MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **69** (2013) 463-486

ISSN 0340 - 6253

On Order–Isomorphisms of Stochastic Orders Generated by Partially Ordered Sets with Applications to the Analysis of Chemical Components of Seaweeds

María Concepción López-Díaz[a], Miguel López-Díaz[b]*

[a]Departamento de Matemáticas, Universidad de Oviedo
C/ Calvo Sotelo s/n. E-33007 Oviedo, Spain. cld@uniovi.es

[b]Departamento de Estadística e I.O. y D.M., Universidad de Oviedo
C/ Calvo Sotelo s/n. E-33007 Oviedo, Spain. mld@uniovi.es

(Received April 17, 2012)

Abstract

Given $\mathcal{X}$ a set endowed with a partial order $\preceq$, we can consider the class of $\preceq$-preserving real functions on $\mathcal{X}$ characterized by $x \preceq y$ implies $f(x) \leq f(y)$. Such a class of functions generates a stochastic order $\preceq_g$ on the set of probabilities associated with $\mathcal{X}$ by means of $P \preceq_g Q$ when $\int_{\mathcal{X}} f\, dP \leq \int_{\mathcal{X}} f\, dQ$ for all f $\preceq$-preserving functions. In this paper we analyze if the property of being order-isomorphic is transferred from partially ordered sets to the corresponding generated stochastic orders and conversely. We obtain that if two posets are order-isomorphic, the posets which are generated by means of the class of measurable preserving functions are also order-isomorphic. We prove that the converse is not true in general, and we obtain particular conditions under which the converse holds. The mathematical results in the paper are applied to the comparison of maritime areas with respect to chemical components of seaweeds. Moreover we show how the solution of the above comparison for specific components can lead to the solution of the comparison when we consider other components of seaweeds by applying the results on order-isomorphisms.

*Corresponding Author: Pho. Nu.: (34)985103362, Fax Nu.: (34)985103354

1 Introduction

Ordered sets have a great importance in many mathematical areas as lattice theory, graph theory, combinatorics, etc, as well as being by itself a remarkable field of research.

The study of ordered sets of probabilities plays a very important role in the mathematical context, involving the analysis of both, theoretical and applied problems. Partial orders on sets of probabilities are also known as stochastic orders, specially in the probabilistic/statistic framework, in which they have a great interest. Stochastic orders have been applied successfully in fields like medicine, genetics, ophthalmology, ecology, veterinary science, physics, economics, quality control theory, etc (see for instance [10], [1], [4], [5] and [9]).

In [7], a method to extend a partial order on a finite set to a stochastic order on the class of probabilities associated with such a set, is proposed. That method is based on the so-called preserving functions and it can be extended to any set. Thus, if $(\mathcal{X}, \mathcal{A})$ is a measurable space and $\preceq$ is a partial order on $\mathcal{X}$, one can take the class of all measurable $\preceq$-preserving real functions, that is, the set of measurable functions $f : \mathcal{X} \to \mathbb{R}$ such that if $x, y \in \mathcal{X}$ satisfy $x \preceq y$, then $f(x) \leq f(y)$. The partial order $\preceq$ generates a binary relation $\preceq_g$ on the set of probabilities on $(\mathcal{X}, \mathcal{A})$ defined by

$$P \preceq_g Q \quad \text{when} \quad \int_{\mathcal{X}} f\, dP \leq \int_{\mathcal{X}} f\, dQ$$

for all f measurable $\preceq$-preserving functions.

In this paper we will focus our attention in analyzing if the property of being order-isomorphic is transferred from partially ordered sets to the stochastic orders generated by such partially ordered sets and conversely. Moreover we apply the mathematical results developed in the paper to the search of maritime areas with elevated values of specific chemical components of seaweeds, namely some amino acids, and we show how the solution of such a problem can derive easily the solution when we consider other components.

The structure of the paper is the following: in Section 2 we include the preliminaries of the manuscript. Section 3 contains the mathematical results of the paper. Finally, we apply the results to the analysis of chemical components of seaweeds in Section 4.

2 Preliminaries

In this section we include the concepts and results which are necessary for our study.

Let $\mathcal{X}$ be a set, a binary relation $\preceq_{\mathcal{X}}$ on $\mathcal{X}$ which satisfies the reflexivity, transitivity and antisymmetric properties is called a *partial order*. The pair $(\mathcal{X}, \preceq_{\mathcal{X}})$ is said to be a *poset (partially ordered set)*.

A subset $U \subset \mathcal{X}$ is said to be an *upper set* if given $x_1, x_2 \in \mathcal{X}$ with $x_1 \in U$ and $x_1 \preceq_{\mathcal{X}} x_2$, then $x_2 \in U$.

An *upper quadrant set* is a subset of $\mathcal{X}$ of the form $Q_x^{\preceq_{\mathcal{X}}} = \{z \in \mathcal{X} \mid x \preceq_{\mathcal{X}} z\}$, with $x \in \mathcal{X}$. We will denote by $\mathcal{Q}^{\preceq_{\mathcal{X}}}$ the class of upper quadrant sets determined by the partial order $\preceq_{\mathcal{X}}$ on $\mathcal{X}$. Note that any upper quadrant set is an upper set.

Let $x_1, x_2 \in \mathcal{X}$. We will say that x_2 *covers* x_1 if $x_1 \preceq_{\mathcal{X}} x_2$, $x_1 \neq x_2$ and there is not $x_3 \in \mathcal{X}$, $x_1 \neq x_3 \neq x_2$, with $x_1 \preceq_{\mathcal{X}} x_3 \preceq_{\mathcal{X}} x_2$.

Let $\mathcal{X}$ be a finite set, the *Hasse diagram* of the poset $(\mathcal{X}, \preceq_{\mathcal{X}})$ is a directed graph with vertices set $\mathcal{X}$ and an edge from x to y if y covers x.

In order to construct a Hasse diagram we will draw the points of $\mathcal{X}$ in the plane such that if $x_1 \preceq_{\mathcal{X}} x_2$, the point for x_2 has a larger y-(vertical) coordinate than the point for x_1.

Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ be posets. A mapping $\phi : \mathcal{X} \to \mathcal{Y}$ is said to be *order-preserving* if for any $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq_{\mathcal{X}} x_2$, we have that $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$.

Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ be a poset. A mapping $f : \mathcal{X} \to \mathbb{R}$ is said to be $\preceq_{\mathcal{X}}$*-preserving* if for any $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq_{\mathcal{X}} x_2$, we have that $f(x_1) \leq f(x_2)$.

Note that the class of mappings which are $\preceq_{\mathcal{X}}$-preserving is the class of order-preserving mappings when we consider the posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathbb{R}, \leq)$, $\leq$ being the usual order on the real line.

Given $U \subseteq \mathcal{X}$, I_U will stand for the *indicator function* of U, that is,

$$I_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if U is an upper set then I_U is $\preceq_{\mathcal{X}}$-preserving.

Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ be posets. A mapping $\phi : \mathcal{X} \to \mathcal{Y}$ is said to be an *order-isomorphism* if:

$i)$ ϕ is order-preserving,

$ii)$ there exists $\phi^{-1} : \mathcal{Y} \to \mathcal{X}$ inverse of ϕ,

$iii)$ ϕ^{-1} is order-preserving.

Clearly a mapping $\phi : \mathcal{X} \to \mathcal{Y}$ is an order-isomorphism if and only if

$i)$ ϕ is bijective,

$ii)$ for all $x_1, x_2 \in \mathcal{X}$ it holds that $x_1 \preceq_{\mathcal{X}} x_2$ if and only if $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$.

Two posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ are said to be *order-isomorphic* if there exists an order-isomorphism $\phi : \mathcal{X} \to \mathcal{Y}$.

Note that if two ordered sets are order-isomorphic, they are indistinguishable for the purposes of order theory because they have the same order structure.

The reader is referred, for instance, to [11] and [12] for an introduction to the theory of ordered sets.

Given $(\mathcal{X}, \preceq_{\mathcal{X}})$ a poset, we will denote by $\mathcal{B}_{\mathcal{X}}$ the σ-algebra generated by the class of upper quadrant sets, that is, $\mathcal{B}_{\mathcal{X}} = \sigma(\mathcal{Q}^{\preceq_{\mathcal{X}}})$.

