



Independence concepts in evidence theory

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ABSTRACT

We study three conditions of independence within evidence theory framework. The first condition refers to the selection of pairs of focal sets. The remaining two ones are related to the choice of a pair of elements, once a pair of focal sets has been selected. These three concepts allow us to formalize the ideas of lack of interaction among variables and among their (imprecise) observations. We illustrate the difference between both types of independence with simple examples about drawing balls from urns. We show that there are no implication relationships between both of them. We also study the relationships between the concepts of “independence in the selection” and “random set independence”, showing that they cannot be simultaneously satisfied, except in some very particular cases.

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1. Introduction

The concept of stochastic independence is essential in probability theory. Factorization allows us to decompose complex problems into simpler components. When generalizing to imprecise probabilities, the concept of independence, which is unique in probability theory, can be extended in different ways. Different definitions of independence for imprecise probabilities are studied and compared in [4,5,15].

Evidence theory [13] falls within the theory of imprecise probabilities. This way, definitions of independence for imprecise probabilities can be transferred to this context. The concept of independence is basic for important problems as local computation in graphical models. The different concepts of independence give rise to different forms of constructing a joint representation of information and, therefore the algorithms to compute in these models are also different [1,3]. So, it is very important to clarify the relationships between these notions and the conditions under which they can be applied. In [8], for instance, sets of joint probability measures associated to joint mass assignments are constructed. Different ways of choosing the weights of the joint focal sets and the probability measures inside these sets are considered. Depending on these conditions, different sets of joint probability measures are obtained. The author shows that some of these cases lead to types of independence described in [5] such as strong independence, random set independence and unknown interaction. The author initially considers the class of all probability measures on a product space whose marginals are dominated by a pair of plausibility measures. Next he establishes three rules to construct probabilities within that class. Each rule is related to a particular aspect of independence and it determines a subclass in the initial set of probability measures. The first rule refers to the choice of weights of the joint focal sets, and it is related to the concept of random set independence. The second and the third rules are referred to the choice of the probability measures inside the focal sets. The author shows that the class of probability measures based on these three rules satisfies independence in the selection. We will go further on this study. First, we will recall these notions under a different framework. Then we will give an intuitive meaning for each rule, by

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means of simple examples about drawing balls from urns. Our main goal is showing that none of these rules is strictly necessary to get independence in the selection. In fact, we will construct product probabilities without using any of these rules. This will be possible because the same probability measure can be constructed by using different procedures. In fact, we can choose weights of the joint focal sets and/or the probability measures inside the focal sets and finally get the same probability measure.

We will also go into further details about the relationships among random set independence [5] with independence in the selection and type 1 independence [4]. It is well known that the class of probability measures associated to random set independence includes the class of probability measures satisfying type 1 independence (see [5], for instance). We will check in the paper that this is a strict inclusion, except for trivial situations (precise probabilities). We will also show that the inclusion is also strict for the set of extreme points of these sets in most of the cases, but with more exceptions (at least one of the masses has pairwise disjoint focal elements).

Our analysis does not apply to all interpretations of evidence theory, but only when the pair of plausibility and belief functions is regarded as a family of probability measures. Different interpretations of evidence theory as the transferable belief model [14] lead to different approaches (see [16], for instance) to the concept of independence.

The paper is organized as follows. Section 2 provides the necessary technical background about upper probabilities, evidence theory and independence notions for imprecise probabilities. Section 3 is devoted to different representations of the class of probability measures dominated by a particular plausibility function. Section 4 studies random set independence and its relationships with the concepts of independence in the selection and type-1 independence. We end the paper with some general concluding remarks and open problems.

2. Preliminary concepts and notation

Let us introduce some notation and recall some definitions needed in the rest of the paper.

