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When is an integral stochastic order generated by a poset?

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Abstract

Given a partial order $\preceq$ on a set $\mathcal{X}$, one can consider the class of $\preceq$-preserving real functions on $\mathcal{X}$ characterized by $x \preceq y$ implies $f(x) \leq f(y)$. Such a class of functions allows us the generation of a binary relation $\preceq_g$ on the set of probabilities associated with $\mathcal{X}$ by means of $P \preceq_g Q$ when $\int f\,dP \leq \int f\,dQ$ for all $\preceq$-preserving functions f. In this paper we characterize when for an integral stochastic order $\precsim$ on the set of probabilities associated with $\mathcal{X}$, there exists a partial order $\preceq$ on $\mathcal{X}$ such that the relation $\preceq_g$ generated by the class of $\preceq$-preserving functions is equal to $\precsim$. The above characterization is related to the maximal generator of $\precsim$, a result which can be applied for the search of maximal generators of stochastic orders generated by posets.

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1 Introduction

Stochastic orders play a key role in many areas. They have been applied successfully in such fields as reliability theory, economics, decision theory, queueing systems, scheduling problems, medicine, genetics, *etc.* A stochastic order is defined as a partial order relation on a set of probabilities associated with a certain measurable space, although in some contexts the antisymmetric condition is not considered.

A method for generating binary relations on a class of probabilities associated with a set by means of partial order relations on such a set is proposed in [1]. That method is based on the so-called order-preserving functions. Thus, if $(\mathcal{X}, \mathcal{A})$ is a measurable space and $\preceq$ is a partial order on $\mathcal{X}$, one can take the class of all measurable $\preceq$-preserving real functions, that is, the set of measurable functions $f : \mathcal{X} \to \mathbb{R}$ such that if $x, y \in \mathcal{X}$ satisfy $x \preceq y$, then $f(x) \leq f(y)$. The partial order $\preceq$ generates a binary relation $\preceq_g$ on the set of probabilities on $(\mathcal{X}, \mathcal{A})$ defined by

$$P \preceq_g Q \quad \text{when} \int f\,dP \leq \int f\,dQ$$

for all measurable $\preceq$-preserving functions f.

In this paper we characterize when, for a stochastic order $\precsim$, there exists a partial order $\preceq$ on $\mathcal{X}$ such that $\preceq_g$ is $\precsim$. The above characterization will be related to the maximal generators of integral stochastic orders. Different examples of stochastic orders generated and not generated by partially ordered sets will be developed.

2 Preliminaries

Let $\mathcal{X}$ be a set. A binary relation $\preceq$ on $\mathcal{X}$ which is reflexive, transitive and antisymmetric is called a *partial order*. The pair $(\mathcal{X}, \preceq)$ is said to be a *poset*.

A mapping $f : \mathcal{X} \to \mathbb{R}$ is said to be $\preceq$*-preserving* if given $x, y \in \mathcal{X}$ with $x \preceq y$, then $f(x) \leq f(y)$.

The reader is referred, for instance, to [2] and [3] for an introduction to the theory of ordered sets.

Let $\mathcal{A}$ be a σ-algebra on $\mathcal{X}$. It will be assumed throughout the paper that for all $x \in \mathcal{X}$ we have that $\{x\} \in \mathcal{A}$.

Let $\mathcal{F}^{\preceq} = \{f : \mathcal{X} \to \mathbb{R} \mid f$ is measurable and $\preceq$ -preserving$\}$.

Let us denote by $\mathcal{P}$ the set of probabilities on the measurable space $(\mathcal{X}, \mathcal{A})$.

A binary relation $\precsim$ on $\mathcal{P}$ is said to be a *stochastic order* if $(\mathcal{P}, \precsim)$ is a poset.