The usual Borel σ-algebra on $\mathbb{R}$ will be denoted by $\mathcal{B}$.

The symbol $\mathcal{F}^{\mathcal{X}}$ will represent the set of mappings $f : \mathcal{X} \to \mathbb{R}$ which are measurable with respect to $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{B}$, and $\preceq_{\mathcal{X}}$-preserving.

On the other hand, $\mathcal{P}_{\mathcal{X}}$ will stand for the set of probability measures on the measurable space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$. Moreover, $\mathcal{P}^0_{\mathcal{X}}$ will denote the subset of $\mathcal{P}_{\mathcal{X}}$ composed by degenerated probabilities, that is,

$$\mathcal{P}^0_{\mathcal{X}} = \{P_x \in \mathcal{P}_{\mathcal{X}} \mid x \in \mathcal{X}, P_x(B) = 1 \text{ if } x \in B, P_x(B) = 0 \text{ otherwise}, B \in \mathcal{B}_{\mathcal{X}}\}.$$

A binary relation $\widetilde{\preceq}$ on $\mathcal{P}$ is said to be a *stochastic order* if $(\mathcal{P}, \widetilde{\preceq})$ is a poset.

The reader is referred to [10] and [13] for a rigorous introduction to stochastic orders.

A method to generate a stochastic order on the class of probabilities associated with a finite set endowed with a partial order is proposed in [7]. Such a method is based on the class of preserving mappings and it can be extended to general sets in the following way.

Given a poset $(\mathcal{X}, \preceq_{\mathcal{X}})$, the class $\mathcal{F}^{\mathcal{X}}$ of all measurable $\preceq_{\mathcal{X}}$-preserving mappings generates a binary relation on $\mathcal{P}_{\mathcal{X}}$, denoted by $\preceq_{\mathcal{X}g}$, as follows: if $P_1, P_2 \in \mathcal{P}_{\mathcal{X}}$, then

$$P_1 \preceq_{\mathcal{X}g} P_2 \quad \text{when} \quad \int_{\mathcal{X}} f\, dP_1 \leq \int_{\mathcal{X}} f\, dP_2$$

for all $f \in \mathcal{F}^{\mathcal{X}}$ for which both integrals exist.

Note that the above relation is a partial order for any poset $(\mathcal{X}, \preceq_{\mathcal{X}})$, that is, $(\mathcal{P}_{\mathcal{X}}, \preceq_{\mathcal{X}g})$ is a poset. Reflexivity and transitivity are obvious. On the other hand, if $P_1 \preceq_{\mathcal{X}g} P_2$ and $P_2 \preceq_{\mathcal{X}g} P_1$, since $I_U \in \mathcal{F}^{\mathcal{X}}$ then $P_1(U) = P_2(U)$ for all $U \in \mathcal{A}^{\preceq}$, where $\mathcal{A}^{\preceq}$ is the class of all finite intersections of upper quadrant sets. Since $\sigma(\mathcal{A}^{\preceq}) = \sigma(\mathcal{Q}^{\preceq})$ and $\mathcal{A}^{\preceq}$ is a π-system (that is, $U_1 \cap U_2 \in \mathcal{A}^{\preceq}$ for any $U_1, U_2 \in \mathcal{A}^{\preceq}$), then $P_1 = P_2$ (see for instance [2], p.42), that is, $\preceq_{\mathcal{X}g}$ also satisfies the antisymmetric property.

The following result can be found in [7]. We should note that the finiteness of $\mathcal{X}$ is essential.

Proposition 2.1. *Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ be a poset with $\mathcal{X}$ finite. Let P_1, $P_2 \in \mathcal{P}_{\mathcal{X}}$, then $P_1 \preceq_{\mathcal{X}g} P_2$ if and only if $P_1(U) \leq P_2(U)$ for any U upper set.*

If $\mathcal{X}$ is not finite, an upper set does not necessarily belong to $\mathcal{B}_{\mathcal{X}}$ and so the above result does not hold.

It is interesting to remark that considering σ-algebras generated by the class of upper sets instead of σ-algebras generated by the class of upper quadrant sets, leads in general to σ-algebras too large. For instance, if we consider in $\mathbb{R}^d$ $(d > 1)$ the usual componentwise order, the σ-algebra generated by the class of upper quadrant sets is the usual Borel σ-algebra. In this case, the σ-algebra generated by the upper sets is larger than the usual Borel σ-algebra, which makes many probability distributions induced by random vectors not to be defined on the σ-algebra generated by the upper sets.

Obviously if $\mathcal{X}$ is finite both σ-algebras are equal.

Throughout the paper we will consider sets endowed with partial orders which are not necessarily finite.

3 On generated stochastic orders and order-isomorphisms

In this section we will see that if two posets are order-isomorphic, the posets which are generated by means of the class of measurable preserving functions are also order-isomorphic. Moreover, we will see that the converse is not true in general, and we will obtain particular conditions under which the converse holds.

Some technical results necessary for our purposes are firstly developed.

Proposition 3.1. *Let* $(\mathcal{X}, \preceq_{\mathcal{X}})$ *and* $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ *be posets and let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be an order-isomorphism. Then* ϕ *is measurable with respect to the* σ*-algebras* $\mathcal{B}_{\mathcal{X}}$ *and* $\mathcal{B}_{\mathcal{Y}}$*.*

Proof. Let $y \in \mathcal{Y}$. There is a unique $x \in \mathcal{X}$, with $y = \phi(x)$. Since ϕ is an order-isomorphism, if $z \in Q_x^{\preceq_{\mathcal{X}}}$ we have that $y \preceq_{\mathcal{Y}} \phi(z)$ and so $z \in \phi^{-1}(Q_y^{\preceq_{\mathcal{Y}}})$, thus $Q_x^{\preceq_{\mathcal{X}}} \subseteq \phi^{-1}(Q_y^{\preceq_{\mathcal{Y}}})$.

Now let $w \in \phi^{-1}(Q_y^{\preceq_{\mathcal{Y}}})$. Since ϕ is an order-isomorphism we have that $x \preceq_{\mathcal{X}} w$, and so $\phi^{-1}(Q_y^{\preceq_{\mathcal{Y}}}) \subseteq Q_x^{\preceq_{\mathcal{X}}}$.

Thus we have that $\phi^{-1}(Q_y^{\preceq_{\mathcal{Y}}}) \in \mathcal{Q}^{\preceq_{\mathcal{X}}}$ for all $y \in \mathcal{Y}$, but $\mathcal{B}_{\mathcal{Y}}$ is the σ-algebra generated by $\mathcal{Q}^{\preceq_{\mathcal{Y}}}$, which implies that $\phi^{-1}(U) \in \mathcal{B}_{\mathcal{X}}$ for all $U \in \mathcal{B}_{\mathcal{Y}}$ (see for instance Theorem 13.1 in [2]). $\square$

Proposition 3.2. *Let* $(\mathcal{X}, \preceq_{\mathcal{X}})$ *and* $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ *be posets and let* $\phi : \mathcal{X} \to \mathcal{Y}$ *be a bijective mapping. Then* ϕ *is an order-isomorphism if and only if the following conditions hold*

i) $\mathcal{F}^{\mathcal{X}} = \{f \circ \phi \mid f \in \mathcal{F}^{\mathcal{Y}}\}$,

ii) $\mathcal{F}^{\mathcal{Y}} = \{f \circ \phi^{-1} \mid f \in \mathcal{F}^{\mathcal{X}}\}$.

Proof. Suppose that ϕ is an order-isomorphism. Let us prove part i).

Let $\mathcal{F} = \{f \circ \phi \mid f \in \mathcal{F}^{\mathcal{Y}}\}$. By means of Proposition 3.1 the maps of $\mathcal{F}$ are measurable. Now consider $f \circ \phi$ with $f \in \mathcal{F}^{\mathcal{Y}}$. Since f and ϕ are order-preserving mappings, $f \circ \phi$ satisfies the same property. Thus $\mathcal{F} \subset \mathcal{F}^{\mathcal{X}}$.

