Conglomerable Natural Extension

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Abstract

We study the weakest conglomerable model that is implied by desirability or probability assessments: the *conglomerable natural extension*. We show that taking the natural extension of the assessments while imposing conglomerability—the procedure adopted in Walley's theory—does not yield, in general, the conglomerable natural extension (but it does so in the case of the marginal extension). Iterating this process produces a sequence of models that approach the conglomerable natural extension, although it is not known, at this point, whether it is attained in the limit. We give sufficient conditions for this to happen in some special cases, and study the differences between working with coherent sets of desirable gambles and coherent lower previsions. Our results indicate that it might be necessary to re-think the foundations of Walley's theory of coherent conditional lower previsions for infinite partitions of conditioning events.

Keywords. Conglomerability, natural extension, desirable gambles, coherent lower previsions.

1 Introduction

You are offered a *gamble* f (that is, a bounded real-valued function representing an uncertain reward) on a possibility space Ω. You assess that, whatever event B you consider in a certain partition $\mathcal{B}$ of Ω, you would desire f conditional on B. Does this imply that you should *unconditionally* desire f?

Common axioms of desirability, such as those in Refs. [11, Section 3.7] or [12], imply that this should indeed be the case, at least when $\mathcal{B}$ is finite. When $\mathcal{B}$ is infinite, some authors have proposed to impose the above requirement through an axiom of so-called *conglomerability*. In fact, conglomerability is a key founding axiom for Walley's theory of *coherent lower previsions* in the conditional case with infinite partitions of conditioning events.

Conglomerability was introduced by de Finetti [2, 3] as a property that a finitely—but not countably—additive probability need not satisfy. In fact, de Finetti was also the first to reject the idea that conglomerability should be required as an axiom of rationality. The concept was studied later by Dubins [5], who established a connection with *disintegrability*. The property of conglomerability was also studied by Seidenfeld, Schervisch and Kadane (e.g., in Refs. [9, 10]). They show in particular [9] that when a probability is defined on all events and takes infinitely many values, countable additivity is equivalent to *full* conglomerability, that is, for conglomerability to hold with respect to all the possible partitions of Ω. See Ref. [4] for an interesting connection with imprecise probability models.

Requiring conglomerability, even only with respect to a single partition $\mathcal{B}$, comes at the expense of some undesirable mathematical properties: for example, a conglomerable coherent lower prevision may not be the lower envelope of conglomerable linear previsions. Perhaps also because of this, conglomerability was rejected in some extensions of de Finetti's work, such Williams's [12] (see also Ref. [8]). In our view, what appears to be mostly controversial is in particular the idea of full conglomerability, as opposed to conglomerability only for the partitions that are actually used for updating beliefs.[1] This is for instance the approach taken by De Cooman and Hermans in Ref. [1] when they require modes to be 'cut conglomerable'.

Here, we do not take any philosophical position about whether models should be conglomerable. Our aim is to perform a technical study of the impact of conglomerability on the possible extensions of an initial set of assessments. We focus in particular on what we call the *conglomerable natural extension*: loosely speaking, this is the weakest (least committal) conglomerable model that is implied by the initial assessments. A related concept is the *natural extension*, which is defined

[1] This is also called *partial conglomerability*. Here, when we talk about conglomerability, we mean partial conglomerability.

in a similar way except for not requiring the extension to be conglomerable.

We start in Section 2 by introducing some basic notions: desirability, coherent lower previsions and the connections between them. We introduce conglomerability in a few different forms: for desirable gambles, in the traditional form and in a weaker variant; and for coherent lower previsions, in the traditional way and in a strengthened form. We uncover the relationships between these notions, which allows us to convert problems formulated for one model into the other.

In Section 3, we focus on desirability. We show that, if it exists, then the conglomerable natural extension $\mathcal{F}$ of a set $\mathcal{R}$ of desirable gambles with respect to a partition $\mathcal{B}$ is the intersection of all conglomerable sets including $\mathcal{R}$. Moreover, we relate $\mathcal{F}$ to the natural extension: we start from $\mathcal{R}$, take its natural extension, and close it with respect to conglomerability, obtaining $\mathcal{E}_1$; we reiterate this process, yielding the sequence $\mathcal{E}_2$, …, $\mathcal{E}_n$, …. We show that $\mathcal{E}_n \subseteq \mathcal{F}$ for all n, and that the sequence stabilises if and only if one if its elements coincides with $\mathcal{F}$. We provide some sufficient conditions for this, as well as a few examples to illustrate the situation. One of them, in particular, shows that the gambles in $\mathcal{R}$ that do not satisfy conglomerability may be only in the border of the set, and yet the closure with respect to conglomerability may extend the set beyond this border.

In Section 4, we study the conglomerable natural extension $\underline{F}$ of a coherent lower prevision $\underline{P}$ with respect to a partition $\mathcal{B}$. Here, too, we consider a sequence: we start from $\underline{P}$, compute its conditional natural extension $\underline{P}(\cdot|\mathcal{B})$, and then the natural extension of the two of them together, $\underline{E}_1$; we iterate the process, yielding the sequence $\underline{E}_2$, …, $\underline{E}_n$, …. We show that $\underline{E}_n \leq \underline{F}$ for all n, and again that the sequence stabilises if and only if one of its elements coincides with $\underline{F}$. We then provide what is arguably the most important result of this paper: we show in Example 5 that $\underline{E}_1$ may not equal $\underline{F}$. This is interesting because, when it comes to natural extension (as well as coherence), Walley's theory is implicitly based on stopping at the first element of the sequence: $\underline{E}_1$. We show that this is not enough to fully capture the implications of conglomerability, and give sufficient conditions for $\underline{E}_1 = \underline{F}$.

In Section 5, we relate the results obtained for desirable gambles and coherent lower previsions: we start from the set $\mathcal{R}$ and induce from this a coherent lower prevision $\underline{P}$. We then create the sequences of sets $\mathcal{E}_n$, on the one hand, and the sequences of coherent lower previsions $\underline{E}_n$, on the other. We investigate the relationship between the elements of these sequences. This allows us, in Example 7, to exploit Example 5 to show that $\mathcal{E}_1$ may not coincide with $\mathcal{F}$: this means that taking the one-step conglomerable closure falls short of the mark for desirable gambles as well. We give sufficient conditions for $\mathcal{E}_1 = \mathcal{F}$, as well as for the two sequences to be made up of equivalent models.

To conclude, we consider in Section 6 the problem of dealing with more than one partition. We show that under the assumptions of the *Marginal Extension Theorem* (see Refs. [11, Theorem 6.7.2] and [6]), it does hold that $\mathcal{E}_1 = \mathcal{F}$.

Due to lack of space, we must assume the reader has a working knowledge of the basics of the theory of coherent lower previsions [11]. We refrain from giving proofs of most technical results for the same reason.

2 Introductory Notions

Consider a possibility space Ω. In this paper Ω will frequently be $\mathbb{N}$, the set of natural numbers without zero, but our results will be applicable to more general spaces. A *gamble* is a map $f\colon \Omega \to \mathbb{R}$. The set of all gambles defined on Ω is denoted by $\mathcal{L}(\Omega)$, or simply $\mathcal{L}$ when there is no ambiguity about the possibility space we are working with. In particular, we use '$f \lneq 0$' to mean '$f \leq 0$ and $f \neq 0$' (and we then say that the gamble f is *negative*), and we write $f \gneq 0$ if $-f \lneq 0$.

Given a set of gambles $\mathcal{R}$, we consider the following axioms of desirability:[2]

D1. $f \gneq 0 \Rightarrow f \in \mathcal{R}$;

D2. $0 \notin \mathcal{R}$;

D3. $f \in \mathcal{R}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{R}$;

D4. $f, g \in \mathcal{R} \Rightarrow f + g \in \mathcal{R}$.

Let us define

$$\operatorname{posi}(\mathcal{R}) := \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon f_k \in \mathcal{R}, \lambda_k > 0, n \geq 1 \right\}.$$

We call $\mathcal{R}$ a *convex cone* if it is closed under positive linear combinations, meaning that $\operatorname{posi}(\mathcal{R}) = \mathcal{R}$. This is equivalent to $\mathcal{R}$ satisfying conditions D3 and D4.

Given a partition $\mathcal{B}$ of Ω, $\mathcal{R}$ is called $\mathcal{B}$-*conglomerable* when it satisfies the following axiom:

D5. if $f \neq 0$ and $(\forall B \in \mathcal{B}' \subseteq \mathcal{B}) Bf \in \mathcal{R}$ then $\sum_{B \in \mathcal{B}'} Bf \in \mathcal{R}$.