A stochastic order $\precsim$ on $\mathcal{P}$ is said to be *integral* if there exists a class $\mathcal{R}$ of measurable mappings from $\mathcal{X}$ to $\mathbb{R}$ satisfying that $P \precsim Q$ if and only if

$$\int f\,dP \leq \int f\,dQ$$

for all $f \in \mathcal{R}$ such that the above integrals exist. The class $\mathcal{R}$ is said to be a *generator* of $\precsim$. It is well known that there could be different generators of the same stochastic order.

The reader is referred to [4, 5] and [6] for a rigorous introduction to stochastic orders and integral stochastic orders.

The class of measurable $\preceq$-preserving functions generates a binary relation on $\mathcal{P}$, denoted by $\preceq_g$, as follows: given $P, Q \in \mathcal{P}$, then $P \preceq_g Q$ when

$$\int f\,dP \leq \int f\,dQ$$

for all measurable $\preceq$-preserving functions f for which both integrals exist.

3 Main results

Given a stochastic order $\precsim$ on $\mathcal{P}$, we will obtain a sufficient and necessary condition for the existence of a partial order $\preceq$ on $\mathcal{X}$ such that $\preceq_g$ and $\precsim$ are the same order. It is immediately seen that for such a condition, the stochastic order $\precsim$ should be integral. Moreover, we connect the above result with maximal generators of integral stochastic orders.

In the first place, we construct a partial order on $\mathcal{X}$ by means of a generator of the integral stochastic order $\precsim$.

Proposition 1 *Let $\precsim$ be an integral stochastic order on $\mathcal{P}$ and $\mathcal{R}$ be a generator of the order. We define the binary relation $\preceq^{\mathcal{R}}$ on $\mathcal{X}$ as follows. Let $x, y \in \mathcal{X}$. Then $x \preceq^{\mathcal{R}} y$ when $f(x) \leq f(y)$ for all $f \in \mathcal{R}$. It holds that $(\mathcal{X}, \preceq^{\mathcal{R}})$ is a poset.*

Proof Obviously, $\preceq^{\mathcal{R}}$ is reflexive and transitive. Let $x, y \in \mathcal{X}$ with $x \preceq^{\mathcal{R}} y$ and $y \preceq^{\mathcal{R}} x$. The first condition implies that $f(x) \leq f(y)$ for all $f \in \mathcal{R}$, or equivalently,

$$\int f\,dP_x \leq \int f\,dP_y \quad \text{for all } f \in \mathcal{R},$$

where $P_w \in \mathcal{P}$ denotes the probability distribution degenerated at the point $w \in \mathcal{X}$. As a consequence, we obtain that $P_x \mathrel{\widetilde{\preceq}} P_y$. In a similar way, we have that $P_y \mathrel{\widetilde{\preceq}} P_x$. Since $\widetilde{\preceq}$ is antisymmetric, it holds that $P_x = P_y$. Now, note that $\{x\} \in \mathcal{A}$, which implies that $x = y$. □

Now, we show that the relation defined in Proposition 1 does not depend on a particular generator of the order, but on the order itself.

Proposition 2 *Let $\widetilde{\preceq}$ be an integral stochastic order on $\mathcal{P}$ and $\mathcal{R}_1$ and $\mathcal{R}_2$ be generators of the order. Then $\preceq^{\mathcal{R}_1}$ and $\preceq^{\mathcal{R}_2}$ are the same partial order.*

Proof Let us suppose that $x \preceq^{\mathcal{R}_1} y$. Thus, $f(x) \leq f(y)$ for all $f \in \mathcal{R}_1$, or equivalently,

$$\int f\,dP_x \leq \int f\,dP_y \quad \text{for all } f \in \mathcal{R}_1,$$

that is, $P_x \mathrel{\widetilde{\preceq}} P_y$. Since $\mathcal{R}_2$ is a generator of $\widetilde{\preceq}$, we have that

$$\int g\,dP_x \leq \int g\,dP_y \quad \text{for all } g \in \mathcal{R}_2.$$

Therefore, $g(x) \leq g(y)$ for all $g \in \mathcal{R}_2$, which means that $x \preceq^{\mathcal{R}_2} y$. So, $x \preceq^{\mathcal{R}_1} y$ implies $x \preceq^{\mathcal{R}_2} y$. Reasoning in a similar way, we obtain that $x \preceq^{\mathcal{R}_2} y$ implies $x \preceq^{\mathcal{R}_1} y$, and so the result is proved. □