2.1. Sets of probability measures

Consider a finite universe Ω . We will denote \mathcal{P}_Ω the class of all probability measures we can define on the power set of Ω , $\wp(\Omega)$. Let $\mathcal{P} \subseteq \mathcal{P}_\Omega$ an arbitrary subset. It induces upper and lower probability functions, respectively defined by

$$P^*(A) = \sup_{Q \in \mathcal{P}} Q(A); P_*(A) = \inf_{Q \in \mathcal{P}} Q(A) \quad (1)$$

The set of probability measures dominated by an upper probability P^* is denoted by $\mathcal{P}(P^*) = \{Q : Q(A) \leq P^*(A), \forall A \subseteq \Omega\}$. If the upper probability measure P^* is generated by the family \mathcal{P} , then $\mathcal{P}(P^*)$ is generally a proper superset of \mathcal{P} . Specifically, $\mathcal{P}(P^*)$ is always the convex hull of \mathcal{P} : $\text{CH}(\mathcal{P}) = \mathcal{P}(P^*)$.

Two sets of probabilities \mathcal{P}_1 and \mathcal{P}_2 are said to be equivalent if and only if they have the same convex hull. Two equivalent sets of probabilities define the same pair of upper and lower probabilities. If \mathcal{P} is a set of probabilities, its largest equivalent set is its convex hull, $\text{CH}(\mathcal{P})$, and the smallest equivalent set is the set of its extreme points, $\text{Ext}(\mathcal{P})$.

Mathematical evidence theory of Shafer extends classical probability theory. In this framework, a *basic mass assignment*, m , is a mass of probability defined over the power set of Ω . It assigns a positive mass to a family of subsets of Ω called the set \mathcal{F}_m of focal subsets. Generally, $m(\emptyset) = 0$ and $\sum_{E \in \mathcal{F}_m} m(E) = 1$. This mass assignment induces set functions called plausibility and belief measures, respectively denoted by Pl and Bel , and defined by Shafer [13] as follows:

$$\text{Pl}(A) = \sum_{E \cap A \neq \emptyset} m(E), \quad \text{Bel}(A) = \sum_{E \subseteq A} m(E).$$

2.2. Independence concepts for imprecise probabilities

Consider two variables or uncertain values which may be regarded as the outcomes of two experiments. Assume that the two outcomes are known to belong to the universes Ω_1 and Ω_2 which are finite. Assume that the set of possible joint outcomes is the Cartesian product $\Omega_1 \times \Omega_2$. Let us respectively represent by $\mathcal{P}_1 \subseteq \mathcal{P}_{\Omega_1}$ and $\mathcal{P}_2 \subseteq \mathcal{P}_{\Omega_2}$ our knowledge about the true distribution of probability that models each marginal experiment. Let $\mathcal{P} \subseteq \mathcal{P}_{\Omega_1 \times \Omega_2}$ represent our (imprecise) knowledge about the joint probability distribution associated to the joint experiment. Given a joint probability measure, P on $\Omega_1 \times \Omega_2$ we will respectively denote P_1 and P_2 its marginals on Ω_1 and Ω_2 , i.e., $P_1(A) = P(A \times \Omega_2)$, and $P_2(B) = P(\Omega_1 \times B)$, $\forall A \subseteq \Omega_1, B \subseteq \Omega_2$.

We say that there is *independence in the selection* [5] when every extreme joint probability $P \in \mathcal{P}$ factorizes as $P = P_1 \otimes P_2$, i.e., $P(A \times B) = P_1(A)P_2(B), \forall A \subseteq \Omega_1, B \subseteq \Omega_2$. In other words, when

$$\text{Ext}(\mathcal{P}) \subseteq \{P_1 \otimes P_2 : P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2\}.$$

This concept coincides with the notion of *type-2 independence* studied in [4]. In this paper, another related concept was considered: if \mathcal{P} is a general set of probabilities (non necessarily convex), we say that there is *type-1 independence* when the factorization property is satisfied for every $P \in \mathcal{P}$, and not only for the extreme points.

