$(b, a, mu) = [b, a](mu). \tag{26}$$

Proof. First, consider $v \in V_c$, $m \in M$, $a \in H$. By (14), $(vm)a - (av)m = (vm)a - (v\overline{a})m = (v, m, a) - (v, \overline{a}, m) + v(ma - \overline{a}m) = (v, m, a) + (v, a, m) = 0$, and similarly $(mv)a - (am)v = 0$.

Now, let $u \in V_a$, $m \in M$, $a, b \in H$. Then $(um)a - (u\overline{a})m = (u, m, a) - (u, \overline{a}, m) + u(ma - \overline{a}m) = 0$, and similarly $a(mu) - (\overline{a}u)m = 0$, which proves (21) and (22). Furthermore, by (21), $(um)a \cdot b = (u\overline{a} \cdot m)b = (u\overline{a} \cdot \overline{b})m = (u \cdot \overline{ba})m = (um)(ba)$, which proves (23). Similarly, by (22), one gets (24). Finally, (25) and (26) follow easily from (23) and (24). ∎

Lemma 3.2. *Let $V = V_a \oplus V_c$ be a unital alternative superbimodule over $B(4,2) = H + M$. Then, V_aM, MV_a, V_cM and MV_c are H-invariant subspaces. Moreover $V_aM + MV_a \subseteq V_c$ and $V_cM + MV_c \subseteq V_a$.*

Proof. Since V_a, V_c, and M are H-invariant, it suffices to prove, for the first part of the lemma, that the product of any H-invariant subspaces U and W in the split extension superalgebra $E = B + V$ is again H-invariant.

We have $(UW)H \subseteq U(WH) + (U,W,H) \subseteq UW + (U,H,W) \subseteq UW$, and similarly $H(UW) \subseteq UW$.

Now, let us prove that $V_aM + MV_a \subseteq V_c$. Recall that, by Proposition 3.2, $V_c = (V,H,H)$. Choose $a,b \in H$ such that $[a,b]^2 \neq 0$. Then $0 \neq [a,b]^2 \in F$, and, by (26),

$$MV_a = [a,b]^2(MV_a) \subseteq [a,b](MV_a) \subseteq (a,b,MV_a) \subseteq (H,H,V) = V_c,$$

and similarly $V_aM \subseteq V_c$.

Finally, for any $v \in V_c,\ m \in M,\ a \in H$, we have by (19) and (15)

$$((vm)a)b = ((av)m)b = (b(av))m = ((ab)v)m = (vm)(ab),$$

which proves that $V_cM \subseteq V_a$. Similarly, by (20) and (15), $MV_c \subseteq V_a$. □

Corollary 3.1. *In the notation of the lemma, $V_a \neq 0$.*

Really, if $V_a = 0$, then $V = V_c$ and $VM = MV = 0$, which yields, for any $v \in V$,

$$v = v \cdot (m_1m_2 - m_2m_1) = (vm_1)m_2 - (vm_2)m_1 = 0,$$

a contradiction. □

Lemma 3.3. *Let V be a unital alternative superbimodule over $B = B(4,2) = H + M$, and let $Z_a = Z_a(V) = \{v \in V_a \mid [v,H] = 0\}$. Then, $Z_a \neq 0$ and satisfies the following conditions:*

i) $[Z_a, B] = 0$,
ii) $(Z_a, B, B) = 0$.

Proof. By Corollary 3.1, V_a is a nonzero unital bimodule over H. The category of unital H-bimodules is equivalent to the category of right unital $H^\circ \otimes H$-modules [4], where H° is the algebra anti-isomorphic to H. Since $H^\circ \otimes H \cong M_4(F)$, this means that every unital H-bimodule is completely reducible and that any two unital irreducible H-bimodules are isomorphic. The regular H-bimodule $Reg\,H$ is unital and irreducible; therefore, the bimodule $V_a = \bigoplus_i W_i$, where each W_i is isomorphic to $Reg\,H$. It is now clear that $Z_a \neq 0$.