Now let $f \in \mathcal{F}^{\mathcal{X}}$, we should see that $f = g \circ \phi$ for certain $g \in \mathcal{F}^{\mathcal{Y}}$.

Let $g = f \circ \phi^{-1} : \mathcal{Y} \to \mathbb{R}$. Since ϕ^{-1} and f are order-preserving mappings, g is $\preceq_{\mathcal{Y}}$-preserving. Thus i) is proved.

Part ii) is proved by means of the case i).

Now suppose that conditions i) and ii) are satisfied. Let us see that the bijective mapping ϕ is an order-isomorphism.

Let $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$ with $x_1 \preceq_{\mathcal{X}} x_2$, thus for any $f \in \mathcal{F}^{\mathcal{Y}}$ we have that $f \circ \phi(x_1) \leq f \circ \phi(x_2)$. Since $I_{Q_{\phi(x_1)}^{\preceq_{\mathcal{Y}}}} \in \mathcal{F}^{\mathcal{Y}}$, $1 \leq I_{Q_{\phi(x_1)}^{\preceq_{\mathcal{Y}}}}(\phi(x_2))$, which implies that $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$.

Now let $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$, with $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$, it can be seen that $x_1 \preceq_{\mathcal{X}} x_2$ following the same method as before using condition ii). $\square$

Now we obtain that if two posets are order-isomorphic, the stochastic orders generated by means of the classes of measurable preserving mappings are also order-isomorphic.

Proposition 3.3. *Let* $(\mathcal{X}, \preceq_{\mathcal{X}})$ *and* $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ *be order-isomorphic posets. Then the posets* $(\mathcal{P}_{\mathcal{X}}, \preceq_{\mathcal{X}g})$ *and* $(\mathcal{P}_{\mathcal{Y}}, \preceq_{\mathcal{Y}g})$ *are order-isomorphic.*

Proof. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be an order-isomorphism. We define $\nabla : \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$ in the following way: given $P \in \mathcal{P}_{\mathcal{X}}$ we construct the mapping $\nabla(P) : \mathcal{B}_{\mathcal{Y}} \to \mathbb{R}$ with

$$\nabla(P)(B) = P(\phi^{-1}(B)) \text{ for all } B \in \mathcal{B}_{\mathcal{Y}}.$$

Note that Proposition 3.1 yields that ϕ is measurable, therefore $\phi^{-1}(B) \in \mathcal{B}_{\mathcal{X}}$ for all $B \in \mathcal{B}_{\mathcal{Y}}$. Since P is a probability, $\nabla(P)$ is a probability and so ∇ is well-defined.

Let P_1, $P_2 \in \mathcal{P}_{\mathcal{X}}$ with $P_1 \neq P_2$, therefore there must exist $A \in \mathcal{B}_{\mathcal{X}}$ with $P_1(A) \neq P_2(A)$, equivalently, $\nabla(P_1)(\phi(A)) \neq \nabla(P_2)(\phi(A))$, which implies that $\nabla(P_1) \neq \nabla(P_2)$. Thus the mapping ∇ is injective.

Now let $Q \in \mathcal{P}_{\mathcal{Y}}$. We define $P : \mathcal{B}_{\mathcal{X}} \to \mathbb{R}$ with $P(A) = Q(\phi(A))$. Since ϕ^{-1} is measurable (see Proposition 3.1) and bijective, and Q is a probability, we have that P is a probability.

Note that for all $B \in \mathcal{B}_{\mathcal{Y}}$, $\nabla(P)(B) = P(\phi^{-1}(B)) = Q(B)$, which proves that the mapping ∇ is surjective.

As a consequence we obtain that $\nabla : \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$ is bijective.

Now suppose that $P_1 \preceq_{\mathcal{X}g} P_2$, therefore

$$\int_{\mathcal{X}} f\, dP_1 \leq \int_{\mathcal{X}} f\, dP_2$$

for all $f \in \mathcal{F}^{\mathcal{X}}$ such that both integrals exist.

Let $Q_i = \nabla(P_i) \in \mathcal{P}_{\mathcal{Y}}$, $1 \leq i \leq 2$. By means of the Change of Variable Theorem (see for instance [8], Theorem C, p.163) we obtain that

$$\int_{\mathcal{X}} f\, dP_i = \int_{\mathcal{Y}} f \circ \phi^{-1}\, dQ_i, \quad 1 \leq i \leq 2,$$

for all $f \in \mathcal{F}^{\mathcal{X}}$.

Now note that $ii)$ in Proposition 3.2 provides that

$$\int_{\mathcal{Y}} g\, dQ_1 \leq \int_{\mathcal{Y}} g\, dQ_2$$

for all $g \in \mathcal{F}^{\mathcal{Y}}$ such that both integrals exist, that is,

$$\nabla(P_1) \preceq_{\mathcal{Y}g} \nabla(P_2).$$

In the same manner, using in this case part $i)$ of Proposition 3.2, we can see that $\nabla(P_1) \preceq_{\mathcal{Y}g} \nabla(P_2)$ implies $P_1 \preceq_{\mathcal{X}g} P_2$.

Therefore we have obtained that the mapping $\nabla : \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$ is an order-isomorphism, and so the result is proved. □

We should remark that the order-isomorphism ∇ is defined by means of the order-isomorphism ϕ. If we know ϕ then ∇ is known. This fact will be used in the applications developed in Section 4.

Example 3.4. Consider the posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ whose Hasse diagrams (taken from [12]) are given in Figure 1.

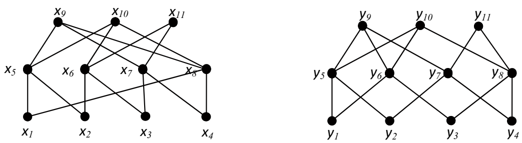

Figure 1: Hasse diagrams in Example 3.4

Let $\phi : \mathcal{X} \to \mathcal{Y}$ be the mapping such that $\phi(x_1) = y_1$, $\phi(x_2) = y_2$, $\phi(x_3) = y_4$, $\phi(x_4) = y_3$, $\phi(x_5) = y_5$, $\phi(x_6) = y_7$, $\phi(x_7) = y_8$, $\phi(x_8) = y_6$, $\phi(x_9) = y_{10}$, $\phi(x_{10}) = y_9$ and $\phi(x_{11}) = y_{11}$. Then ϕ is an order-isomorphism and the posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ are order-isomorphic.

By means of Proposition 3.3 we conclude that the posets $(\mathcal{P}_{\mathcal{X}}, \preceq_{\mathcal{X}g})$ and $(\mathcal{P}_{\mathcal{Y}}, \preceq_{\mathcal{Y}g})$ are order-isomorphic and ϕ determines an order-isomorphism between such posets.

We should indicate that the converse of Proposition 3.3 is not true in general as the following example shows.

Example 3.5. Consider the sets $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2\}$, and the posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ determined by the Hasse diagrams given in Figure 2.

It is obvious that both posets are not order-isomorphic since $|\mathcal{X}| \neq |\mathcal{Y}|$.

Figure 2: Hasse diagrams in Example 3.5

On the other hand, given P_1, $P_2 \in \mathcal{P}_\mathcal{X}$ with $P_1 \neq P_2$, by means of Proposition 2.1 (note that $\mathcal{X}$ is finite) we have that neither $P_1 \preceq_{\mathcal{X}g} P_2$ nor $P_2 \preceq_{\mathcal{X}g} P_1$, and the same happens with the probabilities of $\mathcal{P}_\mathcal{Y}$. Therefore, if we construct a bijection between $\mathcal{P}_\mathcal{X}$ and $\mathcal{P}_\mathcal{Y}$, such a bijection will be an order-isomorphism.

The set $\mathcal{P}_\mathcal{X}$ can be characterized by the subset of $\mathbb{R}^3$

$$\{(p_1, p_2, p_3) \mid \sum_{i=1}^{3} p_i = 1,\, p_i \geq 0\},$$

where each tuple (p_1, p_2, p_3) represents an element of $\mathcal{P}_\mathcal{X}$, p_i being the probability given for such an element to the point x_i.

In a similar way $\mathcal{P}_\mathcal{Y}$ is characterized by the subset of $\mathbb{R}^2$

$$\{(q_1, q_2) \mid \sum_{i=1}^{2} q_i = 1,\, q_i \geq 0\},$$

with the same meaning as in the above case.