Suppose that $\mathcal{P}_1 = \mathcal{P}(\text{Pl}_{m_1})$ and $\mathcal{P}_2 = \mathcal{P}(\text{Pl}_{m_2})$. We say that there is random set independence if $\mathcal{P} = \mathcal{P}(\text{Pl}_m)$, where $m = m_1 \odot m_2$, i.e.,

$$m(A \times B) = m_1(A) m_2(B), \quad \forall A \subseteq \Omega_1, \quad B \subseteq \Omega_2.$$

and $m(C) = 0$ if $C \neq A \times B$ for any $A \subseteq \Omega_1, B \subseteq \Omega_2$.

3. Probability measures dominated by a plausibility function

In this section we will deal with representations of the class of probability measures dominated by a particular plausibility function. Let Ω represent the (finite) universe of discourse and let $\mathcal{F}_m = \{A_1, \dots, A_q\}$ be the class of focal sets associated to a basic mass assignment m . Let Pl_m denote the associated plausibility measure. Grabisch et al. [9] consider the family of tuples $Z(\mathcal{F}_m) = \{\vec{\alpha} = (\alpha_1, \dots, \alpha_q) : \alpha_i : A_i \rightarrow [0, 1], \sum_{\omega \in A_i} \alpha_i(\omega) = m(A_i), i = 1, \dots, q\}$. For each particular tuple $\vec{\alpha} \in Z(\mathcal{F}_m)$, they consider the associated probability measure $P_{\vec{\alpha}} : \wp(\Omega) \rightarrow [0, 1]$ such that $P_{\vec{\alpha}}(\{\omega\}) = \sum_{i:A_i \ni \omega} \alpha_i(\omega), \forall \omega \in \Omega$. Under this construction, they easily check that each $P_{\vec{\alpha}}$ is dominated by Pl_m . Furthermore, for each $A \subseteq \Omega$, there exists $\vec{\alpha}^* \in Z(\mathcal{F}_m)$ such that $P_{\vec{\alpha}^*}(A) = \text{Pl}_m(A)$. Let the reader notice that these conditions are sufficient to check that the class $\mathcal{J}_m = \{P_{\vec{\alpha}} : \vec{\alpha} \in Z(\mathcal{F}_m)\}$ coincides with $\mathcal{P}(\text{Pl}_m)$, since their extreme points do coincide and both of them are convex.

Fetz independently considers in [8] the class of probability measures

$$\mathcal{K}_m := \left\{ \sum_{i=1}^q m(A_i) P^i : P^i \in \mathcal{K}^i \right\}, \text{ where}$$

$$\mathcal{K}^i = \{P^i \in \mathcal{P}_\Omega, : P^i(A_i) = 1, \forall i = 1, \dots, q\}$$

In other words, each probability measure in \mathcal{K}_m is a linear convex combination of q probability measures, P^1, \dots, P^q . Each P^i is a probability measure on the focal set A_i .

The family \mathcal{K}_m coincides with \mathcal{J}_m . In fact, each tuple $\vec{\alpha} = (\alpha_1, \dots, \alpha_q)$ is associated to the tuple of probability measures (P^1, \dots, P^q) defined as

$$P^i(\{\omega\}) = \frac{\alpha_i(\omega)}{m(A_i)}, \quad \forall \omega \in A_i, \quad \forall i = 1, \dots, q.$$

We can give an additional alternative description of the class \mathcal{K}_m . Given a mass function m with focal elements (A_1, \dots, A_q) , we are going to consider the family of all vectors $(m, (P^i)_{i=1}^q)$ where P^i is a probability on Ω satisfying that $P^i(A_i) = 1$. Each of these vectors $(m, (P^i)_{i=1}^q)$ defines a probability measure on $\wp(\mathcal{S}), \mathbb{P} : \wp(\mathcal{S}) \rightarrow [0, 1]$, where $\mathcal{S} = \{(A_i, \omega) : A_i \in \wp(\Omega), \omega \in A_i\}$ and given on the elementary events by:

$$\mathbb{P}(\{(A_i, \omega)\}) = m(A_i) \cdot P^i(\{\omega\}),$$

if A_i is a focal element and $\omega \in A_i$, and 0 otherwise.