Given $\widetilde{\preceq}$ an integral stochastic order, we will denote by $\preceq$ the partial order on $\mathcal{X}$ determined by $\widetilde{\preceq}$, that is, $x \preceq y$ when $f(x) \leq f(y)$ for all mappings of a generator of $\widetilde{\preceq}$. Proposition 2 says that such an order is well defined.

Next, we prove that $P \preceq_g Q$ implies $P \mathrel{\widetilde{\preceq}} Q$.

Proposition 3 *Let $\widetilde{\preceq}$ be an integral stochastic order on $\mathcal{P}$. Let $\preceq$ be the partial order on $\mathcal{X}$ determined by $\widetilde{\preceq}$. Then $P \preceq_g Q$ implies $P \mathrel{\widetilde{\preceq}} Q$.*

Proof Let $P, Q \in \mathcal{P}$ with $P \preceq_g Q$, that is,

$$\int f\,dP \leq \int f\,dQ$$

for all measurable $\preceq$-preserving functions f. Let $\mathcal{R}$ be a generator of $\widetilde{\preceq}$. In accordance with the definition of the partial order $\preceq$, any mapping $g \in \mathcal{R}$ is $\preceq$-preserving; as a consequence,

$$\int g\,dP \leq \int g\,dQ$$

for any $g \in \mathcal{R}$, that is, $P \mathrel{\widetilde{\preceq}} Q$. □

We should note that the converse of the above result is not true in general, that is, $\preceq_g$ and $\widetilde{\preceq}$ are not equal in general. Let us consider the following example.

Example 1 Consider the measurable space $(\mathbb{R}, \mathcal{B}_1)$, $\mathcal{B}_1$ being the usual Borel σ-algebra on $\mathbb{R}$. Let $\precsim$ be the integral stochastic order given by the generator

$$\mathcal{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is convex}\}$$

usually referred to as the convex order. Thus, the partial order $\preceq$ on $\mathbb{R}$ determined by $\precsim$ is given by $x \preceq y$ when $f(x) \leq f(y)$ for all $f \in \mathcal{R}$. It is seen that $\{z \in \mathbb{R} \mid x \preceq z\} = \{x\}$ for any $x \in \mathbb{R}$. Therefore, any measurable map $h : \mathbb{R} \to \mathbb{R}$ is $\preceq$-preserving, and so $\preceq_g$ and $\precsim$ are not the same order. In fact, $P \preceq_g Q$ if and only if $P = Q$.

The next proposition shows that if there exists a partial order which generates the stochastic order $\precsim$, then the partial order $\preceq$ on $\mathcal{X}$ determined by $\precsim$ also generates $\precsim$.

Proposition 4 *Let $\precsim$ be an integral stochastic order on $\mathcal{P}$ and $\preceq$ be the partial order on $\mathcal{X}$ determined by $\precsim$. Let $\preceq'$ be a partial order on $\mathcal{X}$ such that $\preceq'_g$ and $\precsim$ are equal. Then $\preceq_g$ and $\precsim$ are the same stochastic order.*

Proof We have that $\mathcal{F}^{\preceq'}$ is a generator of $\precsim$. In accordance with Proposition 2, it holds that $x \preceq y$ when $f(x) \leq f(y)$ for all $f \in \mathcal{F}^{\preceq'}$.