[2] This axiomatic definition is related to *strict* and *almost*-desirability, see Ref. [11, Section 3.7]. The differences between these concepts lie mostly in the topological properties of the set of desirable gambles and in whether the zero gamble is considered to be desirable.

Axiom D5 is a consequence of D4 when $\mathcal{B}$ is finite. It can be easily checked that D5 is equivalent to:

D5'. if $f \neq 0$ and $(\forall B \in \mathcal{B}) Bf \in \mathcal{R} \cup \{0\}$ then $\sum_{B \in \mathcal{B}} Bf \in \mathcal{R}$.

A *lower prevision* is a real-valued functional defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}$. When $\mathcal{K}$ is a linear space, $\underline{P}$ is called *coherent* when it satisfies the following conditions:

C1. $\underline{P}(f) \geq \inf f$ for all $f \in \mathcal{K}$;

C2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ for all $f \in \mathcal{K}$ and $\lambda > 0$;

C3. $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ for all $f, g \in \mathcal{K}$.

When $\mathcal{K} = \mathcal{L}$ and $\underline{P}$ satisfies C3 with equality, it is called a *linear prevision*. The set of linear previsions that dominate a coherent lower prevision $\underline{P}$ on its domain is denoted by $\mathcal{M}(\underline{P})$.

Given a partition $\mathcal{B}$ of Ω, a *conditional lower prevision* $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$ is a functional such that for every $B \in \mathcal{B}$, $\underline{P}(\cdot|B)$ is a lower prevision on $\mathcal{L}$. It is called *separately coherent* when $\underline{P}(\cdot|B)$ is coherent and $\underline{P}(B|B) = 1$ for every $B \in \mathcal{B}$. For a lower prevision $\underline{P}$ and a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, we use the notation

$$G_{\underline{P}}(f) \coloneqq f - \underline{P}(f), \; G_{\underline{P}}(f|B) \coloneqq B(f - \underline{P}(f|B)$$
$$G_{\underline{P}}(f|\mathcal{B}) \coloneqq f - \underline{P}(f|\mathcal{B}) = \sum_{B \in \mathcal{B}} G_{\underline{P}}(f|B).$$

When both $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ are defined on $\mathcal{L}$, they are called *coherent* if and only if $\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ and

$$\underline{P}(G_{\underline{P}}(f|B)) = 0 \qquad \text{(GBR)}$$

for every gamble f and every $B \in \mathcal{B}$. This last condition is called the *Generalised Bayes Rule.*

Definition 1. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and $\mathcal{B}$ a partition of Ω. $\underline{P}$ is called *$\mathcal{B}$-conglomerable* when the following condition holds:

WC. $\underline{P}(\sum_{n \in N} B_n f) \geq 0$ for any $f \in \mathcal{L}$ and any countable number of distinct sets B_n in $\mathcal{B}$ such that $\underline{P}(B_n) > 0$ and $\underline{P}(B_n f) \geq 0$ for all $n \in N$.

Again, WC holds trivially when N is finite, and in particular when the partition $\mathcal{B}$ is finite, because of the super-additivity of coherent lower previsions.

Let us shed more light on the relation between the coherence and conglomerability concepts for lower previsions and sets of desirable gambles. On the one hand, given a coherent lower prevision $\underline{P}$, we define its associated set of *strictly desirable gambles* by

$$\underline{\mathcal{R}} \coloneqq \{f \in \mathcal{L} \colon f \gneq 0 \text{ or } \underline{P}(f) > 0\}, \qquad (1)$$

and its set of *almost-desirable gambles* by

$$\overline{\mathcal{R}} \coloneqq \{f \in \mathcal{L} \colon \underline{P}(f) \geq 0\}. \qquad (2)$$

$\underline{\mathcal{R}}$ satisfies the axioms D1–D4 considered above, and $\overline{\mathcal{R}}$ is a convex cone that includes all non-negative gambles. Moreover, it follows from the equations above that $\underline{\mathcal{R}} \subseteq \overline{\mathcal{R}}$, and that $\underline{\mathcal{R}}$ contains all positive gambles and is closed under dominance.

Conversely, given a set $\mathcal{R}$ of gambles satisfying D1–D4, we can define the corresponding lower prevision by

$$\underline{P}(f) \coloneqq \sup\{\mu \colon f - \mu \in \mathcal{R}\}. \qquad (3)$$

It follows from Theorem 6 in Ref. [7] that $\underline{P}$ is a coherent lower prevision. Moreover, if we consider the sets $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ given by Eqs. (1) and (2), it follows from Theorem 3.8.1 in Ref. [11] that

$$\sup\{\mu \colon f - \mu \in \underline{\mathcal{R}}\} = \underline{P}(f) = \sup\{\mu \colon f - \mu \in \overline{\mathcal{R}}\}.$$

As a consequence, any set $\mathcal{R}$ such that $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$ induces the same lower prevision $\underline{P}$ through Equation (3) [11, Theorem 3.8.1].

The set $\overline{\mathcal{R}}$ is the closure of $\underline{\mathcal{R}}$ (and as a consequence also of any $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$) in the topology of uniform convergence [7, Proposition 4]. In addition,

$$\underline{\mathcal{R}} \coloneqq \{f \in \mathcal{R} \colon f \gneq 0 \text{ or } f - \varepsilon \in \mathcal{R} \text{ for some } \varepsilon > 0\},$$

for all $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$.

We now establish a conglomerability condition for sets of desirable gambles that is equivalent to WC.

Theorem 1. *Let $\mathcal{R}$ be a set of desirable gambles that satisfies* D1–D4, *and $\underline{P}$ be the coherent lower prevision it induces through Equation* (3). *Then $\underline{P}$ satisfies* WC *if and only if $\mathcal{R}$ satisfies the following condition:*

WD5. *if* $(\forall B \in \mathcal{B}) Bf \in \underline{\mathcal{R}} \cup \{0\}$ *then* $f \in \overline{\mathcal{R}}$.

Since $\underline{\mathcal{R}} \subseteq \mathcal{R} \subseteq \overline{\mathcal{R}}$, D5 implies WD5. On the other hand, when we consider a coherent set of almost-desirable gambles $\overline{\mathcal{R}}$ (see Ref. [11, Section 3.7.3] for the definition), condition D5 is equivalent to:

D5". if $(\forall B \in \mathcal{B}) Bf \in \overline{\mathcal{R}}$ then $f \in \overline{\mathcal{R}}$.

By definition, condition D5" is a consequence of D5. To see that they are equivalent when we work with a coherent set of almost-desirable gambles, note that the zero gamble belongs to $\overline{\mathcal{R}}$, and as a consequence if $Bf \in \overline{\mathcal{R}}$ for all $B \in \mathcal{B}' \subseteq \mathcal{B}$, then also $B_1 \sum_{B \in \mathcal{B}'} Bf$ belongs to $\overline{\mathcal{R}}$ for all $B_1 \in \mathcal{B}$; using D5" we then deduce that $\sum_{B \in \mathcal{B}'} Bf$ belongs to $\overline{\mathcal{R}}$.

We next show that D5 can also be related to a notion of conglomerability for coherent lower previsions:

Definition 2. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and $\mathcal{B}$ a partition of Ω. $\underline{P}$ is called *strongly $\mathcal{B}$-conglomerable* when the following condition holds:

SC. if $f \in \mathcal{L}$ and $(\forall B \in \mathcal{B}' \subseteq \mathcal{B})\underline{P}(Bf) \geq 0$, then $\underline{P}(\sum_{B\in\mathcal{B}'} Bf) \geq 0$.

Theorem 2. *Let $\underline{P}$ be a coherent lower prevision, and let $\overline{\mathcal{R}}$ be its associated set of almost-desirable gambles. Then $\underline{P}$ is strongly $\mathcal{B}$-conglomerable if and only if $\overline{\mathcal{R}}$ satisfies* D5. *Conversely, a coherent set of almost-desirable gambles satisfies* D5 *if and only if the coherent lower prevision $\underline{P}$ it induces satisfies* SC.

We deduce from Theorems 1 and 2 that if a coherent lower prevision is strongly $\mathcal{B}$-conglomerable, then it is also $\mathcal{B}$-conglomerable.