Let us prove first that

$$(Z_a, H, M) = 0. \tag{27}$$

By Lemma 3.2, for any $u \in Z_a,\ a \in H,\ m \in M$ we have

$$(a,u,m) = (au)m - a(um) \overset{(14)}{=} (au)m - (um)\overline{a} \overset{(21)}{=} (au)m - (ua)m = [a,u]m = 0,$$

which proves (27). Furthermore, consider the identity

$$([x,y],y,z) = [y,(x,y,z)], \tag{28}$$

which holds in any alternative algebra. Using its superized linearization, we have for any $u \in Z_a,\ m \in M,\ a,b \in H$

$$([u,m],a,b) = -([u,a],m,b) + (-1)^{d(m)d(u)}[m,(u,a,b)] + [a,(u,m,b)] = 0,$$

since $[u, a] = (u, a, b) = 0$ and $(u, m, b) = 0$. Therefore, $([Z_a, M], H, H) = 0$.

By (15),

$$0 = ([u, m], a, b) = ([u, m]a)b - [u, m](ab) = [u, m](ba) - [u, m](ab) = [u, m][b, a].$$

Therefore, $[Z_a, M][H, H] = 0$, which yields $[Z_a, M] = 0$, proving i).

Consider now the identity

$$2[(x, y, z), t] = ([x, y], z, t) + ([y, z], x, t) + ([z, x], y, t), \tag{29}$$

which holds in every alternative algebra (see [7], Lemma 3.2). Using the corresponding superidentity, we have for any $u \in Z_a,\ m, n \in M,\ a \in H$,

$$2[(u, m, n), a] = ([u, m], n, a) + ([m, n], u, a) - (-1)^{d(u)}([n, u], m, a) = 0,$$

by i) and (27). Therefore, $[(Z_a, M, M), H] = 0$, and by superized linearization of (28) we have

$$0 = [a, (u, m, n)] = -(-1)^{d(u)}[m, (u, a, n)] + (u, m, [n, a]) - (u, a, [n, m]).$$

By (27) and the fact that $Z_a \subseteq V_a$, this implies the equality $(Z_a, M, [M, H]) = 0$. But it is easy to see that $[M, H] = M$; hence $(Z_a, M, M) = 0$, yielding ii). □

Proof of Theorem 3.1. Let $V = V_a \oplus V_c$ be a unital irreducible alternative superbimodule over $B = B(4, 2) = H + M$. By Lemma 3.3, $Z_a \neq 0$; so we can choose some homogeneous element $0 \neq u \in Z_a$. The conditions i) and ii) of Lemma 3.3 show that the subspace $u \cdot B$ is a B-subbimodule of V and the mapping $\varphi : a \mapsto u \cdot a$ is a B-bimodule homomorphism of $Reg\, B$ onto uB, in the case where u is even, or of $(Reg\, B)^{op}$ onto uB, in the case where u is odd. Since both $Reg\, B$ and $(Reg\, B)^{op}$ are irreducible, and $\varphi(1) = u \neq 0$, we have that $uB = V$ is isomorphic to $Reg\, B$ or to $(Reg\, B)^{op}$. □

Proof of Theorem 3.2. Let $U = U_a + U_c$ be a unital superbimodule over $B = B(4, 2) = H + M$. It was shown in the proof of Lemma 3.3 that the bimodule U_a is isomorphic to a direct sum of regular H-bimodules: $U_a = \bigoplus_i U_i$, where, for every i, $U_i = u_i H$, and $u_i \in Z_a(U_i)$ is the image of the unit 1 under the isomorphism of $Reg\, H$ onto U_i. In particular, $[u_i, H] = 0$; hence, by Lemma 3.3, $u_i \in Z_a(U)$.

Consider $W = \sum_i u_i B$. Evidently, W is a B-subbimodule of U and $U_a \subseteq W$. Let $v \in U_c$. Then $v = v(m_1 \circ m_2) = (vm_1)m_2 - (vm_2)m_1$. By Lemma 3.2, $vm_i \in U_a \subseteq W$; so $v \in W$ as well, and $U = W$. Since every bimodule $u_i \cdot B$ is irreducible, $U = W$ is completely reducible. □

4. Bimodules with superinvolution

Recall that a linear even mapping $* : A \longrightarrow A$ is called a *superinvolution* of a superalgebra A, if it satisfies the conditions

$$(a^*)^* = a, \quad (ab)^* = (-1)^{d(a)d(b)} b^* a^*,$$

for any homogeneous elements $a, b \in A$.