Let us suppose that $x \preceq' y$. It implies that $f(x) \leq f(y)$ for all $f \in \mathcal{F}^{\preceq'}$ and so $x \preceq y$. Now, note that if $f \in \mathcal{F}^{\preceq}$ and $x \preceq' y$, then $x \preceq y$ and so $f(x) \leq f(y)$, that is, $f \in \mathcal{F}^{\preceq'}$. As a consequence, we have that $\mathcal{F}^{\preceq} \subset \mathcal{F}^{\preceq'}$, which derives that $P \preceq'_g Q$ implies $P \preceq_g Q$. Now, Proposition 3 provides the result. □

We introduce the following definition which will be key for our purposes.

Definition 1 Let $(\mathcal{X}, \preceq)$ be a poset. The mappings $f, g : \mathcal{X} \to \mathbb{R}$ are said to be $\preceq$-*comonotonic* if there are no two elements $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq x_2$, $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$.

Now, we obtain sufficient and necessary conditions to guarantee that $\preceq_g$ and $\precsim$ are the same stochastic order.

Theorem 1 *Let $\precsim$ be an integral stochastic order on $\mathcal{P}$ and $\preceq$ be the partial order on $\mathcal{X}$ determined by $\precsim$. Then $\precsim$ and $\preceq_g$ are the same order if and only if there exists a generator $\mathcal{R}$ of $\precsim$ satisfying that if $g : \mathcal{X} \to \mathbb{R}$ is a measurable mapping such that g and f are $\preceq$-comonotonic for all $f \in \mathcal{R}$, then $g \in \mathcal{R}$.*

Proof In the first place, let us see that $\precsim$ and $\preceq_g$ are the same order under the existence of a generator satisfying the above condition.

In Proposition 3 we have shown that $P \preceq_g Q$ implies $P \precsim Q$. Now, let $P, Q \in \mathcal{P}$ with $P \precsim Q$. Let g be a measurable $\preceq$-preserving function and $f \in \mathcal{R}$. Clearly, f and g are $\preceq$-comonotonic, thus $g \in \mathcal{R}$, and so

$$\int g \, dP \leq \int g \, dQ$$

for any measurable $\preceq$-preserving function g; that is, $P \preceq_g Q$, which shows that $P \precsim Q$ implies $P \preceq_g Q$.

Let us prove the converse. If $\preceq_g$ and $\widetilde{\preceq}$ are the same order, we have that $\mathcal{R} = \{f : \mathcal{X} \to \mathbb{R} \mid f$ is measurable and $\preceq$-preserving$\}$ is a generator of $\widetilde{\preceq}$. Let us see that this generator satisfies the condition of the statement.

Let $g : \mathcal{X} \to \mathbb{R}$ be a measurable mapping satisfying that g and f are $\preceq$-comonotonic for all $f \in \mathcal{R}$. Let us see that given $x, y \in \mathcal{X}$ with $x \preceq y$ ($x \neq y$) there exists a mapping $l \in \mathcal{R}$ with $l(x) < l(y)$, which implies that $g(x) \leq g(y)$, and so g is $\preceq$-preserving, that is, $g \in \mathcal{R}$.

If the above result is false, there exist $x, y \in \mathcal{X}$ with $x \neq y$ such that $x \preceq y$ and $l(x) = l(y)$ for all $l \in \mathcal{R}$. This implies that

$$\int l\, dP_x = \int l\, dP_y \quad \text{for all } l \in \mathcal{R}.$$

The antisymmetric property of $\widetilde{\preceq}$ implies that $P_x = P_y$, which is a contradiction with $x \neq y$ since $\{x\} \in \mathcal{A}$. □

Let us consider examples of integral stochastic orders which are in fact generated by posets.

Example 2 Consider the measurable space $(\mathbb{R}, \mathcal{B}_1)$. Let $\widetilde{\preceq}$ be the usual stochastic order, that is, $P \widetilde{\preceq} Q$ when

$$\int I_{(t,\infty)}\, dP \leq \int I_{(t,\infty)}\, dQ$$

for any $t \in \mathbb{R}$, where I_A stands for the indicator function of the set A. According to Proposition 2, the partial order $\preceq$ on $\mathbb{R}$ determined by $\widetilde{\preceq}$ is given by $x \preceq y$ when $x \leq y$.