3 Conglomerability for Sets of Desirable Gambles

Let us consider a set of gambles $\mathcal{R}$, and look for the smallest superset $\mathcal{F}$ (if it exists) that satisfies D1–D5 with respect to a fixed partition $\mathcal{B}$. This set is called the *$\mathcal{B}$-conglomerable natural extension* of $\mathcal{R}$. A first characterisation of this set is given in the following:

Proposition 1. *If there is some set of gambles including $\mathcal{R}$ and satisfying* D1–D4 *and* D5 *(resp.* WD5*), then $\mathcal{F}$ is the intersection of all such sets.*

From now on, we assume that $\mathcal{R}$ satisfies conditions D1–D4; D2 is necessary for the existence of a $\mathcal{B}$-conglomerable natural extension, and D1, D3 and D4 can be satisfied by replacing $\mathcal{R}$ with the convex cone $\operatorname{posi}(\mathcal{R} \cup \{f : f \gneq 0\})$.

The existence of a superset of $\mathcal{R}$ that satisfies D1–D5 does not guarantee that there is a half-space that includes $\mathcal{R}$ and satisfies these axioms. The example that establishes this is a reformulation of [11, Example 6.6.9]:

Example 1. Let Ω be the set of integers without zero, and consider the partition $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$ given by $B_n := \{-n, n\}$.

Let P_1 be a linear prevision on $\mathcal{L}$ satisfying $P_1(\{n\}) = \frac{1}{2^{n+1}}$ and $P_1(\{-n\}) = 0$ for all $n \in \mathbb{N}$, and $P_1(\mathbb{N}) = \frac{1}{2}$. Also consider a linear prevision P_2 satisfying $P_2(\{-n\}) = \frac{1}{3^n}$, $P_2(\{n\}) = 0$ for all $n \in \mathbb{N}$, and $P_2(\mathbb{N}) = \frac{1}{2}$. Let $\underline{P} := \min\{P_1, P_2\}$.

Consider $\underline{\mathcal{R}} := \{f : f \gneq 0 \text{ or } \underline{P}(f) > 0\}$, the set of strictly desirable gambles associated with $\underline{P}$. Then $\underline{\mathcal{R}}$ satisfies D1–D4. To see that it also satisfies D5, note that if for a gamble $0 \neq f$, $B_n f \in \underline{\mathcal{R}} \cup \{0\}$ for all $n \in \mathbb{N}$, then either $\underline{P}(B_n f) > 0$ or $B_n f \geq 0$, and in the latter case $\underline{P}(B_n f) \geq 0$. But since $\underline{P}(B_n f) > 0$ implies that both $P_1(B_n f) > 0$ and $P_2(B_n f) > 0$, and since this in turn means that both $f(-n)$ and $f(n)$ are non-negative, we also deduce that $\underline{P}(B_n f) > 0$ implies that $B_n f \gneq 0$. As a consequence, if $B_n f \in \underline{\mathcal{R}} \cup \{0\}$ for all $B_n \in \mathcal{B}$, then $f \geq 0$, and since $f \neq 0$ we deduce that $f \in \underline{\mathcal{R}}$.

Let us now show that there is no half-space including $\underline{\mathcal{R}}$ and satisfying WD5 (and as a consequence neither D5). Assume *ex absurdo* that $\mathcal{D}$ is such a space. Let P be the associated linear prevision, given by $P(f) := \sup\{\mu : f - \mu \in \mathcal{D}\}$. Since $\underline{\mathcal{R}} \subseteq \mathcal{D}$, we deduce that P dominates $\underline{P}$. But Walley has shown in Ref. [11, Example 6.6.9] that no dominating linear prevision satisfies WC, and using Theorem 1, we deduce that $\mathcal{D}$ does not satisfy WD5, and as a consequence it does not satisfy D5 either. ♦

Our next goal is to derive a more practical expression for $\mathcal{F}$. In order to do this, let us define the following sequence of sets of desirable gambles, starting with:

$$\begin{aligned}\mathcal{R}^* &:= \{f \neq 0 : (\forall B \in \mathcal{B}) Bf \in \mathcal{R} \cup \{0\}\}\\ \mathcal{E}_1 &:= \operatorname{posi}(\mathcal{R} \cup \mathcal{R}^*)\end{aligned}$$

and for all $n \geq 2$:

$$\begin{aligned}\mathcal{E}^*_{n-1} &:= \{f \neq 0 : (\forall B \in \mathcal{B}) Bf \in \mathcal{E}_{n-1} \cup \{0\}\}\\ \mathcal{E}_n &:= \operatorname{posi}(\mathcal{E}_{n-1} \cup \mathcal{E}^*_{n-1}).\end{aligned} \tag{4}$$

We will also use $\mathcal{E}_0 := \mathcal{R}$ and $\mathcal{E}^*_0 := \mathcal{R}^*$.

Lemma 1. *Let $\mathcal{F}' \supseteq \mathcal{R}$ and suppose that $\mathcal{F}'$ satisfies* D1–D5. *Then $\mathcal{F}' \supseteq \mathcal{E}_n$ for all $n \in \mathbb{N}$.*

It follows that the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$, if it exists, must include $\bigcup_n \mathcal{E}_n$. As a consequence, in that case we can also express the sets $\mathcal{E}$ as

$$\begin{aligned}\mathcal{E}_1 &= \{f + g : f \in \mathcal{R} \cup \{0\}, g \in \mathcal{R}^* \cup \{0\}\} \setminus \{0\},\\ \mathcal{E}_n &= \{f + g : f \in \mathcal{E}_{n-1} \cup \{0\}, g \in \mathcal{E}^*_{n-1} \cup \{0\}\} \setminus \{0\}.\end{aligned}$$

We next investigate which desirability axioms are satisfied by the sets $\mathcal{E}_n$ and $\mathcal{E}^*_n$.

Proposition 2. *Assume that there is some superset $\mathcal{F}$ of $\mathcal{R}$ satisfying* D1–D5. *Then:*

1. *$\mathcal{E}_n$ satisfies* D1–D4 *for all $n \in \mathbb{N}$.*
2. *$\mathcal{E}^*_n$ satisfies* D1–D5 *for all $n \in \mathbb{N}$.*

We can now characterise under which conditions $\mathcal{E}_n$ coincides with the $\mathcal{B}$-conglomerable natural extension, in terms of the desirability axioms:

Proposition 3. *The following conditions are equivalent for any natural number $n \geq 0$:*

1. *$\mathcal{E}^*_n \subseteq \mathcal{E}_n$.*
2. *$\mathcal{E}_n$ satisfies* D5.
3. *$\mathcal{F} = \mathcal{E}_n$.*

This simple result has interesting consequences: on the one hand, if $\mathcal{E}_n$ is not the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$, then there must be some gamble f in $\mathcal{E}_n^* \setminus \mathcal{E}_n$, and as a consequence $\mathcal{E}_n$ is a proper subset of $\mathcal{E}_{n+1}$. In other words, the sequence $\mathcal{E}_n$ does not stabilise unless we get to the $\mathcal{B}$-conglomerable natural extension. On the other hand, if $\mathcal{E}_n^* = \mathcal{E}_{n+1}^*$ for some n then $\mathcal{E}_{n+1}^*$ is included in $\mathcal{E}_{n+1}$, and Proposition 3 implies that $\mathcal{E}_{n+1}$ is the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$. Hence, we can use both sequences to determine at which step we get to $\mathcal{F}$: $\mathcal{E}_n = \mathcal{F}$ if $\mathcal{E}_{n-1}^* = \mathcal{E}_n^*$, and also if and only if $\mathcal{E}_n = \mathcal{E}_{n+1}$.

Next we provide a sufficient condition for $\mathcal{E}_1$ to coincide with $\mathcal{F}$:

Proposition 4. *Let $\mathcal{R}$ be a set of desirable gambles satisfying* D1–D4*, and assume that its $\mathcal{B}$-conglomerable natural extension $\mathcal{F}$ exists.*

1. $(\forall f \in \mathcal{R})(\forall B \in \mathcal{B}) Bf \in \mathcal{R} \cup \{0\} \Leftrightarrow \mathcal{R}^* = \mathcal{F} \Leftrightarrow \mathcal{R} \subseteq \mathcal{R}^*$.
2. *If there is some superset $\mathcal{Q}$ of $\mathcal{R}$ satisfying* D1–D5 *and such that $\mathcal{Q}^* = \mathcal{R}^*$, then $\mathcal{E}_1 = \mathcal{F}$.*

As a consequence, when $\mathcal{R}$ is included in $\mathcal{R}^*$ the sequence $\mathcal{E}_n$ stabilises in the first step: $\mathcal{E}_1 = \mathcal{F}$.