Now, let V be a superbimodule over a superalgebra $(A, *)$ with superinvolution. By analogy with the non-graded case (see [2]), we will call V an *A-bimodule with superinvolution*, if there exists a linear mapping $- : V \longrightarrow V$ such that the mapping

$$a + v \mapsto a^* + \overline{v}$$

is a superinvolution of the split null extension superalgebra $E = A + V$. Evidently, for a superalgebra with superinvolution A, the bimodules $Reg\,A$ and $(Reg\,A)^{op}$ have the superinvolutions induced by that of A.

It was shown in [5] that the superalgebras $B(1,2)$ and $B(4,2)$ admit the following superinvolutions:

In $B(1,2)$, $a_0 + a_1 \mapsto a_0 - a_1$; and in $B(4,2)$, $a_0 + a_1 \mapsto \overline{a_0} - a_1$, where the mapping $a \mapsto \overline{a}$ is the symplectic involution of the matrix algebra $M_2(F)$.

Now, we will study the structure of superbimodules with superinvolution over $B(1,2)$ and $B(4,2)$. Our first objective is to prove that every irreducible superbimodule with superinvolution over these superalgebras is of the type $Reg\ B$ or $(Reg\,B)^{op}$.

In fact, we will consider the superbimodules with involution that satisfy the additional condition of so-called *J-admissibility* (see [2]). A superbimodule with superinvolution $(V, -)$ over a superalgebra with superinvolution $(A, *)$ is called *J-admissible* if all the symmetric elements of the superalgebra with superinvolution $E = A+V$ lie in the associative center (the nucleus) of E. In fact, only J-admissible bimodules are needed for applications to Jordan algebras.

Theorem 4.1. *Every irreducible unital J-admissible superbimodule V with superinvolution over $B = B(1,2)$ is isomorphic to $Reg\,B$ or to $(Reg\,B)^{op}$.*

Proof. Let V be a superbimodule under consideration, with a superinvolution $v \mapsto \overline{v}$. Observe first that for any $a \in B,\ v \in V$, we have

$$\overline{[a,v]} = \overline{av} - (-1)^{d(v)d(a)}\overline{va} = (-1)^{d(a)d(v)}\overline{v}\,\overline{a} - \overline{a}\,\overline{v} = -[\overline{a},\overline{v}].$$

This means that the subspace $[V,a]$ is invariant with respect to the superinvolution and so is a subbimodule with superinvolution. Now, all the arguments of the proof of Proposition 2.1 are applied to our case, and we conclude that V is a commutative B-supermodule.

It is clear that $V = Sym\,V \oplus Skew\,V$, where, for any $h \in Sym\,V,\ k \in Skew\,V$, we have $\overline{h} = h,\ \overline{k} = -k$. Assume first that $Sym\,V \neq 0$ and choose some $0 \neq h \in Sym\,V$. By J-admissibility, $(h,B,B) = 0$, and so we have

$$\begin{aligned}
&(hx)x = (h,x,x) + h(xx) = 0, \quad (hy)y = 0,\\
&(hx)y = (h,x,y) + h(xy) = h(xy) = h, \quad (hy)x = -h,\\
&\overline{hx} = (-1)^{d(h)}\overline{x}\,\overline{h} = -(-1)^{d(h)}x\,h = -hx, \quad \overline{hy} = -hy.
\end{aligned}$$

Therefore, the subspace $U = Fh + F(hx) + F(hy)$ is a B-subbimodule with involution of V, and hence $U = V$. It is clear that $U \cong Reg\,B$ for even h, and $U \cong (Reg\,B)^{op}$ for odd h.