It is known that the class of non-decreasing functions is a generator of $\widetilde{\preceq}$. Let $\mathcal{R}$ be that class. If g is measurable and $\preceq$-comonotonic with any map of $\mathcal{R}$, then $g \in \mathcal{R}$, and in accordance with Theorem 1, $\preceq_g$ and $\widetilde{\preceq}$ are the same order.

Example 3 Consider the space $(\mathbb{R}, \mathcal{B}_1)$. Let $\widetilde{\preceq}$ be the bidirectional order (see [7] and [8]). It is known that $P \widetilde{\preceq} Q$ if and only if

$$\int I_{(t,\infty)}\, dP \leq \int I_{(t,\infty)}\, dQ \quad \text{and} \quad \int I_{(-\infty,-t)}\, dP \leq \int I_{(-\infty,-t)}\, dQ$$

for any $t \in (0, \infty)$. The definition of the order provides the partial order $\preceq$ on $\mathbb{R}$ determined by $\widetilde{\preceq}$. We obtain that $x \preceq y$ when

$$\begin{cases} x \leq y & \text{if } x > 0, \\ y \leq x & \text{if } x < 0, \\ y \in \mathbb{R} & \text{if } x = 0. \end{cases}$$

In [7] it is shown that the class $\mathcal{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is non-decreasing in $(0, \infty)$, non-increasing in $(-\infty, 0)$, and with a minimum at $0\}$ is a generator of the order. A $\preceq$-comonotonic function with the maps of $\mathcal{R}$ belongs to it. Thus, $\preceq_g$ and $\widetilde{\preceq}$ are the same order.

We will analyze some examples of multivariate stochastic orders after Theorem 2.

Note that, in general, if $\mathcal{R}$ is a generator of a stochastic order and g and f are $\preceq$-comonotonic functions for all $f \in \mathcal{R}$, g is not necessarily an element of $\mathcal{R}$. Consider the generator of Example 1, there are comonotonic functions with the maps of the class of convex functions which are not convex. In fact, any mapping $g : \mathbb{R} \to \mathbb{R}$ is $\preceq$-comonotonic with all functions of $\mathcal{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is convex}\}$.

Now we are proving that under an appropriate framework, a generator satisfying the condition of Theorem 1 is the maximal generator of the order $\tilde{\preceq}$.

We briefly describe the concept of a maximal generator of an integral stochastic order (see [4] or [5]). Such a concept is associated with the so-called *weight function*. That function is a measurable mapping $b : \mathcal{X} \to [1, \infty)$. The weight function determines the space of mappings in which we are looking for the maximal generator. Such a space will be the class of measurable mappings with bounded b-norm, where the *b-norm* of a mapping $f : \mathcal{X} \to \mathbb{R}$ is

$$\|f\|_b = \sup_{x \in \mathcal{X}} \frac{|f(x)|}{b(x)}.$$

For our purpose, it is sufficient to consider $b = 1$. In this way, the b-norm of any mapping $f : \mathcal{X} \to \mathbb{R}$ is equal to

$$\|f\|_b = \sup_{x \in \mathcal{X}} |f(x)|.$$

So, if $\mathcal{B}$ denotes the set of measurable functions $f : \mathcal{X} \to \mathbb{R}$ with the finite b-norm, $\mathcal{B}$ is the set of all measurable and bounded mappings.

The *maximal generator* of an integral stochastic order $\tilde{\preceq}$ is the set of all functions $f \in \mathcal{B}$ such that

$$P \tilde{\preceq} Q \quad \text{implies} \quad \int_{\mathcal{X}} f \, dP \leq \int_{\mathcal{X}} f \, dQ.$$

We will assume that all functions in the following results belong to the class $\mathcal{B}$.