As vector $(m, (P^i)_{i=1}^q)$ univocally determines probability measure \mathbb{P} , from now on, we will write $\mathbb{P} \equiv (m, (P^i)_{i=1}^q)$.

Each probability \mathbb{P} defines a probability $P \in \mathcal{K}_m$, given by:

$$P(\{\omega\}) = \sum_{A \in \wp(\Omega)} \mathbb{P}(\{(A, \omega)\}) = \sum_{i=1}^q m(A_i) \cdot P^i(\{\omega\}).$$

This probability P will be called the probability associated to \mathbb{P} .

Each probability measure \mathbb{P} can be seen as describing the uncertainty associated to a two steps procedure: first, a subset $A_i \subseteq \Omega$ is selected (according to probability m), and afterwards, an element $\omega \in A_i$ is chosen with probability P^i . This second step determines a precise probability P among all the possibilities associated to m .

Each probability $P \in \mathcal{K}_m$ is defined by at least one probability \mathbb{P} . In fact, given a tuple $\vec{\alpha} = (\alpha_1, \dots, \alpha_q)$, we can define one probability \mathbb{P} given by $(m, (P^i)_{i=1}^q)$, where P^i is the probability measure given by $P^i(\{\omega\}) = \alpha_i(\omega)$, if $\omega \in A_i$ and $P^i(\{\omega\}) = 0$, otherwise.

Remark 1. For an arbitrary $Q \in \mathcal{P}(\text{Pl})$, there exists at least one tuple $\vec{\alpha}$ such that $Q = P_{\vec{\alpha}}$. But this association is not necessarily unique. Let us consider, for instance, the universe $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and the mass assignment $m : \wp(\Omega) \rightarrow [0, 1]$ such that $\mathcal{F}_m = \{A_1, A_2\}$ where $A_1 = \{\omega_1, \omega_2\}, A_2 = \{\omega_1, \omega_2, \omega_3\} = \Omega$, and $m(A_1) = 0.5 = m(A_2)$. Let us now consider the probability measure $P : \wp(\Omega) \rightarrow [0, 1]$ such that $P(\{\omega_1\}) = P(\{\omega_2\}) = 5/12$ and $P(\{\omega_3\}) = 1/6$. Let $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$ the tuples of mappings defined as follows:

$$\alpha_1(\omega_1) = \alpha_1(\omega_2) = 0.25,$$

$$\alpha_2(\omega_1) = \alpha_2(\omega_2) = \alpha_2(\omega_3) = 1/6.$$

$$\beta_1(\omega_1) = 5/12, \beta_1(\omega_2) = 1/12,$$

$$\beta_2(\omega_1) = 0, \beta_2(\omega_2) = 1/3, \beta_2(\omega_3) = 1/6.$$

We easily check that

$$m(A) = \sum_{\omega \in A} \alpha_A(\omega) = \sum_{\omega \in A} \beta_A(\omega), \quad \forall A \text{ and}$$

$$P(\{\omega\}) = \sum_{A \ni \omega} \alpha_A(\omega) = \sum_{A \ni \omega} \beta_A(\omega), \quad \forall \omega \in \Omega.$$

4. Independence concepts in evidence theory

The notion of independence in evidence theory is studied from different points of view in the literature. In [16], for instance, the ideas of decomposability and irrelevance are studied and compared within the theory of evidence. In this paper, we will distinguish between independence of variables and independence of their observations. The first one is related to the concept of “type 1 independence” [4] and the second one is associated to “random set independence” [5].

In [8], Fetz establishes three different restrictions to the elements in $\mathcal{P}(Pl_m)$. Each one of them is related to some aspect of independence. Fetz shows some relationships between these restrictions and some other notions of independence considered in [5]. In this section, we will continue these investigations. First of all, we will recall the notions given by Fetz, but we will use a different nomenclature. For each definition, we will give an intuitive interpretation and an example of an urn model to which the definition is applied.