Let us give an example showing that the inclusion $\mathcal{R} \subseteq \mathcal{R}^*$ does not imply that $\mathcal{R} = \mathcal{R}^*$, or, equivalently, that we may have $\mathcal{R} \subsetneq \mathcal{E}_1 = \mathcal{F}$:

Example 2. Consider $\Omega = \mathbb{N}$, $B_n := \{2n-1, 2n\}$ and $\mathcal{B} := \{B_n \colon n \in \mathbb{N}\}$. Let $\mathcal{R}$ be the set of gambles f for which there is some $n_f \in \mathbb{N}$ such that

$$\begin{aligned}&f(n_f \rightarrow) \gneq 0 \text{ and}\\ &f(2n) + f(2n-1) \geq 0 \text{ and } f(2n) \geq 0 \text{ for all } n \in \mathbb{N},\end{aligned}$$

where $(n_f \rightarrow) := \{n_f, n_f+1, \dots\}$. Then $\mathcal{R}$ satisfies D1–D4:

D1. Any $f \gneq 0$ belongs to $\mathcal{R}$ by definition: take $n_f = 1$.

D2. $0 \notin \mathcal{R}$ by definition.

D3. Let $f \in \mathcal{R}$ and $\lambda > 0$. Then there is some $n_f \in \mathbb{N}$ such that $f(n_f \rightarrow) \gneq 0$, $f(2n) + f(2n-1) \geq 0$ and $f(2n) \geq 0$ for all $n \in \mathbb{N}$, whence $(\lambda f)(n_f \rightarrow) = \lambda(f(n_f \rightarrow)) \gneq 0$, $(\lambda f)(2n) + (\lambda f)(2n-1) = \lambda(f(2n) + f(2n-1)) \geq 0$ and $\lambda(f(2n)) \geq 0$ for all $n \in \mathbb{N}$. Since moreover $\lambda f \neq 0$ because $f \neq 0$ and $\lambda > 0$, we conclude that $\lambda f \in \mathcal{R}$.

D4. Let $f, g \in \mathcal{R}$. Then there are $n_f, n_g \in \mathbb{N}$ such that $f(n_f \rightarrow) \geq 0$ and $g(n_g \rightarrow) \geq 0$, whence given $n^* := \max\{n_f, n_g\}$, we infer that $(f+g)(n^* \rightarrow) \gneq 0$. On the other hand, $(f+g)(2n) + (f+g)(2n-1) = f(2n) + g(2n) + f(2n-1) + g(2n-1) \geq 0$ and $(f+g)(2n) \geq 0$ for all $n \in \mathbb{N}$, whence also $f + g \in \mathcal{R}$.

To see that $\mathcal{R} \subseteq \mathcal{R}^*$, observe that given a gamble $f \in \mathcal{R}$ and $B_n \in \mathcal{B}$, $B_n(f(2m) + f(2m-1)) \geq 0$ and $B_n(f(2m)) \geq 0$ for all $m \in \mathbb{N}$. Moreover, if $B_n f = 0$ then automatically $B_n f \in \mathcal{R} \cup \{0\}$; and if $B_n f \neq 0$ then either $f(2n) > 0$, in which case $B_n f \in \mathcal{R}$ by letting $n_{B_n f} = 2n$, or $f(2n) = 0$, in which case $f(2n-1) > 0$ and $B_n f \in \mathcal{R}$ by letting $n_{B_n f} = 2n-1$.

However, $\mathcal{R}$ does not satisfy D5, and as a consequence it does not coincide with $\mathcal{R}^*$: the gamble g given by $g(2n) = 1, g(2n-1) = -1$ for all n does not belong to $\mathcal{R}$ because there is no natural number n_g for which $g(n_g \rightarrow) \gneq 0$. On the other hand, for every natural number n, $B_n g$ does belong to $\mathcal{R}$: consider $n_{B_n g} = 2n$. Therefore $g \in \mathcal{R}^*$. ♦

This example also allows us to show that conditions D5 and WD5 are not equivalent:

Example 3. Consider the set $\mathcal{R}$ from Example 2. We have already shown there that $\mathcal{R}$ does not satisfy D5. To see that it satisfies WD5, observe that given a gamble f and $B_n \in \mathcal{B}$, $B_n f$ belongs to $\underline{\mathcal{R}} \cup \{0\}$ if and only if $B_n f \geq 0$, because there is no $\delta > 0$ such that $B_n f - \delta \in \mathcal{R}$. As a consequence, $(\forall B_n \in \mathcal{B}) B_n f \in \underline{\mathcal{R}} \cup \{0\}$ implies that $0 \leq f \in \overline{\mathcal{R}}$. ♦

The same example shows us something else: even if the gambles that violate D5 are only on the border of $\mathcal{R}$, taking the closure of $\mathcal{R}$ with respect to D5 will require us in general to enlarge the set beyond its border.

Example 4. Consider set $\mathcal{R}$ and gamble g from Example 2. Taking into account the observations in Example 3, there is no $\delta > 0$ such that $B_n g - \delta \in \mathcal{R}$, because this gamble is not positive, and on the other hand, we know that $B_n g \in \mathcal{R}$. This means that $B_n g \in \mathcal{R} \setminus \underline{\mathcal{R}} \subseteq \overline{\mathcal{R}} \setminus \underline{\mathcal{R}}$ for all $B_n \in \mathcal{B}$. Now consider any $\delta \in (-1, 0)$, and observe that $g - \delta \notin \mathcal{R}$: in fact, $g(2n-1) - \delta < 0$ for all $n \geq 1$, so there is no $n_g \in \mathbb{N}$ such that $(g - \delta)(n_g \rightarrow) \geq 0$. On the other hand, $g + 1 \gneq 0$ and hence belongs to $\mathcal{R}$. This means that

$$\sup\left\{\mu \colon g - \mu \in \overline{\mathcal{R}}\right\} = \sup\left\{\mu \colon g - \mu \in \mathcal{R}\right\} = -1,$$

and therefore $g \notin \overline{\mathcal{R}}$. ♦

It is an open problem whether the sequence $\mathcal{E}_n$ always stabilises in a finite number of steps, and, if it does not, whether the sequence limit $\bigcup_{n \in \mathbb{N}} \mathcal{E}_n$ always coincides with the $\mathcal{B}$-conglomerable natural extension $\mathcal{F}$ of $\mathcal{R}$.

4 Conglomerability for Coherent Lower Previsions

We now turn to the relationship between the natural extension studied in Ref. [11, Chapter 8] and the conglomerable natural extension, which we define next. Throughout this section, $\mathcal{B}$ is a partition of Ω.

Definition 3. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{K}$. Its *$\mathcal{B}$-conglomerable natural extension* is the smallest coherent lower prevision $\underline{F}$ on $\mathcal{L}$ that dominates $\underline{P}$ and is $\mathcal{B}$-conglomerable.

There may be no dominating $\mathcal{B}$-conglomerable coherent lower prevision, and then the $\mathcal{B}$-conglomerable natural extension will not exist. On the other hand, if there is some dominating $\mathcal{B}$-conglomerable coherent lower prevision, then there is a $\mathcal{B}$-conglomerable natural extension, because $\mathcal{B}$-conglomerability is preserved by taking lower envelopes.

We may assume without loss of generality that the domain $\mathcal{K}$ of $\underline{P}$ is the set $\mathcal{L}$ of all gambles: otherwise, it suffices to consider the natural extension $\underline{E}$ of $\underline{P}$ to $\mathcal{L}$. To see that the $\mathcal{B}$-conglomerable natural extensions of $\underline{P}$ and $\underline{E}$ coincide, denote these by $\underline{F}_1$ and $\underline{F}_2$, respectively. Trivially $\underline{F}_2 \geq \underline{F}_1$. Conversely, $\underline{F}_1$ is by definition a $\mathcal{B}$-conglomerable coherent lower prevision that dominates $\underline{P}$ on $\mathcal{K}$, and therefore also dominates its natural extension $\underline{E}$. Hence $\underline{F}_1 \geq \underline{F}_2$ as well.

Given a coherent lower prevision $\underline{P}$, Walley defines its *conditional natural extension* as

$$\underline{P}(f|B) := \begin{cases} \sup\{\mu\colon \underline{P}(B(f-\mu)) \geq 0\} & \text{if } \underline{P}(B) > 0 \\ \inf_{\omega\in B} f(\omega) & \text{otherwise} \end{cases} \tag{5}$$

for every $f \in \mathcal{L}$ and $B \in \mathcal{B}$. In fact, when $\underline{P}(B) > 0$ then $\underline{P}(f|B)$ is to the unique value of μ such that $\underline{P}(B(f-\mu)) = 0$, i.e., for which (GBR) is satisfied.