Since the sets $\{(p_1, p_2, p_3) \mid \sum_{i=1}^{3} p_i = 1,\, p_i \geq 0\}$ and $\{(q_1, q_2) \mid \sum_{i=1}^{2} q_i = 1,\, q_i \geq 0\}$ have the same cardinality (see for instance [6]), there exists a bijection between both sets, and so $(\mathcal{P}_\mathcal{X}, \preceq_{\mathcal{X}g})$ and $(\mathcal{P}_\mathcal{Y}, \preceq_{\mathcal{Y}g})$ are order-isomorphic.

We obtain conditions under which an order-isomorphism between $(\mathcal{P}_\mathcal{X}, \preceq_{\mathcal{X}g})$ and $(\mathcal{P}_\mathcal{Y}, \preceq_{\mathcal{Y}g})$ implies the existence of an order-isomorphism between $(\mathcal{X}, \preceq_\mathcal{X})$ and $(\mathcal{Y}, \preceq_\mathcal{Y})$.

The following result shows one of such conditions, in this case involving degenerated probabilities. If the image of $\mathcal{P}^0_\mathcal{X}$ by an order-isomorphism is equal to $\mathcal{P}^0_\mathcal{Y}$, then $(\mathcal{X}, \preceq_\mathcal{X})$ and $(\mathcal{Y}, \preceq_\mathcal{Y})$ are order-isomorphic.

Proposition 3.6. *Let $(\mathcal{X}, \preceq_\mathcal{X})$ and $(\mathcal{Y}, \preceq_\mathcal{Y})$ be posets, consider the posets $(\mathcal{P}_\mathcal{X}, \preceq_{\mathcal{X}g})$ and $(\mathcal{P}_\mathcal{Y}, \preceq_{\mathcal{Y}g})$. Let $\nabla : \mathcal{P}_\mathcal{X} \to \mathcal{P}_\mathcal{Y}$ be an order-isomorphism such that $\nabla(\mathcal{P}^0_\mathcal{X}) = \mathcal{P}^0_\mathcal{Y}$, then $(\mathcal{X}, \preceq_\mathcal{X})$ and $(\mathcal{Y}, \preceq_\mathcal{Y})$ are order-isomorphic.*

Proof. We define the mapping $\phi : \mathcal{X} \to \mathcal{Y}$ with $\phi(x) = y$ such that $\nabla(P_x) = P_y$. Note that the condition $\nabla(\mathcal{P}^0_\mathcal{X}) = \mathcal{P}^0_\mathcal{Y}$ implies that ϕ is well-defined.

Let $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$. If $P_{x_1} = P_{x_2}$ then $1 = P_{x_2}(Q_{x_1}^{\preceq_{\mathcal{X}}})$ which implies that $x_1 \preceq_{\mathcal{X}} x_2$, and $1 = P_{x_1}(Q_{x_2}^{\preceq_{\mathcal{X}}})$ which leads to $x_2 \preceq_{\mathcal{X}} x_1$ and so $x_1 = x_2$. Therefore $P_{x_1} \neq P_{x_2}$ and thus $\nabla(P_{x_1}) \neq \nabla(P_{x_2})$, which implies that $\phi(x_1) \neq \phi(x_2)$, therefore ϕ is injective.

Given $y \in \mathcal{Y}$, let $x \in \mathcal{X}$ with $\nabla(P_x) = P_y$. Since $\nabla(\mathcal{P}_{\mathcal{X}}^0) = \mathcal{P}_{\mathcal{Y}}^0$ and ∇ is bijective we have that $\phi(x) = y$ and so the mapping ϕ is surjective.

Let $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq_{\mathcal{X}} x_2$. For any $f \in \mathcal{F}^{\mathcal{X}}$ we have that $f(x_1) \leq f(x_2)$ and so

$$f(x_1) = \int_{\mathcal{X}} f \, dP_{x_1} \leq \int_{\mathcal{X}} f \, dP_{x_2} = f(x_2),$$

which proves that $P_{x_1} \preceq_{\mathcal{X}g} P_{x_2}$.

Since ∇ is an order-isomorphism, we obtain that $\nabla(P_{x_1}) \preceq_{\mathcal{Y}g} \nabla(P_{x_2})$.

Now let $y_i \in \mathcal{Y}$, $1 \leq i \leq 2$, such that $P_{y_i} = \nabla(P_{x_i})$, therefore $P_{y_1} \preceq_{\mathcal{Y}g} P_{y_2}$. Thus $1 \leq P_{y_2}(Q_{y_1}^{\preceq_{\mathcal{Y}}})$, which implies that $y_2 \in Q_{y_1}^{\preceq_{\mathcal{Y}}}$, that is, $y_1 \preceq_{\mathcal{Y}} y_2$, equivalently, $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$.

In a similar way it can be proved that if $x_1, x_2 \in \mathcal{X}$ with $\phi(x_1) \preceq_{\mathcal{Y}} \phi(x_2)$ then $x_1 \preceq_{\mathcal{X}} x_2$.

Thus the mapping ϕ is an order isomorphism, which proves the result. □

The following result provides other conditions, which in conjunction with an order-isomorphism between $(\mathcal{P}_{\mathcal{X}}, \preceq_{\mathcal{X}g})$ and $(\mathcal{P}_{\mathcal{Y}}, \preceq_{\mathcal{Y}g})$, lead to an order-isomorphism between $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$.

For such an analysis we introduce the following concept.

Definition 3.7. *Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ be a poset and $I \subset \mathcal{X}$. A mapping $\Upsilon : I \to \mathcal{X}$ is said to conserve $\preceq_{\mathcal{X}}$ in each point separately when given $x_1, x_2 \in I$, if one of the following relations holds: $x_1 \preceq_{\mathcal{X}} x_2$, $\Upsilon(x_1) \preceq_{\mathcal{X}} x_2$, $x_1 \preceq_{\mathcal{X}} \Upsilon(x_2)$, then the three relations are satisfied simultaneously.*

Example 3.8. Let us consider the poset $(\mathcal{X}, \preceq_{\mathcal{X}})$ whose Hasse diagram (taken from [12]) is in Figure 3. Let $\Upsilon : \mathcal{X} \to \mathcal{X}$ with $\Upsilon(x_1) = x_4, \Upsilon(x_2) = x_3, \Upsilon(x_3) = x_2, \Upsilon(x_4) = x_1, \Upsilon(x_5) = x_6, \Upsilon(x_6) = x_5, \Upsilon(x_7) = x_8, \Upsilon(x_8) = x_7, \Upsilon(x_9) = x_{11}, \Upsilon(x_{10}) = x_{10}$ and $\Upsilon(x_{11}) = x_9$.

It is seen that such a map conserves $\preceq_{\mathcal{X}}$ in each point separately.

Given $(\mathcal{X}, \preceq_{\mathcal{X}})$ a poset and S a subset of $\mathcal{X}$, we will denote by $\overline{S}$ the complement of S in $\mathcal{X}$, that is, $\overline{S} = \mathcal{X} \setminus S$.

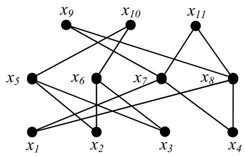

Figure 3: Hasse diagram in Example 3.8

In order to clarify the following result, we introduce Figure 4 which describes some of the subsets involved in Proposition 3.9.