Now, assume that $Sym\,V = 0$, that is, $\overline{v} = -v$ for any $v \in V$. Then we have

$$\overline{vx} = (-1)^{d(v)}\overline{x}\,\overline{v} = (-1)^{d(v)}xv = vx;$$

hence $vx \in Sym\,V = 0$. Similarly, $vy = 0$, and finally $v = v(xy - yx) = (vx)y - (vy)x = 0$, a contradiction. □

Theorem 4.2. *Every unital J-admissible alternative superbimodule V with superinvolution over the superalgebra $B = B(1,2)$ is completely reducible.*

Proof. It suffices to prove that V is a sum of irreducible subbimodules with involution, or, equivalently, that every element $v \in V$ lies in a sum of irreducible subbimodules with involution. Assume first that $v = h \in Sym\, V$. We know that $(h, B, B) = 0$. Now let us show that also $[h, B] = 0$. Consider

$$(xhy)x = (xh \cdot y)x = (xh, y, x) + (xh)(yx) = (y, x, xh) - xh = -xh - xh = xh.$$

On the other hand,

$$\begin{aligned}(xhy)x &= (x \cdot hy)x = x(hy \cdot x) + (x, hy, x) = -xh + (-1)^{d(h)}(hy, x, x) \\ &= -xh - (-1)^{d(h)}hx.\end{aligned}$$

Hence, $[x, h] = xh - (-1)^{d(h)}hx = 0$. Similarly, $[y, h] = 0$, and so $[B, h] = 0$.

We can now apply the arguments from the proof of Theorem 4.1 which show that the elements h, hx, hy span an irreducible subbimodule with involution of V. So, in this case we are done.

Now, let $v = k \in Skew\, V$. By the previous arguments, the subbimodule $(Sym\, V)B$ generated by symmetric elements of V is completely reducible; so it suffices to prove that $k \in (Sym\, V)B$. Below, for $v \in V$ we will write $v \equiv 0$ if $v \in (Sym\, V)B$.

It is easy to see that

$$Skew\, V \circ B_1 \subseteq Sym\, V, \quad [Skew\, V, B_1] \subseteq Skew\, V; \tag{30}$$

hence $k \circ z \equiv 0$ for any $z \in B_1$. Moreover, we have

$$\begin{aligned}0 &\equiv (k \circ z)z = kz \cdot z + (-1)^{d(k)}zk \cdot z = (k, z, z) + (-1)^{d(k)}zk \cdot z \\ &= -(-1)^{d(k)}(z, k, z) + (-1)^{d(k)}zk \cdot z = (-1)^{d(k)}z \cdot kz.\end{aligned}$$

Linearizing this relation on z, we have

$$x \cdot ky + y \cdot kx \equiv 0. \tag{31}$$

Now, consider the element $(k \circ x)y \in (Skew\, V \circ B_1)B_1 \subseteq (Sym\, V)B_1 = (Sym\, V) \circ B_1 \subseteq Skew\, V$. We have

$$(k \circ x)y = k + (k, x, y) + (-1)^{d(k)}xk \cdot y.$$

Since the elements $k, (k, x, y), (k \circ x)y$ are skewsymmetric, so is $xk \cdot y$. We have

$$\overline{xk \cdot y} = (-1)^{d(x)d(k)+d(x)d(y)+d(y)d(k)}\overline{y} \cdot \overline{k}\,\overline{x} = y \cdot kx;$$

hence

$$xk \cdot y = -y \cdot kx.$$

Comparing this relation with (31), we get

$$xk \cdot y = -y \cdot kx \equiv x \cdot ky,$$

which yields $(x, k, y) \equiv 0$. Now, we have by (30),

$$k = k \cdot xy \equiv kx \cdot y \equiv \frac{1}{2}[k, x]y \equiv \frac{1}{4}[[k, x], y] = [[k, x], y].$$

By Corollary 2.1, for any B-superbimodule V, the equality $[[[V, B], B], B] = 0$ holds. Therefore, we have

$$k \equiv [[k, x], y] \equiv [[[[k, x], y], x], y] = 0,$$

which proves the theorem. ∎

Corollary 4.1. *Every unital alternative J-admissible superbimodule with superinvolution over the superalgebra $B(1, 2)$ is commutative.* ∎

Now, we turn to bimodules with superinvolution over $B(4,2)$.