From Theorem 6.8.2 in Ref. [11], $\underline{P}$ is $\mathcal{B}$-conglomerable if and only if it is coherent with the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ derived from $\underline{P}$ by natural extension. In Ref. [11, Section 6.6], Walley gives a number of examples of coherent lower previsions that are not $\mathcal{B}$-conglomerable. We give a sufficient condition for conglomerability:

Proposition 5. *If the conditional natural extension $\underline{P}(\cdot|\mathcal{B})$ of $\underline{P}$ is given by $\underline{P}(f|B) = \inf_{\omega\in B} f(\omega)$ for all $B \in \mathcal{B}$ and $f \in \mathcal{L}$, then $\underline{P}$ is $\mathcal{B}$-conglomerable, and so is any $\underline{Q} \leq \underline{P}$.*

When $\underline{P}$ is not $\mathcal{B}$-conglomerable, we can consider the natural extensions $\underline{E}$, $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$, determined by Theorem 8.1.5 in Ref. [11]:

$$\underline{E}(f) := \sup_{g,h\in\mathcal{L}} \sup\left\{\mu\colon f-\mu \geq G_{\underline{P}}(g) + G_{\underline{P}}(h|\mathcal{B})\right\},$$

and it can be checked that $\underline{E}(\cdot|\mathcal{B})$ coincides with the conditional natural extension of $\underline{E}$: it can be obtained using Eq. (5).

Proposition 6. *The natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ is dominated by the $\mathcal{B}$-conglomerable natural extension $\underline{F}$ of $\underline{P}$. They coincide if and only if $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$ are coherent. Moreover, if we let $\underline{Q} := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, we have*

$$\begin{aligned}\mathcal{M}(\underline{E}) &= \left\{P \in \mathcal{M}(\underline{P})\colon (\forall f \in \mathcal{L}) P(G_{\underline{P}}(f|\mathcal{B})) \geq 0\right\} \\ &= \mathcal{M}(\underline{P}) \cap \mathcal{M}(\underline{Q}).\end{aligned}$$

As a consequence, if $\underline{Q} \geq \underline{P}$, then $\underline{Q}$ coincides with $\underline{E}$ and it is the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$.

Next we show that $\underline{E}$ does not necessarily coincide with the conglomerable natural extension:

Example 5. Consider $\Omega := \mathbb{N} \cup -\mathbb{N}$, $B_n := \{-n, n\}$ and let $\mathcal{B}$ be the partition of Ω given by $\mathcal{B} := \{B_n\colon n \in \mathbb{N}\}$. Let P be a finitely additive probability on $\mathcal{P}(\mathbb{N})$ that satisfies $P(\{n\}) = 0$ for every n (it follows from Ref. [9] that P is not conglomerable), and consider the linear previsions P_1, ..., P_4, where P_1 is the expectation functional associated with the σ-additive probability measure with

$$P_1(\{n\}) = P_1(\{-n\}) = \frac{1}{2^{n+1}} \text{ for all } n \in \mathbb{N}$$

and P_2, P_3 and P_4 are given, by

$$\begin{aligned} P_2(h) &= \frac{1}{2}\sum_{n=1}^{\infty} h(n)\frac{1}{2^n} + \frac{1}{2}P(h_2) \\ P_3(h) &= \frac{3}{4}P(h_1) + \frac{1}{4}P(h_2) \\ P_4(h) &= \frac{1}{2}P_1(h) + \frac{1}{2}P_3(h), \end{aligned}$$

where for every $h \in \mathcal{L}$ the gambles h_1, h_2 are defined on $\mathbb{N}$ by $h_1(n) := h(n)$ and $h_2(n) := h(-n)$ for every $n \in \mathbb{N}$.

First, we consider the coherent lower prevision $\underline{P} := \min\{P_1, P_2, P_4\}$. Since

$$\underline{P}(B_n) = \min\left\{\frac{1}{2^n}, \frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right\} > 0$$

for all $n \in \mathbb{N}$, we see that for every gamble f:

$$\underline{P}(f|B_n) = \min\left\{f(n), \frac{f(n)+f(-n)}{2}\right\}. \tag{6}$$

To see that $\underline{P}$ is not $\mathcal{B}$-conglomerable, consider the gamble f given by

$$f(n) := 1 - \frac{1}{n} \text{ and } f(-n) := -f(n) \text{ for all } n \in \mathbb{N}.$$

It follows from Eq. (6) that $\underline{P}(f|B_n) = 0$ for every n, whence $G_{\underline{P}}(f|\mathcal{B}) = f$. On the other hand,

$$\underline{P}(G_{\underline{P}}(f|\mathcal{B})) \leq P_2(f) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}(1 - \frac{1}{n}) - \frac{1}{2} < 0,$$

taking into account that $P_2(-\mathbb{N}f) := \frac{1}{2}P(f_2) = -\frac{1}{2}$.

Next we show that $P_4(G_{\underline{P}}(h|\mathcal{B})) \geq 0$ for every gamble h. Note first of all that

$$G_{\underline{P}}(h|\mathcal{B})(n) = \begin{cases} 0 & \text{if } h(n) \leq h(-n) \\ \dfrac{h(n)-h(-n)}{2} & \text{otherwise} \end{cases}$$

$$G_{\underline{P}}(h|\mathcal{B})(-n) = \begin{cases} h(-n)-h(n) & \text{if } h(n) \leq h(-n) \\ \dfrac{h(-n)-h(n)}{2} & \text{otherwise.} \end{cases}$$

As a consequence, $G_{\underline{P}}(h|B_n) \geq 0$ when $h(n) \leq h(-n)$, and this means that $P_4(G_{\underline{P}}(h|\mathcal{B})) \geq P_4(G_{\underline{P}}(h|\mathcal{B})C)$, where $C := \bigcup\{B_n\colon h(n) \geq h(-n)\}$. On the other hand, $G_{\underline{P}}(h|\mathcal{B})(n) = -G_{\underline{P}}(h|\mathcal{B})(-n) \geq 0$ for every $n \in C$, so

$$P_4(G_{\underline{P}}(h|\mathcal{B})C) = 0 + \frac{1}{2}P_3(G_{\underline{P}}(h|\mathcal{B})C)$$

and

$$P_3(G_{\underline{P}}(h|\mathcal{B})C) = \frac{3}{4}P(h') - \frac{1}{4}P(h') \geq 0,$$

where h' is the non-negative gamble on $\mathcal{L}(\mathbb{N})$ given by $h'(n) \coloneqq G_{\underline{P}}(h|\mathcal{B})(n)C(n)$, and where the second term on the right-hand side follows from the definition of P_3.

To determine the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, we apply Proposition 6. First of all, for every linear prevision $Q \in \mathcal{M}(\underline{P})$, there are α_1, α_2 and $\alpha_4 \in [0,1]$ such that $\alpha_1 + \alpha_2 + \alpha_4 = 1$ and $Q = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_4 P_4$. We are going to check which of these combinations satisfies $Q(G_{\underline{P}}(f|\mathcal{B})) \geq 0$ for every gamble f. On the one hand, if $\alpha_2 = 0$ then Q belongs to $\mathcal{M}(\underline{E})$, since we have just proven that P_4 dominates $\underline{E}$ and P_1 is conglomerable. Assume now that $\alpha_2 > 0$, and consider an arbitrary gamble f. As before, since $G_{\underline{P}}(f|\mathcal{B}) \geq G_{\underline{P}}(f|\mathcal{B})C$, where $C \coloneqq \bigcup\{B_n \colon f(n) \geq f(-n)\}$, we can concentrate on gambles f such that $f(n) \geq f(-n)$ for every $n \in \mathbb{N}$. In that case, if we denote $h \coloneqq G_{\underline{P}}(f|\mathcal{B})$, it holds that $h_1 \geq 0$ and $h_2 = -h_1$. As a consequence,

$$\begin{aligned} Q(h) &= \alpha_1 P_1(h) + \alpha_2 P_2(h) + \alpha_4 P_4(h) \\ &= \alpha_2 P_1(h\mathbb{N}) + P(h_1)(\frac{1}{4}\alpha_4 - \frac{1}{2}\alpha_2). \end{aligned}$$

When $\alpha_4 \geq 2\alpha_2 > 0$, we deduce from the non-negativity of $h\mathbb{N}$ (and as a consequence of h_1) that $Q(h) \geq 0$ and therefore $Q \in \mathcal{M}(\underline{E})$. When $\alpha_4 < 2\alpha_2$, there is some natural number n^* such that

$$\frac{1}{2^{n^*}} < \frac{\frac{1}{2}\alpha_2 - \frac{1}{4}\alpha_4}{\alpha_2}.$$

We consider the gamble f given by $f(n) \coloneqq 0$ for $n \leq n^*$, $f(n) \coloneqq 1$ for $n > n^*$ and $f(-n) \coloneqq -f(n)$ for all $n \in \mathbb{N}$. Then $h = G_{\underline{P}}(f|\mathcal{B}) = f$, and using the equation above we obtain $P_1(h\mathbb{N}) = \frac{1}{2^{(n^*+1)}}$ and $P(h_1) = 1$. As a consequence, $Q(h) = \alpha_2 P_1(h\mathbb{N}) + P(h_1)(\frac{1}{4}\alpha_4 - \frac{1}{2}\alpha_2) < 0$, since by construction $P_1(h\mathbb{N}) < \frac{\frac{1}{2}\alpha_2 - \frac{1}{4}\alpha_4}{\alpha_2}$.