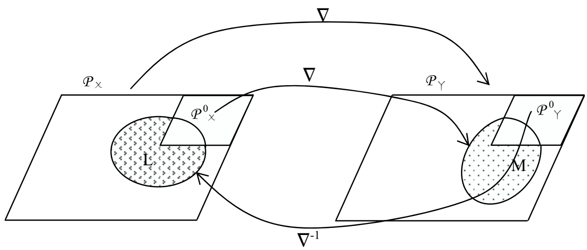

Figure 4: Different sets in Proposition 3.9

Proposition 3.9. *Let $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ be posets, let us consider the posets $(\mathcal{P}_{\mathcal{X}}, \preceq_{\mathcal{X}g})$ and $(\mathcal{P}_{\mathcal{Y}}, \preceq_{\mathcal{Y}g})$. Let $\nabla : \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$ be an order-isomorphism. Let us define the sets $L = \nabla^{-1}(\mathcal{P}^0_{\mathcal{Y}}) \cap \overline{\mathcal{P}^0_{\mathcal{X}}}$ and $M = \nabla(\mathcal{P}^0_{\mathcal{X}}) \cap \overline{\mathcal{P}^0_{\mathcal{Y}}}$. If L and M are nonempty sets and if there exists an order-isomorphism $\Omega : L \to M$ such that $\Omega^{-1} \circ \nabla : \nabla^{-1}(M) \to \mathcal{P}_{\mathcal{X}}$ and $\nabla^{-1} \circ \Omega : L \to \mathcal{P}_{\mathcal{X}}$ conserve $\preceq_{\mathcal{X}g}$ in each point separately, then the posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ are order-isomorphic.*

Proof. We define the mapping $\widehat{\nabla} : \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$ with

$$\widehat{\nabla}(P) = \begin{cases} \nabla(P) & \text{if } P \in \mathcal{P}^0_{\mathcal{X}} \text{ and } \nabla(P) \in \mathcal{P}^0_{\mathcal{Y}} \text{ (condition } C_1\text{)}, \\ \nabla(P) & \text{if } P \notin \mathcal{P}^0_{\mathcal{X}} \text{ and } \nabla(P) \notin \mathcal{P}^0_{\mathcal{Y}} \text{ (condition } C_2\text{)}, \\ \nabla \circ \Omega^{-1} \circ \nabla(P) & \text{if } P \in \mathcal{P}^0_{\mathcal{X}} \text{ and } \nabla(P) \notin \mathcal{P}^0_{\mathcal{Y}} \text{ (condition } C_3\text{)}, \\ \Omega(P) & \text{if } P \notin \mathcal{P}^0_{\mathcal{X}} \text{ and } \nabla(P) \in \mathcal{P}^0_{\mathcal{Y}} \text{ (condition } C_4\text{)}. \end{cases}$$

Note that $\widehat{\nabla}$ is well-defined.

To prove the proposition we will see that $\widehat{\nabla}$ is an order-isomorphism, and that $\widehat{\nabla}(\mathcal{P}^0_{\mathcal{X}}) = \mathcal{P}^0_{\mathcal{Y}}$, which leads to the final result by means of Proposition 3.6.

Let us see that $\widehat{\nabla}$ is a bijection. In the first place we will check that $\widehat{\nabla}$ is injective. Let $P_1,\ P_2 \in \mathcal{P}_{\mathcal{X}}$ with $P_1 \neq P_2$.

- If P_1 and P_2 satisfy the same condition C_i, $i \in \{1, 2, 3, 4\}$, $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$ since ∇ and Ω are bijective mappings.

- If P_1 satisfies condition C_i and P_2 satisfies condition C_j where $i \neq j$ and $i, j \in \{1, 2\}$, then $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$ since ∇ is injective.

- Consider the case where P_1 satisfies condition C_1.
  - If P_2 satisfies C_3, we have that $P_1 \in \mathcal{P}^0_{\mathcal{X}}$ and $\Omega^{-1} \circ \nabla(P_2) \notin \mathcal{P}^0_{\mathcal{X}}$, thus $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$, since ∇ is injective.
  - If P_2 satisfies C_4, we have that $\nabla(P_1) \in \mathcal{P}^0_{\mathcal{Y}}$ and $\Omega(P_2) \notin \mathcal{P}^0_{\mathcal{Y}}$, thus $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$.

- Consider the case where P_1 satisfies condition C_2.
  - If P_2 satisfies C_3, we have that $\nabla(P_1) \notin \mathcal{P}^0_{\mathcal{Y}}$ and $\nabla \circ \Omega^{-1} \circ \nabla(P_2) \in \mathcal{P}^0_{\mathcal{Y}}$, thus $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$.
  - If P_2 satisfies C_4, we have that $P_1 \notin \mathcal{P}^0_{\mathcal{X}}$ and $\nabla^{-1} \circ \Omega(P_2) \in \mathcal{P}^0_{\mathcal{X}}$, thus $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$.

- Consider P_1 satisfying condition C_3.
  - If P_2 satisfies C_4, we have that $\Omega^{-1} \circ \nabla(P_1) \notin \mathcal{P}^0_{\mathcal{X}}$ and $\nabla^{-1} \circ \Omega(P_2) \in \mathcal{P}^0_{\mathcal{X}}$, thus $\widehat{\nabla}(P_1) \neq \widehat{\nabla}(P_2)$.

We have proved the injectivity of $\widehat{\nabla}$.

Now let us see that the mapping $\widehat{\nabla}$ is onto. Let $Q \in \mathcal{P}_{\mathcal{Y}}$ and $P = \nabla^{-1}(Q)$.

- Consider in first place the case $Q \in \mathcal{P}^0_{\mathcal{Y}}$.
  - If $P \in \mathcal{P}^0_{\mathcal{X}}$, then $\widehat{\nabla}(P) = \nabla(P) = Q$.

  - If $P \notin \mathcal{P}^0_{\mathcal{X}}$, then $P \in L$ and so $\Omega(P) \in M$ which implies that there exists $P' \in \mathcal{P}^0_{\mathcal{X}}$ with $\nabla(P') = \Omega(P) \notin \mathcal{P}^0_{\mathcal{Y}}$. Thus $\widehat{\nabla}(P') = \nabla \circ \Omega^{-1} \circ \nabla(P') = \nabla(P) = Q$.

- Now consider the case $Q \notin \mathcal{P}^0_{\mathcal{Y}}$.
  - If $P \in \mathcal{P}^0_{\mathcal{X}}$, then $Q \in \nabla(\mathcal{P}^0_{\mathcal{X}}) \cap \overline{\mathcal{P}^0_{\mathcal{Y}}} = M$. Since $\Omega : L \to M$ is bijective, there exists $P' \in L$ with $\Omega(P') = Q$, and so $\widehat{\nabla}(P') = \Omega(P') = Q$.
  - If $P \notin \mathcal{P}^0_{\mathcal{X}}$, then $\widehat{\nabla}(P) = \nabla(P) = Q$.

As a consequence we have obtained that the mapping $\widehat{\nabla}$ is bijective.

Let us see that $\widehat{\nabla}$ is an order-isomorphism. For such a purpose we will see that given $P_1, P_2 \in \mathcal{P}_{\mathcal{X}}$, we have that

$$P_1 \preceq_{\mathcal{X}g} P_2 \Longleftrightarrow \widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2).$$

We analyze the different possibilities for the expressions of $\widehat{\nabla}(P_i)$, $1 \leq i \leq 2$.

- If P_1 and P_2 satisfy the same condition C_i, $i \in \{1, 2, 3, 4\}$, we obtain the result since ∇ and Ω are order-isomorphisms.

- If P_1 satisfies condition C_i and P_2 satisfies condition C_j where $i \neq j$ and $i, j \in \{1, 2\}$, then we obtain the result since ∇ is an order-isomorphism.

- Consider the cases where P_1 satisfies condition C_1 or C_2.
  - If P_2 satisfies C_3, we have that $\widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2)$ if and only if $\nabla(P_1) \preceq_{\mathcal{Y}g} \nabla \circ \Omega^{-1} \circ \nabla(P_2)$, which is equivalent to $P_1 \preceq_{\mathcal{X}g} \Omega^{-1} \circ \nabla(P_2)$, and this to $P_1 \preceq_{\mathcal{X}g} P_2$, because of the conservation in each point separately of the mapping $\Omega^{-1} \circ \nabla$.
  - If P_2 satisfies C_4, we have that $\widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2)$ if and only if $\nabla(P_1) \preceq_{\mathcal{Y}g} \Omega(P_2)$, which is equivalent to $P_1 \preceq_{\mathcal{X}g} \nabla^{-1} \circ \Omega(P_2)$, and this to $P_1 \preceq_{\mathcal{X}g} P_2$ because of the conservation in each point separately of $\nabla^{-1} \circ \Omega$.