Theorem 4.3. *Every unital J-admissible superbimodule with superinvolution V over the superalgebra with superinvolution $B = B(4,2)$ is completely reducible and is a direct sum of irreducible bimodules with superinvolution isomorphic to $Reg\, B$ or to $(Reg\, B)^{op}$.*

Proof. By Theorem 3.2, $V = \bigoplus_i Bu_i$ for certain elements $u_i \in Z_a = Z_a(V)$. In particular, we always have $Z_a \neq 0$. Let us show that $Z_a \subseteq Sym\, V$. First, it is easy to see that Z_a is invariant under the superinvolution; so $Z_a = (Sym\, V \cap Z_a) \oplus (Skew\, V \cap Z_a)$. Assume that there exists $0 \neq u \in Z_a$ such that $\overline{u} = -u$. Consider the element $s = um_1 = \frac{1}{2} u \circ m_1$ (recall that $[u, B] = 0$), where m_1 is one of the two basic elements of M. It is easy to check that $\overline{s} = s$; hence, by J-admissibility of V, we should have $(s, B, B) = 0$. But, by Lemma 3.3, $(um_1, m_2, m_1) = -um_1$. Hence $s = 0$, a contradiction.

Now, if V is irreducible then, for any homogeneous $0 \neq u \in Z_a$ we have $V = uB$, which is isomorphic to $Reg\, B$ or to its opposite, according to the parity of u, under the isomorphism $b \mapsto ub$.

In the general case, it suffices to notice that every u_i generates an irreducible subsuperbimodule which is invariant under the superinvolution and is isomorphic to $Reg\, B$ or to its opposite. $\square$

5. Factorization theorems

In this section, we will prove for the superalgebras $B(1,2)$ and $B(4,2)$ some analogue of the Kronecker factorization theorem for Cayley algebras from [3].

Theorem 5.1. *Let B be an alternative superalgebra with J-admissible superinvolution (that is, every symmetric element lies in the nucleus of B) such that B contains $B(1,2)$ as a unital subsuperalgebra with superinvolution. Then $B \cong U \widetilde{\otimes} B(1,2)$ for a certain commutative associative superalgebra U, where $\widetilde{\otimes}$ denotes a graded tensor product, that is,*

$$(u \widetilde{\otimes} a)(v \widetilde{\otimes} b) = (-1)^{d(a)d(v)} (uv) \widetilde{\otimes} (ab) \tag{32}$$

for any homogeneous $u, v \in U,\ a, b \in B(1,2)$. In particular, the superalgebra B is commutative.

Proof. Consider B as a $B(1,2)$-superbimodule with superinvolution. By Theorem 4.2 and J-admissibility, we conclude that $B = \sum_i u_i B(1,2)$, where $\overline{u_i} = u_i, (u_i, B, B) = 0$. Moreover, $[B, B(1,2)] = 0$, by Corollary 4.1. Let $U = Sym\, B = \{u \in B | \overline{u} = u\}$. Then $B = UB(1,2)$, and we will show that this product is isomorphic to the tensor product we are looking for.

Consider the following identity, which is valid in any alternative algebra (see [7]):

$$[a, b](a, b, c) - (a, b, (a, b, c)) = 0. \tag{33}$$

Superlinearizing it, we have for any $u, v \in U,\ a, b, c \in B(1,2)$

$$\begin{aligned} [u, v](a, b, c) \quad &= \quad \pm[a, v](u, b, c) \pm [u, b](a, v, c) \pm [a, b](u, v, c) \pm (u, v, (a, b, c)) \\ &\quad \pm(a, v, (u, b, c)) \pm (u, b, (a, v, c)) \pm (a, b, (u, v, c)) = 0. \end{aligned}$$

It is easy to see that $(B(1,2), B(1,2), B(1,2)) = (B(1,2))_1 = Fx + Fy$; hence $[u, v]x = [u, v]y = 0$ and $[u, v] = -[u, v](xy - yx) = -([u, v]x)y + ([u, v]y)x = 0$.

Therefore, $[U,U]=0$. Since $U\circ U\subseteq U$, this proves that U is a commutative (and associative) subsuperalgebra of B.