We deduce from all this that $\underline{E}$ is the lower envelope of the set $\{P_1, P_4, \frac{1}{3}P_2 + \frac{2}{3}P_4\}$, and as a consequence it induces the conditional lower prevision $\underline{E}(\cdot|\mathcal{B})$ determined by

$$\underline{E}(f|B_n) = \min\left\{\frac{f(n)+f(-n)}{2}, \frac{2f(n)+f(-n)}{3}\right\}. \quad (7)$$

To see that $\underline{E}$ is not $\mathcal{B}$-conglomerable, consider any gamble g such that $g(n) \leq g(-n)$ for all $n \in \mathbb{N}$, then Eq. (7) yields

$$\underline{E}(g|B_n) = \frac{2g(n)+g(-n)}{3},$$

and consequently

$$\begin{aligned} G_{\underline{E}}(g|B_n)(n) &= \frac{g(n)-g(-n)}{3}, \\ G_{\underline{E}}(g|B_n)(-n) &= \frac{2g(-n)-2g(n)}{3}. \end{aligned}$$

Thus, given $h \coloneqq G_{\underline{E}}(g|\mathcal{B})$ we obtain $h_2 = -2h_1 \geq 0$, whence

$$\begin{aligned} P_4(h) &= \frac{1}{2}P_1(h) + \frac{3}{8}P(h_1) + \frac{1}{8}P(h_2) \\ &= \frac{1}{2}(P_1(h\mathbb{N}) + P_1(h{-}\mathbb{N})) + \frac{1}{8}P(h_1) \\ &= -\frac{1}{2}P_1(h\mathbb{N}) + \frac{1}{8}P(h_1). \end{aligned}$$

Now, if we make for instance $P(h_1) < 4P_1(h\mathbb{N})$, as is the case for $g(n) \coloneqq g(-n) \coloneqq 0$ for $n = 1, 2$ and $g(n) \coloneqq -1$ and $g(-n) \coloneqq 1$ for $n > 2$, then we get $P_4(G_{\underline{E}}(g|\mathcal{B})) < 0$, whence $\underline{E}(G_{\underline{E}}(g|\mathcal{B})) < 0$. Hence, $\underline{E}$ is not $\mathcal{B}$-conglomerable, and therefore it does not coincide with the conglomerable natural extension $\underline{F}$, which exists because $P_1 \geq \underline{P}$ is $\mathcal{B}$-conglomerable. ♦

On the other hand, we can give a number of sufficient conditions for $\underline{E}$ to be $\mathcal{B}$-conglomerable.

Proposition 7. *If the conditional natural extension derived from $\underline{P}$ is linear and the $\mathcal{B}$-conglomerable natural extension $\underline{F}$ exists, then it coincides with the natural extension $\underline{E}$ of $\underline{P}$ and $P(\cdot|\mathcal{B})$. More generally, if there is a coherent lower prevision $\underline{Q} \geq \underline{P}$ that is coherent with the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ derived from $\underline{P}$ using natural extension, then the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ coincides with the $\mathcal{B}$-conglomerable natural extension $\underline{F}$.*

Hence, if $\underline{P}$ is not $\mathcal{B}$-conglomerable, we can consider the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. If then $\underline{E}$ is not $\mathcal{B}$-conglomerable, we can consider the natural extension $\underline{E}_1$ of $\underline{E}$ and $\underline{E}(\cdot|\mathcal{B})$, and so on. Our next result shows that the resulting sequence $\underline{E}_n$ of coherent lower previsions does not stabilise unless we get to a $\mathcal{B}$-conglomerable coherent lower prevision.

Proposition 8. *If $\underline{P}$ is not $\mathcal{B}$-conglomerable, then it does not coincide with the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. On the other hand, if $\underline{E}(\cdot|\mathcal{B}) = \underline{P}(\cdot|\mathcal{B})$ then $\underline{E}$ is $\mathcal{B}$-conglomerable.*

The sequence $\underline{E}_n$ is increasing and therefore converges to a coherent lower prevision $\underline{E}_\infty$, which by construction is dominated by the $\mathcal{B}$-conglomerable natural extension $\underline{F}$ of $\underline{P}$: it suffices to use induction on n and to take into account that at each step n, $\underline{E}_{n+1}$ is a lower bound of any coherent extension of $\underline{E}_n$ and $\underline{E}_n(\cdot|\mathcal{B})$, and is therefore bounded by the $\mathcal{B}$-conglomerable natural extension $\underline{F}$. It is an open problem whether the two coherent lower previsions $\underline{E}_\infty$ and $\underline{F}$ coincide, and also to find an example where $\underline{E}_n$ does not coincide with $\underline{F}_\infty$ for any n, i.e., where we cannot get to the $\mathcal{B}$-conglomerable natural extension in a finite number of steps.

5 Connecting the Two Approaches

The correspondence between sets of desirable gambles and coherent lower previsions we have summarised in Section 2 does not extend towards the notion of $\mathcal{B}$-conglomerable natural extension we have discussed in Sections 3 and 4. The reason is that in our definition of the $\mathcal{B}$-conglomerable natural extension of a set of gambles we are using condition D5, while the $\mathcal{B}$-conglomerable natural extension for coherent lower previsions is based on condition WC, which is equivalent to WD5, and therefore weaker than D5 in general. We now exhibit all this in more detail.

Let $\mathcal{R}$ be a set of desirable gambles satisfying D1–D4, and let $\underline{P}$ be its associated coherent lower prevision, given by Eq. (3). If $\mathcal{R}$ does not satisfy D5, then we can consider the increasing sequence of sets of desirable gambles $\mathcal{E}_n$, defined by means of Eq. (4). From each of these sets of desirable gambles we can induce a coherent lower prevision $\underline{P}_n$, again by means of Eq. (3). At the same time, we can consider the sequence $\underline{E}_n$ of coherent lower previsions derived from $\underline{P}$ in the manner discussed in Section 4: $\underline{E}_1$ is the natural extension of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$, where $\underline{P}(\cdot|\mathcal{B})$ is the conditional natural extension of $\underline{P}$; $\underline{E}_2$ is the natural extension of $\underline{E}_1$ and $\underline{E}_1(\cdot|\mathcal{B})$; and so on.

Proposition 9. *$\underline{E}_n(f) \le \underline{P}_n(f)$ for all $f \in \mathcal{L}$.*

However, $\underline{E}_n$ and $\underline{P}_n$ do not coincide in general:

Example 6. Consider the set of desirable gambles $\mathcal{R}$ from Example 2, and let $\underline{P}$ be its associated coherent lower prevision. We have shown in Example 3 that $\mathcal{R}$ satisfies WD5, so Theorem 1 implies that $\underline{P}$ is $\mathcal{B}$-conglomerable, and in particular $\underline{E}_1(f) = \underline{P}(f)$ for every f. On the other hand, we have seen in Example 2 that $\mathcal{R}$ does not satisfy D5, and in particular that the gamble $g = \text{even} - \text{odd}$ belongs to $\mathcal{R}^* \setminus \mathcal{R}$. Moreover, we have seen in Example 4 that $\sup\{\mu\colon g - \mu \in \mathcal{R}\} = -1$. From all this, we infer that

$$\underline{P}_1(g) \ge 0 > -1 = \sup\{\mu\colon g - \mu \in \mathcal{R}\} = P(f) = \underline{E}_1(f).$$

This shows that the inequality in Proposition 9 may be strict. ♦

The reason for this lies in the next result:

Proposition 10. *$\underline{P}_n$ is the natural extension of $\underline{P}_{n-1}$ and $\underline{P}'_{n-1}(\cdot|\mathcal{B})$, where $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ is derived from the set $\mathcal{E}_{n-1}$ by*

$$\underline{P}'_{n-1}(f|B) \coloneqq \sup\{\mu\colon B(f-\mu) \in \mathcal{E}_{n-1}\} \tag{8}$$

for all $f \in \mathcal{L}$ and $B \in \mathcal{B}$.