- Consider P_1 satisfying condition C_3.
  - If P_2 satisfies C_1 or C_2, we have that $\widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2)$ if and only if $\nabla \circ \Omega^{-1} \circ \nabla(P_1) \preceq_{\mathcal{Y}g} \nabla(P_2)$, this being equivalent to $\Omega^{-1} \circ \nabla(P_1) \preceq_{\mathcal{X}g} P_2$, which is equivalent to $P_1 \preceq_{\mathcal{X}g} P_2$.

- If P_2 satisfies C_4, we have that $\widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2)$, equivalently $\nabla \circ \Omega^{-1} \circ \nabla(P_1) \preceq_{\mathcal{Y}g} \Omega(P_2)$, which is the same as $\Omega^{-1} \circ \nabla(P_1) \preceq_{\mathcal{X}g} \nabla^{-1} \circ \Omega(P_2)$, and this is the same as $P_1 \preceq_{\mathcal{X}g} P_2$.

• Consider P_1 satisfying condition C_4.

  - If P_2 satisfies C_1 or C_2, then $\widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2)$ is the same as $\nabla^{-1} \circ \Omega(P_1) \preceq_{\mathcal{X}g} P_2$, which is equivalent to $P_1 \preceq_{\mathcal{X}g} P_2$.

  - If P_2 satisfies C_3, we have that $\widehat{\nabla}(P_1) \preceq_{\mathcal{Y}g} \widehat{\nabla}(P_2)$ is the same as $\Omega(P_1) \preceq_{\mathcal{Y}g} \nabla \circ \Omega^{-1} \circ \nabla(P_2)$, that is $\nabla^{-1} \circ \Omega(P_1) \preceq_{\mathcal{X}g} \Omega^{-1} \circ \nabla(P_2)$, equivalently $P_1 \preceq_{\mathcal{X}g} P_2$.

Therefore we have proved that $\widehat{\nabla}$ is an order-isomorphism.

Let us see that $\widehat{\nabla}(\mathcal{P}^0_{\mathcal{X}}) = \mathcal{P}^0_{\mathcal{Y}}$.

Let $P \in \mathcal{P}^0_{\mathcal{X}}$.

If P satisfies C_1, then $\widehat{\nabla}(P) = \nabla(P) \in \mathcal{P}^0_{\mathcal{Y}}$.

If P satisfies C_3, then $\widehat{\nabla}(P) = \nabla \circ \Omega^{-1} \circ \nabla(P)$, but $\Omega^{-1}(\nabla(P)) \in L \subset \nabla^{-1}(\mathcal{P}^0_{\mathcal{Y}})$, which implies that $\widehat{\nabla}(P) \in \mathcal{P}^0_{\mathcal{Y}}$.

Thus we have obtained that $\widehat{\nabla}(\mathcal{P}^0_{\mathcal{X}}) \subset \mathcal{P}^0_{\mathcal{Y}}$.

Now let $Q \in \mathcal{P}^0_{\mathcal{Y}}$.

If $\nabla^{-1}(Q) \in \mathcal{P}^0_{\mathcal{X}}$, then $\widehat{\nabla}(\nabla^{-1}(Q)) = \nabla(\nabla^{-1}(Q)) = Q$.

If $\nabla^{-1}(Q) \notin \mathcal{P}^0_{\mathcal{X}}$, we consider the element of $\mathcal{P}_{\mathcal{X}}$ given by $\nabla^{-1} \circ \Omega \circ \nabla^{-1}(Q)$. By construction we have that $\nabla^{-1} \circ \Omega \circ \nabla^{-1}(Q) \in \mathcal{P}^0_{\mathcal{X}}$ and $\nabla(\nabla^{-1} \circ \Omega \circ \nabla^{-1}(Q)) = \Omega \circ \nabla^{-1}(Q) \in \overline{\mathcal{P}^0_{\mathcal{Y}}}$, thus $\widehat{\nabla}(\nabla^{-1} \circ \Omega \circ \nabla^{-1}(Q)) = \nabla \circ \Omega^{-1} \circ \nabla(\nabla^{-1} \circ \Omega \circ \nabla^{-1}(Q)) = Q$, which proves that $\mathcal{P}^0_{\mathcal{Y}} \subset \widehat{\nabla}(\mathcal{P}^0_{\mathcal{X}})$, and so $\widehat{\nabla}(\mathcal{P}^0_{\mathcal{X}}) = \mathcal{P}^0_{\mathcal{Y}}$.

Therefore $\widehat{\nabla} : \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$ is an order-isomorphism such that $\widehat{\nabla}(\mathcal{P}^0_{\mathcal{X}}) = \mathcal{P}^0_{\mathcal{Y}}$. Proposition 3.6 implies that the posets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ are order-isomorphic. □

4 Some applications in relation to the analysis of chemical components of seaweeds

In this section we develop applications of the above theoretical results in relation to the analysis of chemical components of seaweeds. Namely, on the one hand we will apply stochastic orders generated by partially ordered sets to the problem of the search of maritime areas with elevated values of some components of seaweeds. On the other hand we will see how the solution of the above problem for specific components can lead to the solution of the problem when we consider other components of seaweeds by applying the results on order-isomorphisms in Section 3. This is interesting in maritime industry since seaweeds are used, among others, for food, medicines and industrial products such as paper coatings, adhesives, dyes, explosives, etc. Moreover seaweeds are currently under consideration as potential source of bioethanol.

We should indicate that the procedure we develop below can be applied in multiple fields.

Data will be stored in a matrix such that each column will contain values of a quantitative characteristic and each row will contain the values of the studied characteristics in an experimental unit.

We will consider that all columns have the same orientation, that is, all the characteristics contribute to the aim for ranking in the same way, namely, they have a common monotonicity with the aim.

If we have information on n experimental units, let us denote them by s_1, s_2, ... and s_n, and we have analyzed m characteristics, c_1, c_2, ... and c_m, we will store our information in a matrix with the following structure

$$\begin{array}{c|cccc} & c_1 & c_2 & \dots & c_m \\ \hline s_1 & a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_i & a_{i1} & a_{i2} & \dots & a_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_n & a_{n1} & a_{n2} & \dots & a_{nm} \end{array}$$

where a_{ij} is the value of the characteristic c_j for the experimental unit s_i.

The above data matrices lead under very mild conditions to a partial order on the set of experimental units. The partial order given by the characteristics c_1, c_2, ... and c_m, let us denote it by $\preceq_{\{1,2,\dots,m\}}$, is defined by

$$s_i \preceq_{\{1,2,\dots,m\}} s_j \quad \text{when} \quad (a_{i1}, a_{i2}, \dots, a_{im}) \leq (a_{j1}, a_{j2}, \dots, a_{jm}),$$

where $\leq$ stands for the usual componentwise order, that is, $a_{ik} \leq a_{jk}$ for all $k \in \{1, 2, \dots, m\}$.

Clearly the above binary relation is a partial order when there are not two equal rows. Observe that two equal rows means that the corresponding experimental units are equal with respect to the analyzed characteristics, and so we can reduce the number of experimental units since both have the same behavior.

Note that if we consider a subset of the set of characteristics, such a subset defines a new partial order in the same way. For instance, if we consider characteristics c_1 and c_2 we have that

$$s_i \preceq_{\{1,2\}} s_j \quad \text{when} \quad (a_{i1}, a_{i2}) \leq (a_{j1}, a_{j2}).$$

The reader is referred to the smart book [3] for a detailed description of these concepts, and a clear introduction to ranking and prioritization methods.

Now we describe the key ideas of our application. Let us suppose that we analyze m chemical components, c_1, c_2, $\cdots$, c_m in n types of seaweeds, s_1, s_2, $\cdots$, s_n. We want to rank seaweeds in accordance with the values of the components c_1 and c_2, being interested in elevated values of such components. Therefore we should consider the order $\preceq_{\{1,2\}}$. Such an order gives a "ranking" on the set of seaweeds with respect to c_1 and c_2.

It is known that seaweeds appear in different maritime areas in accordance with different proportions or probabilities. How can we rank such areas when we look for elevated values of c_1 and c_2?