Furthermore, we have for any $u,v\in U,\ a,b\in B(1,2)$,

$$\begin{aligned}(ua)(vb) &= u(avb)=u([a,v]b+(-1)^{d(a)d(v)}vab)=(-1)^{d(a)d(v)}u(vab)\\ &= (-1)^{d(a)d(v)}(uv)(ab),\end{aligned}$$

which shows that B is a homomorphic image of $U\widetilde{\otimes}B(1,2)$. Assume that $u+vx+wy=0$ for some $u,v,w\in U$. Then $u\in Sym\,B,\ vx+wy\in Skew\,B$; hence $u=vx+wy=0$. Moreover, we have $0=(vx+wy)x=-w$ and $0=(vx+wy)y=v$. Therefore, $B\cong U\widetilde{\otimes}B(1,2)$.

One can easily see that, since U and $B(1,2)$ are commutative superalgebras, so is B. □

Theorem 5.2. *Let B be an alternative superalgebra such that B contains $B(4,2)$ as a unital subsuperalgebra. Then $B\cong U\widetilde{\otimes}B(4,2)$ for a certain commutative associative superalgebra U.*

Proof. As before, consider B as a $B(4,2)$-superbimodule. By Theorem 4.3, $B=\sum_i u_iB(4,2)$, where $u_i\in Z_a(B)=\{u\in B|[u,B(4,2)]=0\}$. Set $U=Z_a$. Then $B=UB(4,2)$, and we will show that U is the desired superalgebra.

Let us see first that U is a subsuperalgebra of B. Fix arbitrary $u,v,w\in U,\ a,b,c\in B(4,2)$. Then, by (3),

$$[uv,a] \quad = \quad u[v,a]+(-1)^{d(v)d(a)}[u,a]v=0;$$

hence $UU\subseteq U$. Furthermore, by Lemma 3.3, $(U,B(4,2),B(4,2))=0$, and so, by superization of (29),

$$([a,b],u,v)=\pm([b,u],a,v)\pm([u,a],b,v)\pm[(a,b,u),v]=0.$$

Since $B(4,2)=F1+[B(4,2),B(4,2)]$, this yields that $(U,U,B(4,2))=0$.

Furthermore, by superized linearization of (33), we have

$$\begin{aligned}[a,b](u,v,w) &= \pm[a,v](u,b,w)\pm[u,b](a,v,w)\pm[u,v](a,b,w)\pm(a,b,(u,v,w))\\ &\quad\pm(a,v,(u,b,w))\pm(u,b,(a,v,w))\pm(u,v,(a,b,w))=0.\end{aligned}$$

Choose $a,b\in B(4,2)_0=M_2(F)$ such that $[a,b]^2=\alpha\in F,\ \alpha\neq 0$. Then $\alpha(u,v,w)=[a,b]^2(u,v,w)=[a,b]([a,b](u,v,w))=0$ and $(u,v,w)=0$. Thus, U is associative.

Applying again the superized linearization of (33), we get

$$\begin{aligned}[u,v](a,b,c) &= \pm[a,v](u,b,c)\pm[u,b](a,v,c)\pm[a,b](u,v,c)\pm(u,v,(a,b,c))\\ &\quad\pm(a,v,(u,b,c))\pm(u,b,(a,v,c))\pm(a,b,(u,v,c))=0.\end{aligned}$$

Since $m_i=-(e_{ii},e_{ji},m_j),\ i,j=1,2,\ i\neq j$, this implies $[u,v]m_i=0,\ i=1,2$, and finally

$$[u,v]=[u,v](m_1m_2-m_2m_1)=([u,v]m_1)m_2-([u,v]m_2)m_1=0.$$

Therefore, U is a commutative and associative subsuperalgebra of B.

It is clear that B is a homomorphic image of $U\widetilde{\otimes}B(4,2)$. Assume that $w=\sum_{ij}u_{ij}e_{ij}+u_1m_1+u_2m_2=0$ for some $u_i,u_{ij}\in U$. Then we have

$$\begin{aligned}0 &= (e_{11},e_{21},w)=-u_2m_1,\\ 0 &= (e_{22},e_{12},w)=-u_1m_2,\end{aligned}$$

which implies easily that $u_1 = u_2 = 0$. Furthermore,

$$0 = (e_{ii}w)e_{jj} = u_{ij}e_{ij},$$

which yields easily $u_{ij} = 0$ for all i, j. □

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