$\underline{P}'_{n-1}(\cdot|\mathcal{B})$ satisfies (GBR) with respect to $\underline{P}_{n-1}$: given a gamble f and a set $B \in \mathcal{B}$, then for all $\varepsilon > 0$,

$$\begin{aligned}\underline{P}_{n-1}&(G_{\underline{P}'_{n-1}}(f|B) + \varepsilon) \\ &\ge \underline{P}_{n-1}(B(f - \underline{P}'_{n-1}(f|B) + \varepsilon)) \ge 0,\end{aligned}$$

whence $\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B)) \ge -\varepsilon$ for every $\varepsilon > 0$ and therefore $\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B)) \ge 0$. Conversely, if there is some $\varepsilon > 0$ such that $\underline{P}_{n-1}(G_{\underline{P}'_{n-1}}(f|B)) \ge \varepsilon$, then the gamble $G_{\underline{P}'_{n-1}}(f|B)) - \frac{\varepsilon}{2}$ must belong to $\mathcal{E}_{n-1}$, and therefore also the gamble $B(f - \underline{P}'_{n-1}(f|B) - \frac{\varepsilon}{2})$, which is greater. But this means that we can increase the value $\underline{P}'_{n-1}(f|B)$ by $\frac{\varepsilon}{2} > 0$, a contradiction with Eq. (8). As a consequence, $\underline{P}'_{n-1}(\cdot|\mathcal{B})$ can strictly dominate the conditional natural extension $\underline{P}_{n-1}(\cdot|\mathcal{B})$ of $\underline{P}_{n-1}$ only when some of the conditioning events have lower probability zero.

From Proposition 9, we can infer the following:

Proposition 11. *Let $\mathcal{R}$ be a coherent set of strictly desirable gambles, and let $\underline{P}$ be its associated coherent lower prevision. Then $\underline{P}_1 = \underline{E}_1$. As a consequence, if $\mathcal{E}_1$ is the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$, then $\underline{E}_1$ is the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$.*

Note however that the number of steps necessary to compute the $\mathcal{B}$-conglomerable natural extension can be different in the two cases, as Example 6 shows.

As a consequence of Proposition 11, if $\underline{E}_1$ is not $\mathcal{B}$-conglomerable, then $\mathcal{E}_1$ does not satisfy D5, provided we start from a set of strictly desirable gambles. Using this, we give an example where the sequence of sets $\mathcal{E}_n$ does not stabilise at the first step:

Example 7. Consider the coherent lower prevision $\underline{P}$ from Example 5 and let $\mathcal{R}$ be its associated set of strictly desirable gambles. We have shown in Example 5 that the natural extension $\underline{E}$ of $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ is not $\mathcal{B}$-conglomerable, and therefore it does not coincide with the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$. Applying Proposition 11, we deduce that $\mathcal{E}_1$ cannot be the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$, and therefore the sequence $\mathcal{E}_n$ does not stabilise at the first step. ♦

Next we give another sufficient condition for the two sequences of coherent lower previsions to coincide:

Proposition 12. *If $\underline{P}(B) > 0$ for all $B \in \mathcal{B}$, then $\underline{P}_n(f) = \underline{E}_n(f)$ for all $f \in \mathcal{L}$.*

The intuition behind this result is that when the conditioning events have all positive lower probability, then the corresponding conditional lower prevision is uniquely determined by (GBR), and then it necessarily coincides with the natural extension of the unconditional. It implies the following:

Corollary 1. *If $\underline{P}(B) > 0$ for all $B \in \mathcal{B}$ and $\mathcal{E}_n$ is the $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}$, then $\underline{E}_n$ is the $\mathcal{B}$-conglomerable natural extension of $\underline{P}$.*

The condition $\underline{P}(B) > 0$ for every $B \in \mathcal{B}$ does not imply that the sequence stabilises at the first step, as Example 5 shows. On the other hand, the sequences $\underline{E}_n$ and $\mathcal{E}_n$ need not stabilise at the same time: there are examples where $\mathcal{R}$ satisfies WD5, so the associated coherent lower prevision $\underline{P}$ is $\mathcal{B}$-conglomerable, but it does not satisfy D5, so $\mathcal{R}$ is strictly included in $\mathcal{E}_1$.

6 The Case of More Partitions

Next we consider a finite number of sets $\mathcal{R}_1, \dots, \mathcal{R}_m$, where $\mathcal{R}_i$ satisfies D1–D5 with respect to a partition $\mathcal{B}_i$, and we look for the smallest superset $\mathcal{F}$, if it exists, that satisfies D1–D5 with respect to all partitions in $\mathbb{B} := \{\mathcal{B}_1, \dots, \mathcal{B}_m\}$.

We first show that conglomerability with respect to the partitions $\mathcal{B}_1, \dots, \mathcal{B}_m$ is equivalent to conglomerability with respect to all partitions that can be derived from them. Let us define $\mathbb{B}'$ as the (finite) set of partitions $\mathcal{B}$ such that

$$(\forall B \in \mathcal{B})(\exists j \in \{1, \dots, m\}) B \in \mathcal{B}_j.$$

Proposition 13. *Let $\mathcal{R}$ be a set of gambles satisfying* D1–D4. *If it satisfies* D5 *(resp.* WD5*) with respect to all partitions in $\mathbb{B}$, then it also satisfies* D5 *(resp.* WD5*) with respect to all partitions in $\mathbb{B}'$.*

Taking into account Proposition 1, we can show:

Proposition 14. *If there is a set of gambles that includes $\bigcup_{i=1}^m \mathcal{R}_i$ and satisfies* D1–D4 *and* D5 *(resp.,* WD5*) with respect to all partitions in $\mathbb{B}$, then the smallest such set is given by the intersection of all sets that do so.*

On the other hand, if we consider the notion of conglomerability for coherent lower previsions, this time with respect to a finite number of partitions, we can make a link with the property of weak coherence studied in Ref. [11, Section 7.1]:

Proposition 15. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$. The following statements are equivalent:*

1. *$\underline{P}$ is $\mathcal{B}$-conglomerable for all $\mathcal{B} \in \mathbb{B}$.*
2. *$\underline{P}$ is $\mathcal{B}$-conglomerable for all $\mathcal{B} \in \mathbb{B}'$.*
3. *There are conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1)$, $\dots$, $\underline{P}_m(\cdot|\mathcal{B}_m)$ that are weakly coherent with $\underline{P}$.*

6.1 The Marginal Extension Theorem

We next prove that when the partitions are nested, the sequence stabilises after one step. This is a version in terms of sets of desirable gambles of the Marginal Extension Theorem 6.7.2 established in Ref. [11] and generalised to any finite number of partitions in Ref. [6]. In a different context, using different notations, this result was also proved (in a different manner) by De Cooman and Hermans [1, Theorem 3]. To proceed, we need to introduce a number of definitions:

Definition 4. Let $\mathcal{B}$ be a partition of Ω. A gamble f on Ω is called *$\mathcal{B}$-measurable* when it is constant on the elements of $\mathcal{B}$. The set of all $\mathcal{B}$-measurable gambles is denoted by $\mathcal{G}(\mathcal{B})$.

Definition 5. Let $\mathcal{Q}$ be a linear subspace of gambles containing all constant gambles, and let $\mathcal{R} \subseteq \mathcal{Q}$. We say that $\mathcal{R}$ is coherent *relative to* $\mathcal{Q}$ if it satisfies D2–D4 and

D1*. if $f \in \mathcal{Q}$ and $f \gneq 0$ then $f \in \mathcal{R}$.

When $\mathcal{Q} = \mathcal{L}$, this reduces to the usual coherence notion characterised by axioms D1–D4.

We begin by establishing our result for the case of one partition only.