In this point stochastic orders generated by partially ordered sets play a key role. Order preserving functions give greater values to "higher" points of the order, that is, if $x \preceq y$, then $f(x) \leq f(y)$ for any f $\preceq$-preserving mapping. Two probabilities are ordered by the generated stochastic order $\preceq_g$ when

$$\int f\, dP \leq \int f\, dQ$$

for any f $\preceq$-preserving mapping. Roughly speaking $P \preceq_g Q$ means that Q deposits at least as much probability as P in the "higher" parts of the order $\preceq$.

On the one hand, by means of generated stochastic orders we can compare maritime areas of seaweeds with respect to the values of components c_1 and c_2.

On the other hand, we will see that solving the above problem for components c_1 and c_2 means to obtain the solution of the problem for any pair of components c_i and c_j such that the partial orders $\preceq_{\{1,2\}}$ and $\preceq_{\{i,j\}}$ are isomorphic. For such a purpose we will use the results developed in Section 3. Note that if two partial orders are isomorphic, the stochastic orders generated by such partial orders satisfy the same condition (see Proposition 3.3). An order-isomorphism between two posets means that the solution of an optimization problem in a poset, leads to the solution of the problem in the other poset. This is very interesting since solving this kind of problems is in general a costly and time-consuming process from a computational point of view, and so the results in this paper lead to time and effort-saving methods.

First let us describe a method for ranking maritime areas with respect to some chemical components of seaweeds using stochastic orders generated by partially ordered sets.

Let us consider chemical components c_1 and c_2. We have the set $S = \{s_1, s_2, \cdots, s_n\}$ of seaweeds endowed with the partial order $\preceq_{\{1,2\}}$. Suppose that the seaweeds $s_1, s_2, \dots, s_n$ appear with proportions $p_1, p_2, \dots, p_n$ and $q_1, q_2, \dots, q_n$ in two maritime areas respectively. Let P and Q be the probabilities given by the above proportions. We want to find out in which area there are "higher" values of c_1 and c_2. That is, we want to study if some of the relations $P \preceq_{\{1,2\}g} Q$ or $Q \preceq_{\{1,2\}g} P$ are satisfied. According to Proposition 2.1, the above relations are equivalent to $P(U) \leq Q(U)$ or $Q(U) \leq P(U)$ for any upper set U respectively, thus it is necessary to determine the class of upper sets of the partial order $\preceq_{\{1,2\}}$.

Consider a matrix $U_{\preceq_{\{1,2\}}}$ such that there is a univocal correspondence between the rows of $U_{\preceq_{\{1,2\}}}$ and the class of upper sets, that is, if a row corresponds to an upper set U with $U = \{s_{i_1}, s_{i_2}, \dots s_{i_l}\} \subset S$, then there is 1 in the positions $i_1, i_2, \dots, i_l$ of such a row, and 0 in any other case. We will say that the matrix $U_{\preceq_{\{1,2\}}}$ is a *representing matrix of the poset* $(S, \preceq_{\{1,2\}})$.

Observe that we have different representing matrices of $(S, \preceq_{\{1,2\}})$ but any of them can be obtained by means of a permutation of rows of another representing matrix.

The size of a representing matrix is $r \times n$ where n is the cardinal of the set, and r depends on the poset. Note that any point determines a quadrant set and any upper set is a union of quadrant sets, therefore it holds that $n \leq r \leq 2^n$. These bounds can be reached.

Given the probabilities P and Q on S, we denote by $p_i = P(\{s_i\})$ and $q_i = Q(\{s_i\})$, and $\mathbf{p}$ and $\mathbf{q}$ will stand for the vectors

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}.$$

In accordance with Proposition 2.1 we have that $P \preceq_{\{1,2\}g} Q$ is equivalent to

$$U_{\preceq_{\{1,2\}}}\mathbf{p} \leq U_{\preceq_{\{1,2\}}}\mathbf{q}.$$

Observe that the i^{th} row of $U_{\preceq_{\{1,2\}}}\mathbf{p}$ is the probability P of the upper set associated with the i^{th} row of $U_{\preceq_{\{1,2\}}}$.

In this way we have approached the problem of ranking maritime areas with respect to components c_1 and c_2 using stochastic orders generated by partially ordered sets.

Now let us use results in Section 3 for ranking maritime areas with respect to chemical components which determine a poset order-isomorphic to that given by $\preceq_{\{1,2\}}$. Of course we can follow the steps stated for approaching the case when we consider c_1 and c_2, but in the case where we have order-isomorphisms we can apply results in Section 3 and obtain a time and effort-saving method.

Let us consider components c_3 and c_4. If the proportions of seaweeds in two areas are modelled by probabilities P and Q, we need to study if some of the conditions $P \preceq_{\{3,4\}g} Q$ or $Q \preceq_{\{3,4\}g} P$ are satisfied.

Suppose that $\preceq_{\{1,2\}}$ and $\preceq_{\{3,4\}}$ are isomorphic, and ϕ is an order-isomorphism.

By means of Proposition 3.3, $P \preceq_{\{3,4\}g} Q$ is equivalent to $\nabla^{-1}(P) \preceq_{\{1,2\}g} \nabla^{-1}(Q)$, and so the problem is reduced to the analysis of components c_1 and c_2 with probabilities $\nabla^{-1}(P)$ and $\nabla^{-1}(Q)$, previously developed. Note that $\nabla^{-1}(P)$ is immediately known since $\nabla^{-1}(P)(\{s_i\}) = P(\phi(s_i))$ where ϕ is the isomorphism between $\preceq_{\{1,2\}}$ and $\preceq_{\{3,4\}}$.

Let us see applications of the above techniques. Consider Table 1 where we have data obtained in the analysis of four different chemical components (Valine=Va, Leucine=Le, Glutamate=Gl, Threonine=Th) in eight types of red seaweeds (*Acanthophora delillii*=*Ad*, *Acanthophora specifera*=*As*, *Botryocladia leptopoda*=*Bl*, *Gracillaria corticada*=*Gc*, *Hypnea muciformis*=*Hm*, *Sebdania polydactyla*=*Sp*, *Scinia indica*=*Si*, *Laurencia nidifica* = *Ln*).

	Va	Le	Gl	Th
Ad	50.00	63.90	95.90	82.80
As	48.90	65.40	96.10	59.30
Bl	40.90	49.30	57.00	59.90
Gc	25.80	28.70	42.90	34.10
Hm	34.00	50.80	56.90	63.40
Sp	43.70	47.80	57.90	56.20
Si	49.70	48.40	90.80	72.00
Ln	49.10	68.90	104.60	71.50

Table 1: Amino acid composition of red seaweeds. Amino acids are presented in mg g^{-1}.

Let us consider the partial order given by components Va and Le, that is, $\preceq_{\{\mathrm{Va,Le}\}}$. The Hasse diagram of such an order is given in Figure 5.

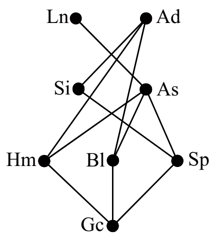

Figure 5: Hasse diagram of the partial order $\preceq_{\{\mathrm{Va,Le}\}}$.

The upper sets determined by the partial order $\preceq_{\{\mathrm{Va,Le}\}}$ are:

$\{Ad\}$

$\{As, Ln\}$

$\{Ad, As, Bl, Ln\}$

$\{Ad, As, Bl, Gc, Hm, Sp, Si, Ln\}$

$\{Ad, As, Hm, Ln\}$

$\{Ad, As, Sp, Si, Ln\}$

$\{Ad, Si\}$

$\{Ln\}$

$\{Ad, As, Ln\}$

$\{Ad, Ln\}$

$\{ Ad, As, Si, Ln \}$

$\{ Ad, As, Bl, Hm, Ln \}$

$\{ Ad, As, Bl, Sp, Si, Ln \}$

$\{ Ad, As, Bl, Si, Ln \}$

$\{ Ad, As, Hm, Sp, Si, Ln \}$

$\{ Ad, As, Hm, Si, Ln \}$

$\{ Ad, Si, Ln \}$

$\{ Ad, As, Bl, Hm, Sp, Si, Ln \}$

$\{ Ad, As, Bl, Hm, Si, Ln \}$

A representing matrix $U_{\preceq_{\{\mathrm{Va,Le}\}}}$ of the partial order is

$$U_{\preceq_{\{\mathrm{Va,Le}\}}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Given two probabilities P and Q on the set of seaweeds, we have that

$$P \preceq_{\{\mathrm{Va,Le}\}g} Q \quad \text{when} \quad U_{\preceq_{\{\mathrm{Va,Le}\}}} \mathbf{p} \le U_{\preceq_{\{\mathrm{Va,Le}\}}} \mathbf{q}.$$

In a geographical zone, two maritime areas contain the red seaweeds Ad, As, Bl, Gc, Hm, Sp, Si and Ln with proportions 0.139, 0.130, 0.061, 0.028, 0.188, 0.223, 0.114, 0.117 and 0.110, 0.154, 0.076, 0.069, 0.173, 0.245, 0.085, 0.088 respectively. Which area is more convenient for the search of high values of Valine and Leucine?