Theorem 2 *Let $\tilde{\preceq}$ be an integral stochastic order on $\mathcal{P}$. Let $\mathcal{R}$ be a generator of $\tilde{\preceq}$ satisfying that if $g : \mathcal{X} \to \mathbb{R}$ is a measurable mapping such that g and f are $\preceq$-comonotonic for all $f \in \mathcal{R}$, then $g \in \mathcal{R}$. It holds that $\mathcal{R}$ is the maximal generator of $\tilde{\preceq}$.*

Proof Let us see that $\mathcal{R}$ is a convex cone containing the constant functions and closed under pointwise convergence.

Let $f \in \mathcal{R}$ and $\lambda > 0$. Suppose that there exists $h \in \mathcal{R}$ such that h and λf are not $\preceq$-comonotonic. Then there exist $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq x_2$, $h(x_1) < h(x_2)$ and $\lambda f(x_1) > \lambda f(x_2)$. This implies that $f(x_1) > f(x_2)$ which is a contradiction with $x_1 \preceq x_2$ since $f \in \mathcal{R}$. Therefore, λf and h are $\preceq$-comonotonic for all $h \in \mathcal{R}$ and so $\lambda f \in \mathcal{R}$. As a consequence, $\mathcal{R}$ is a cone.

Now, consider $f_1, f_2 \in \mathcal{R}$ and $\lambda \in [0,1]$. Suppose that there exists $h \in \mathcal{R}$ such that h and $\lambda f_1 + (1-\lambda) f_2$ are not $\preceq$-comonotonic. We have that there exist $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq x_2$, $h(x_1) < h(x_2)$ and $(\lambda f_1 + (1-\lambda) f_2)(x_1) > (\lambda f_1 + (1-\lambda) f_2)(x_2)$. But $\lambda f_1(x_1) \preceq \lambda f_1(x_2)$ and $(1-\lambda) f_2(x_1) \preceq (1-\lambda) f_2(x_2)$ since $x_1 \preceq x_2$ and $f_1, f_2 \in \mathcal{R}$. Hence, $\mathcal{R}$ is convex.

On the other hand, any constant function and any element of $\mathcal{R}$ are $\preceq$-comonotonic. So, any constant function is in $\mathcal{R}$.

Moreover, let $\{f_n\}_n \subset \mathcal{R}$ such that $\{f_n\}_n$ tends to $f : \mathcal{X} \to \mathbb{R}$ pointwise. Let $h \in \mathcal{R}$. Suppose that h and f are not $\preceq$-comonotonic. We have that there exist $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq x_2$, $h(x_1) < h(x_2)$ and $f(x_1) > f(x_2)$. Then for n large enough, we have that $f_n(x_1) > f_n(x_2)$, which contradicts that $f_n \in \mathcal{R}$. Thus, $\mathcal{R}$ is closed under pointwise convergence.

Now, the result is a consequence of Corollary 2.3.9 in [5] which says that a generator which is a convex cone containing the constant functions and is closed under pointwise convergence is the maximal generator of the order. □

Example 4 Consider $(\mathbb{R}^d, \mathcal{B}_d)$ with $\mathcal{B}_d$ the usual Borel σ-algebra on $\mathbb{R}^d$. Let $\widetilde{\preceq}$ stand for the usual multivariate stochastic order. Thus, two probabilities P and Q on the measurable space $(\mathbb{R}^d, \mathcal{B}_d)$ are ordered in the usual multivariate stochastic order if

$$\int f\,dP \leq \int f\,dQ$$

for any bounded increasing mapping $f : \mathbb{R}^d \to \mathbb{R}$, where by increasing we mean that $f(x) \leq f(y)$ for any $x, y \in \mathbb{R}^d$ with $x \preceq_{\mathrm{cw}} y$, and $\preceq_{\mathrm{cw}}$ denotes the usual componentwise order on $\mathbb{R}^d$.