Proposition 16. *Let $\mathcal{R}_0$ be a set of desirable gambles coherent relative to $\mathcal{G}(\mathcal{B})$. For each $B \in \mathcal{B}$, let $\mathcal{R}|B$ be a coherent set of desirable gambles on $\mathcal{L}(B)$. The $\mathcal{B}$-conglomerable natural extension of $\mathcal{R}_0$ and $\mathcal{R}|B$, $B \in \mathcal{B}$, is the set $\mathcal{F}$ given by*

$$\left\{ f + \sum_{B \in \mathcal{B}} B g_B \colon f \in \mathcal{R}_0 \cup \{0\}, g_B \in \mathcal{R}|B \cup \{0\} \right\} \setminus \{0\}.$$

Proof. Let us show that $\mathcal{F}$ satisfies D1–D5:

D1. Consider $h \gneq 0$. Write it as $h = \sum_{B \in \mathcal{B}\colon Bh \neq 0} Bh = \sum_{B \in \mathcal{B}\colon Bh \neq 0} B g_B$, where gamble $g_B \in \mathcal{L}(B)$ is defined by $g_B(\omega) := h(\omega)$ for all $\omega \in B$. Since $g_B \gneq 0$, it belongs to the coherent set $\mathcal{R}|B$. Hence h belongs to $\mathcal{F}$.

D2. We know that $0 \notin \mathcal{F}$ by definition.

D3. Consider $h \in \mathcal{F}$ and $\lambda > 0$. We know that $\lambda h = \lambda f + \sum_{B \in \mathcal{B}} B \lambda g_B$. Since $\mathcal{G}(\mathcal{B})$ is a linear space containing all constant gambles, and $\mathcal{R}_0$ is coherent relative to it, it follows that $\lambda f \in \mathcal{R}_0 \cup \{0\}$; moreover, $\lambda g_B \in \mathcal{R}|B \cup \{0\}$, because $\mathcal{R}|B$ is a coherent set. It follows that $\lambda h \in \mathcal{F}$.

D4. Consider $h, h' \in \mathcal{F}$. Then $h + h' = f + f' + \sum_{B \in \mathcal{B}} B(g_B + g'_B)$, where $f, f' \in \mathcal{R}_0 \cup \{0\}$ and $g_B, g'_B \in \mathcal{R}|B \cup \{0\}$. For analogous reasons as in the previous step, it holds that $f + f' \in \mathcal{R}_0 \cup \{0\}$ and $g_B + g'_B \in \mathcal{R}|B \cup \{0\}$. From this, we obtain that $h + h' \in \mathcal{F}$, provided that $h + h' \neq 0$. To see that this is indeed the case, assume that $h + h' = 0$; then either $0 = f + f'$ or $f + f' \neq 0$. In the first case, the coherence of $\mathcal{R}_0$ implies that $f = f' = 0$, and similarly since $g_B + g'_B = 0$ for every B we should have that $g_B = g'_B = 0$ for all B. But then $h = h' = 0$, a contradiction. In the second case, $0 \neq f + f' = -\sum_{B \in \mathcal{B}} B(g_B + g'_B)$. Taking into account that $f + f'$ is $\mathcal{B}$-measurable, there must be some $B \in \mathcal{B}$ such that $B(f + f') \gneq 0$: otherwise $f + f' \leq 0$ and $\mathcal{R}_0$ would incur partial loss. But on such a B we obtain that $g_B + g'_B \lneq 0$, so $\mathcal{R}|B$ would incur partial loss, a contradiction.

D5. Consider $0 \neq h \in \mathcal{L}$ such that $Bh \in \mathcal{F} \cup \{0\}$ for all $B \in \mathcal{B}$. We focus on the case $Bh \neq 0$, where it holds that $Bh = f + \sum_{B \in \mathcal{B}} B g_B$. If $f = 0$, then $Bh = B g_B$. If $f \neq 0$, then consider $B' \in \mathcal{B}$ such that $B' \neq B$. Bh is zero on B', and hence $B' f + B' g_{B'} = 0$. Now, recalling that f is $\mathcal{B}$-measurable, it is only possible that $f < 0$ on B': otherwise, $\mathcal{R}|B'$ would incur partial loss. Since we can repeat this reasoning for all $B' \neq B$, we deduce that $f > 0$ on B, as otherwise $\mathcal{R}_0$ would incur partial loss. In other words, f is a positive constant, say k_B, on B. Then $g_B + k_B \in \mathcal{R}|B$, so that if we re-define $g_B \coloneqq g_B + k_B$, we obtain that $Bh = Bg_B$. Thus, $h = \sum_{B\in\mathcal{B}\colon Bh\neq 0} Bh = \sum_{B\in\mathcal{B}\colon Bh\neq 0} Bg_B \in \mathcal{F}$.

Since $\mathcal{F}$ is included in any superset of $\mathcal{R}_0 \cup \mathcal{R}|\mathcal{B}$ satisfying D1–D5, this completes the proof. $\square$

The result also holds for a finite number of partitions.

Proposition 17. *Let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be partitions of Ω such that $\mathcal{B}_{i+1}$ is finer than $\mathcal{B}_i$ for $i = 1, \dots, n-1$. Let $\mathcal{R}_0$ be a set of desirable gambles coherent relative to $\mathcal{G}(\mathcal{B}_1)$. For each $i = 1, \dots, n-1$ and each $B_i \in \mathcal{B}_i$, let $\mathcal{B}_{i+1}|B_i \coloneqq \{B_{i+1} \in \mathcal{B}_{i+1}\colon B_{i+1} \subseteq B_i\}$ and $\mathcal{R}_i|B_i$ be a coherent set of desirable gambles on $\mathcal{L}(B_i)$ relative to $\mathcal{G}(\mathcal{B}_{i+1}|B_i)$. Finally, for each $B_n \in \mathcal{B}_n$, let $\mathcal{R}_n|B_n$ be a coherent set of desirable gambles on $\mathcal{L}(B_n)$. The conglomerable natural extension $\mathcal{F}_n$ of $\mathcal{R}_0$ and $\mathcal{R}_i|B_i$, $B_i \in \mathcal{B}_i$, is given by*

$$\left\{ f_0 + \sum_{i=1}^{n} \sum_{B_i \in \mathcal{B}_i} B_i g_{B_i} \colon f_0 \in \mathcal{R}_0 \cup \{0\}, g_{B_i} \in \mathcal{R}_i|B_i \cup \{0\} \right\} \setminus \{0\}.$$

7 Conclusions

We have studied the extension of desirability and probabilistic assessments under the requirement of conglomerability. Our main finding is that taking the natural extension while imposing conglomerability (which is the procedure adopted in Walley's theory), does not yield the conglomerable natural extension in general (but it does so in the case of the Marginal Extension Theorem); and that although iterating that process yields models ever closer to it, it is an open problem whether or not the conglomerable natural extension is achieved in the limit, or whether the limit is achieved in a finite number of steps. Future work could consist in (i) addressing these problems, and extending everything to the case of multiple partitions; (ii) defining a new coherence notion that follows from the conglomerable natural extension; (iii) investigating the relationship between such an extension and envelope theorems; and (iv) more generally, investigating whether the conglomerable natural extension always allows the most informative conclusions to be drawn.

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References

[1] G. De Cooman and F. Hermans. Coherent immediate prediction: bridging two theories of imprecise probability. *Artificial Intelligence*, 172(11):1400–1427, 2008.

[2] B. de Finetti. Sulla proprietà conglomerativa della probabilità subordinate. *Rendiconti del Reale Instituto Lombardo*, 63:414–418, 1930.

[3] B. de Finetti. *Probability, Induction and Statistics*. Wiley, London, 1972.

[4] S. Doria. Coherent upper and lower previsions defined by hausdorff outer and inner measures. In A. Rauh and E. Auer, editors, *Modeling, Design and Simulation Systems with Uncertainties*, pages 175–195. Springer, 2011.

[5] L. E. Dubins. Finitely additive conditional probabilities, conglomerability and disintegrations. *The Annals of Probability*, 3:88–99, 1975.

[6] E. Miranda and G. de Cooman. Marginal extension in the theory of coherent lower previsions. *International Journal of Approximate Reasoning*, 46(1):188–225, 2007.

[7] E. Miranda and M. Zaffalon. Notes on desirability and conditional lower previsions. *Annals of Mathematics and Artificial Intelligence*, 2011. In press.

[8] R. Pelessoni and P. Vicig. Williams coherence and beyond. *International Journal of Approximate Reasoning*, 50(4):612–626, 2009.

[9] M. Schervisch, T. Seidenfeld, and J. Kadane. The extent of non-conglomerability of finitely additive probabilities. *Zeitschrift fur Wahrscheinlichkeitstheorie und verwandte Gebiete*, 66:205–226, 1984.

[10] T. Seidenfeld, M. Schervisch, and J. Kadane. Non-conglomerability for finite-valued finitely additive probability. *Sankhya*, 60(3):476–491, 1998.

[11] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[12] P. M. Williams. Notes on conditional previsions. Technical report, School of Mathematical and Physical Science, University of Sussex, UK, 1975. Reprinted in [13].

[13] P. M. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44:366–383, 2007. Revised journal version of [12].