Let $\mathbf{p}^t = (0.139, 0.130, 0.061, 0.028, 0.188, 0.223, 0.114, 0.117)$ and $\mathbf{q}^t = (0.110, 0.154, 0.076, 0.069, 0.173, 0.245, 0.085, 0.088)$. Then the minimum value of $U_{\preceq_{\{\mathrm{Va,Le}\}}}(\mathbf{p} - \mathbf{q})$ is equal to 0, and the maximum value is 0.087. As a consequence we obtain that $Q \preceq_{\{\mathrm{Va,Le}\}g} P$, which means that searching for elevated values of Valine and Leucine in the first area is more convenient.

Now let us consider the componentes Glutamate and Threonine. The order $\preceq_{\{\mathrm{Gl,Th}\}}$ has the Hasse diagram in Figure 6.

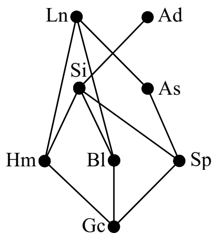

Figure 6: Hasse diagram of the partial order $\preceq_{\{\mathrm{Gl,Th}\}}$.

The posets given by the orders $\preceq_{\{\mathrm{Va,Le}\}}$ and $\preceq_{\{\mathrm{Gl,Th}\}}$ are order-isomorphic. We have that $\phi : S \to S$ such that $\phi(Ad) = Ln$, $\phi(As) = Si$, $\phi(Bl) = Bl$, $\phi(Gc) = Gc$, $\phi(Hm) = Hm$, $\phi(Sp) = Sp$, $\phi(Si) = As$ and $\phi(Ln) = Ad$ is an order-isomorphism. The matrix A which determines the order-isomorphism ϕ is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} \phi(Ad) \\ \phi(As) \\ \phi(Bl) \\ \phi(Gc) \\ \phi(Hm) \\ \phi(Sp) \\ \phi(Si) \\ \phi(Ln) \end{pmatrix} = A \begin{pmatrix} Ad \\ As \\ Bl \\ Gc \\ Hm \\ Sp \\ Si \\ Ln \end{pmatrix}.$$

Now results obtained in Section 3 provide that if P and Q are the probabilities of the considered red seaweeds in two maritime areas, we have that

$$P \preceq_{\{\mathrm{Gl},\mathrm{Th}\}g} Q \iff \nabla^{-1}(P) \preceq_{\{\mathrm{Va},\mathrm{Le}\}g} \nabla^{-1}(Q) \iff P \circ \phi \preceq_{\{\mathrm{Va},\mathrm{Le}\}g} Q \circ \phi$$

$$\iff U_{\preceq_{\{\mathrm{Va},\mathrm{Le}\}}} A\mathbf{p} \leq U_{\preceq_{\{\mathrm{Va},\mathrm{Le}\}}} A\mathbf{q}.$$

Therefore we have obtained a method for ranking maritime areas looking for elevated values of Glutamate and Threonine by using the calculations for the case where we consider components Va and Le.

A second geographical zone has two maritime areas whose proportions of the red seaweeds Ad, As, Bl, Gc, Hm, Sp, Si and Ln are 0.122, 0.149, 0.086, 0.039, 0.178, 0.238, 0.112, 0.076, and 0.160, 0.151, 0.084, 0.026, 0.189, 0.201, 0.093, 0.096 respectively. Note that such proportions are quite similar to the proportions of the first zone. In this case, which area is more convenient for the search of elevated values of Glutamate and Threonine?

Instead of approaching the problem in the same way of the first geographical zone, we take advantage of the isomorphism between $\preceq_{\{\mathrm{Va},\mathrm{Le}\}}$ and $\preceq_{\{\mathrm{Gl},\mathrm{Th}\}}$. Thus, if $\mathbf{p}^t = (0.122, 0.149, 0.086, 0.039, 0.178, 0.238, 0.112, 0.076)$ and $\mathbf{q}^t = (0.160, 0.151, 0.084, 0.026, 0.189, 0.201, 0.093, 0.096)$, we have that

$$P \preceq_{\{\mathrm{Gl},\mathrm{Th}\}g} Q \quad \text{is equivalent to} \quad U_{\preceq_{\{\mathrm{Va},\mathrm{Le}\}}} A\mathbf{p} \leq U_{\preceq_{\{\mathrm{Va},\mathrm{Le}\}}} A\mathbf{q}.$$

In this case we obtain that the minimum value of $U_{\preceq_{\{\mathrm{Va},\mathrm{Le}\}}} A(\mathbf{p} - \mathbf{q})$ is -0.060, and the maximum value is equal to 0. Therefore we obtain the relation $P \preceq_{\{\mathrm{Gl},\mathrm{Th}\}g} Q$, that is, in this case is more appropriate to search for elevated values of Glutamate and Threonine in the second maritime area.

5 Conclusions

In this paper we have proved that the property of being order-isomorphic is transferred from partially ordered sets to generated stochastic orders, but the converse is not true in general. Moreover we have obtained particular conditions under which the converse holds. Such results have been applied to the search of maritime areas with elevated values of specific chemical components of seaweeds, introducing a time and effort-saving method which can lead to the solution of the problem with other components of seaweeds in the case where we have order-isomorphisms. We should indicate that the procedures developed in this paper to the search of maritime areas with elevated values of chemical components, can be applied in multiple fields.

Acknowledgment: We would like to thank the Editor and the Referee for their useful comments and suggestions. The authors are indebted to the Spanish Ministry of Science and Innovation since this research is financed by Grants MTM2010-18370 and MTM2011-22993.

References

[1] G. Ayala, M.C. López–Díaz, M. López–Díaz, L. Martínez–Costa, Studying hypertension in ocular fundus images using Hausdorff dispersion ordering, *Math. Med. Biol.*, accepted.

[2] P. Billingsley, *Probability and Measure*, Wiley, New York, 1995.

[3] R. Brüggemann, G. P. Patil, *Ranking and Prioritization for Multi-indicator Systems*, Springer, New York, 2011.

[4] C. Carleos, M. López–Díaz, An indexed dispersion criterion for testing the sex–biased dispersal of lek mating behavior of capercaillies, *Environ. Ecol. Stat.* **17** (2010) 283–301.

[5] C. Carleos, M. C. López–Díaz, M. López–Díaz, A stochastic order of shape variability with an application to cell nuclei involved in mastitis, *J. Math. Imaging Vis.* **38** (2010) 95–107.

[6] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[7] A. Giovagnoli, H. P. Wynn, Stochastic orderings for discrete random variables, *Stat. Prob. Lett.* **78** (2008) 2827–2835.

[8] P. R. Halmos, *Measure Theory*, Van Nostrand, New York, 1950.

[9] M. López–Díaz, A test for the bidirectional stochastic order with an application to quality control theory, *Appl. Math. Comput.* **217** (2011) 7762–7771.

[10] A. Müller, D. Stoyan, *Comparison Methods for Stochastic Models and Risks*, Wiley, Chichester, 2002.

[11] J. Neggers, H. S. Kim, *Basic Posets*, World Scientific, Singapore, 1998.

[12] B. S. W. Schröder, *Ordered Sets, An Introduction*, Birkhäuser, Boston, 2003.

[13] M. Shaked, J. G. Shanthikumar, *Stochastic Orders*, Springer, New York, 2007.