4.1. Three conditions of independence

Let $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$ and $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$ be two arbitrary basic mass assignments. Let us respectively denote by $\mathcal{F}_{m_1} = \{A_1, \dots, A_q\}$ and $\mathcal{F}_{m_2} = \{B_1, \dots, B_r\}$ their families of focal elements. Let us now consider a basic mass assignment on $\Omega_1 \times \Omega_2$, $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$ satisfying the following conditions:

- The family of focal elements associated to m coincides with (or it is included in) $\mathcal{F}_m = \{A_i \times B_j : i = 1, \dots, q, j = 1, \dots, r\}$. The Cartesian product $A_i \times B_j$ will be called a *rectangle*, and when this condition is fulfilled, we will say that m has rectangles as focal sets.
- $m_1(A_i) = \sum_{j=1}^r m(A_i \times B_j), \quad i = 1, \dots, q.$
- $m_2(B_j) = \sum_{i=1}^q m(A_i \times B_j), \quad j = 1, \dots, r.$

In these conditions, as the family of focal elements is $\mathcal{F}_m = \{A_i \times B_j : i = 1, \dots, q, j = 1, \dots, r\}$, a probability measure \mathbb{P} will be denoted as $(m, (P^{ij})_{i=1}^q, (P^{ij})_{j=1}^r)$, where P^{ij} is a probability measure in $\Omega_1 \times \Omega_2$ verifying $P^{ij}(A_i \times B_j) = 1$ (i.e., we use two superscripts to describe the set of focal elements and their associated probabilities).

In what follows, we are going to consider three definitions of independence for a probability measure $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q, (P^{ij})_{j=1}^r)$. The first of these conditions will be the plain random set independence for the mass m and the other two will be additional conditions under which the associated probability on $\Omega_1 \times \Omega_2$ given by

$$P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$$

factorizes as product of its marginals: $P = P_1 \otimes P_2$. The three definitions can be seen as conditions under which independence in the selection is the right independence concept, instead of random set independence.

The three definitions are closely related to three restrictions established in [8] to the elements in the class \mathcal{K}_m . Each condition reflects a different aspect associated to the notion of independence, as we will check below.

Definition 1. A probability measure $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q, (P^{ij})_{j=1}^r)$ satisfies the *first independence condition* if $m = m_1 \odot m_2$, i.e.,

$$m(A_i \times B_j) = m_1(A_i) \cdot m_2(B_j)$$

$$\forall i = 1, \dots, q, \quad j = 1, \dots, r.$$

This notion is associated to the concept of random set independence recalled in Section 2. As it is expressed in terms of the mass function, then either it is verified by all the probability distributions \mathbb{P} associated to m or by none of them. Let us illustrate this type of independence.

Example 1. Suppose that we have two urns, each of them with 10 balls. The first urn has five red, two white and three unpainted balls. The second urn has three red, three white and 4 unpainted balls. We select one ball from each urn in a stochastically independent way, and if either one between the selected balls are not coloured, then they are painted white or red by a completely unknown procedure. There can be arbitrary correlation between the colours they are finally assigned.

In this example, we are interested in the final colours of the two balls we draw from the urns. So, the universe of discourse is $\Omega_1 \times \Omega_2 = \{r, w\} \times \{r, w\}$. The focal elements associated to both selections are $\mathcal{F}_{m_1} = \{A_1, A_2, A_3\}$ and $\mathcal{F}_{m_2} =$

$\{B_1, B_2, B_3\}$, where $A_1 = B_1 = \{r\}$, $A_2 = B_2 = \{w\}$ and $A_3 = B_3 = \{r, w\}$. The marginal mass assignments for the colours of the selected balls are:

$$\begin{aligned} m_1(A_1) &= 0.5 & m_1(A_2) &= 0.2 & m_1(A_3) &= 0.3 \\ m_2(B_1) &= 0.3 & m_2(B_2) &= 0.3 & m_2(B_3) &= 0.4 \end{aligned}$$

The mass assignment associated to the joint experiment satisfies the equalities:

$$m(A_i \times B_j) = m_1(A_i) m_2(B_j), \quad \forall i, j.$$

The class of probability measures representing our (imprecise) information about the joint experiment is $\mathcal{P}(P|_m) = \mathcal{K}_m$. Each one of them is associated to a probability measure \mathbb{P} satisfying the first condition of independence.