Let $\mathcal{R}$ be the above class of mappings which is a generator of $\widetilde{\preceq}$. Such a generator allows to obtain the partial order $\preceq$ on $\mathbb{R}^d$ determined by $\widetilde{\preceq}$. It can be seen that this order is the order $\preceq_{\mathrm{cw}}$.

To analyze if the integral stochastic order $\widetilde{\preceq}$ is generated by $\preceq$, we apply Theorem 1, but taking into account Theorem 2; that is, if there exists a generator satisfying the condition of Theorem 1, such a generator should be the maximal generator of $\widetilde{\preceq}$. It is well known that the class $\mathcal{R}$ is the maximal generator of $\widetilde{\preceq}$.

It is not hard to prove that any bounded measurable mapping which is $\preceq$-comonotonic with all the mappings of $\mathcal{R}$ belongs to it, and so we conclude that the usual multivariate stochastic order is generated by the componentwise order.

Example 5 Now let $\widetilde{\preceq}$ stand for the upper orthant order on the class of probabilities associated with the measurable space $(\mathbb{R}^d, \mathcal{B}_d)$. Two probabilities P and Q are ordered in the upper orthant order if

$$P\big((t,\infty)\big) \leq Q\big((t,\infty)\big)$$

for any $t \in \mathbb{R}^d$, where (t,∞) denotes the set $(t_1,\infty) \times (t_2,\infty) \times \cdots \times (t_d,\infty)$ with $t = (t_1,\ldots,t_d)$.

Thus, the set $\mathcal{R} = \{I_{(t,\infty)} \mid t \in \mathbb{R}^d\}$ is a generator of $\widetilde{\preceq}$. This generator leads to the partial order $\preceq$ on $\mathbb{R}^d$ determined by $\widetilde{\preceq}$. It can be seen that this order is $\preceq_{\mathrm{cw}}$.

As an immediate consequence, we obtain that there does not exist a partial order generating the upper orthant order, since we have seen in the above example that $\preceq_{\mathrm{cw}}$ generates the usual multivariate stochastic order.

However, let us check that the maximal generator of $\widetilde{\preceq}$ does not satisfy the condition of Theorem 1. That is, there are bounded measurable mappings which are $\preceq$-comonotonic with all the maps of the maximal generator of $\widetilde{\preceq}$ and they do not belong to such a generator.

A mapping $f : \mathbb{R}^d \to \mathbb{R}$ is said to be Δ-monotone if for every subset $J = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, d\}$ and for every $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k > 0$, it holds that

$$\Delta_{i_1}^{\varepsilon_1} \Delta_{i_2}^{\varepsilon_2} \cdots \Delta_{i_k}^{\varepsilon_k} f(x) \geq 0$$

for all $x \in \mathbb{R}^d$, where

$$\Delta_i^{\varepsilon} f(x) = f(x + \varepsilon e_i) - f(x),$$

e_i being the ith unit vector.

In [5] it is proved that the class $\mathcal{L}$ of all bounded Δ-monotone functions is the maximal generator of $\widetilde{\preceq}$.

Let us see that there are $\preceq$-comonotonic mappings with all the mappings of $\mathcal{L}$, which do not belong to such a class.

Consider $d = 2$ and the mapping $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$g(x_1, x_2) = \begin{cases} 3 & \text{if } (x_1, x_2) \in \{(z_1, z_2) \mid (z_1 \geq 1, z_2 \geq 3) \text{ or } (z_1 \geq 3, z_2 \geq 1)\}, \\ 1 & \text{in any other case.} \end{cases}$$

The mapping g is $\preceq$-comonotonic with all the elements of $\mathcal{L}$, but g is not Δ-monotone.

To conclude we should point out that Theorem 2 could be applied to obtain maximal generators of integral stochastic orders which have been generated by means of partially ordered sets, since the maximal generator is the unique generator which satisfies the condition required in Theorem 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MCL-D and ML-D have worked together to obtain the results in this manuscript.

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