Definition 2. A probability measure $\mathbb{P} \equiv \left(m, (P^{ij})_{i=1, j=1}^{q, r} \right)$ is said to satisfy the *second independence condition* if $P^{ij} = P_1^{ij} \otimes P_2^{ij}, \forall i = 1, \dots, q, j = 1, \dots, r$, i.e.,

$$\begin{aligned} P^{ij}(A \times B) &= P_1^{ij}(A) \cdot P_2^{ij}(B), \\ \forall A \subseteq \Omega_1, B \subseteq \Omega_2, \quad \forall i &= 1, \dots, q, \quad \forall j = 1, \dots, r, \end{aligned}$$

where P_1^{ij} and P_2^{ij} are the marginal probability measures of P^{ij} on Ω_1 and Ω_2 , respectively.

Example 2. Consider the same urns as in [Example 1](#) and assume again that we select one ball from each urn in a stochastically independently way. Let us also assume that, when both selected balls are not painted, there is no correlation between the colours they are assigned. If we have no additional information, our knowledge about the joint experiment is described by the class of probability measures of the form $P = \sum_{i=1}^3 \sum_{j=1}^3 m(A_i \times B_j) P^{ij}$, where m is the mass assignment from [Example 1](#), and P^{ij} is a probability measure on $\Omega_1 \times \Omega_2$ satisfying:

- $P^{ij}(A \times B) = P_1^{ij}(A) \times P_2^{ij}(B), \quad \forall A \in \wp(\Omega_1), \quad B \in \wp(\Omega_2),$
- $P^{ij}(A_i \times B_j) = 1$, for each $i = 1, 2, 3$ and each $j = 1, 2, 3$.

Every probability measure $\mathbb{P} \equiv \left(m, (P^{ij})_{i=1, j=1}^{q, r} \right)$ associated to this information satisfies the first and the second independence conditions. As we pointed out above, both balls are selected in a stochastically independent way. Furthermore, when both selected balls have no colour, we use separate procedures to paint them. Nevertheless, there can remain some dependence relation. Let us, for instance assume the following procedure to assign each colour:

- If only one of the selected balls is coloured, we will draw a dice to choose the colour of the other one. If the number in the dice is “5”, we will paint it with the same colour. Otherwise, we will choose the opposite.
- If both selected balls have no colour we will draw two coins, each one for each ball.

The probability measure, $P: \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$, associated to the joint experiment satisfies both conditions given in [Definitions 1 and 2](#). However, it cannot be expressed as a product. In fact, there exists a stochastic dependence between the colours of both balls. Let us notice, for instance, that

- $P(\{(r, r)\}) = 0.15 + 0.2 \cdot \frac{1}{4} + 0.09 \cdot \frac{1}{6} + 0.12 \cdot \frac{1}{4}$
- $P_1(\{r\}) = 0.5 + 0.09 \cdot \frac{1}{6} + 0.09 \cdot \frac{5}{6} + 0.12 \cdot \frac{1}{2}$, and
- $P_2(\{r\}) = 0.3 + 0.2 \cdot \frac{1}{6} + 0.06 \cdot \frac{5}{6} + 0.12 \cdot \frac{1}{2}$

Thus, $P(\{(r, r)\}) = 0.245$ does not coincide with $P_1(\{r\}) \cdot P_2(\{r\}) = 0.65 \cdot 0.46$.

Definition 3. A probability measure $\mathbb{P} \equiv \left(m, (P^{ij})_{i=1, j=1}^{q, r} \right)$ satisfies the *third independence condition* when

$$\begin{aligned} P_1^{i1} &= \dots = P_1^{ir} = P_1^i, \quad \forall i = 1, \dots, q \quad \text{and} \\ P_2^{1j} &= \dots = P_2^{qj} = P_2^j, \quad \forall j = 1, \dots, r. \end{aligned}$$

Example 3. Suppose again we have the urns in [Example 1](#). Let us draw a ball from each urn. If some of the balls is uncoloured, we decide its colour without checking whether the other one is red, white or uncoloured. Nevertheless, there can be some dependence relationship between both colours. Let us, for instance, consider the following procedure to assign each colour:

- We will toss a dice and if any of the balls is uncoloured, then if the number in the dice is “5”, we will paint it red. Otherwise, we will paint it white. In this procedure, we assume that if both balls are uncoloured, then the same dice is used to paint them, so they will have the same colour (we do not need to see the colour of the other ball to paint one of them, but there is dependence in the way the balls are coloured).

The probability measure, $\mathbb{P} \equiv (m, (P^{ij})_{i=1,j=1}^q, r)$, associated to the joint experiment satisfies the conditions given in **Definitions 1 and 3**. In particular, the equalities $P_1^{21} = P_1^{22} = P_1^{23}$ reflect that, when the first ball is uncoloured, the probability of painting it red is 1/6, independently from the initial colour of the second ball (red, white or uncoloured). A similar explanation applies to the equalities $P_2^{12} = P_2^{22} = P_2^{32}$.

Nevertheless, the probability measure that models the joint experiment (the probability measure $P = \sum_{i=1}^3 \sum_{j=1}^3 m(A_i \times B_j) P^{ij}$) cannot be written as the product of its marginals. For instance, the probability of the result (r, r) is, approximately, 0.22. On the other hand $P_1(\{r\}) = 0.55$ and $P_2(\{r\}) \approx 0.37$. Hence, $P(\{(r, r)\})$ does not coincide with the product $P_1(\{r\}) \cdot P_2(\{r\})$.

Summarizing, each condition reflects a different aspect of the notion of independence. The first condition (m verifies random set independence) reflects independence between the procedures used to select both balls from the urns. In the last examples, this condition is satisfied, because each ball is selected from a different urn, in a stochastically independent way. The second condition reflects independence between the procedures to paint both balls, once they have been selected. Finally third condition reflects independence between the procedure used to select one ball from an urn and the procedure used to paint the other ball, once it has been selected.

In **Examples 1–3** we show situations where some, but not all of these conditions are satisfied, and the probability P associated to \mathbb{P} cannot be written as a product. If $\mathbb{P} = (m, (P^{ij})_{i=1,j=1}^q, r)$, satisfies conditions **Definitions 1–3** then the probability measure $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$ can be factorized as $P = P_1 \otimes P_2$, as Fetz checks in [8]. Conversely, we easily check that every product probability $P = P_1 \otimes P_2$, where $P_1 \in \mathcal{P}(PI_{m_1})$ and $P_2 \in \mathcal{P}(PI_{m_2})$, can be written as $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$, where \mathbb{P} satisfies conditions given in **Definitions 1–3**. So, the second and the third independence conditions can be seen as additional restrictions for the joint probabilities in order that independence in the selection is verified. If we have two marginal masses m_1 and m_2 and we know that the joint probabilities P are obtained by a process which can be described as a probability \mathbb{P} verifying conditions **Definitions 1–3**, then independence in the selection is the right independence concept. In the next section we will make a further study about the connection between these conditions and independence in the selection.

4.2. Independence in the selection

As we pointed out in the last subsection, any probability measure $P = P_1 \otimes P_2$ with $P_1 \in \mathcal{P}(PI_{m_1}), P_2 \in \mathcal{P}(PI_{m_2})$ is associated to a probability measure \mathbb{P} satisfying independence conditions given in the last section. In other words, it can be written as a linear convex combination $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$, where $m = m_1 \odot m_2$ and $P^{ij} = P_1^i \otimes P_2^j, \forall i = 1, \dots, q, j = 1, \dots, r$. On the other hand, we can use a different linear convex combinations and get the same probability measure, as we have checked in **Remark 1**. So we can ask ourselves whether we can find an alternative linear convex combination

$$P = \sum_{i=1}^q \sum_{j=1}^r m'(A_i \times B_j) Q^{ij},$$

where $\mathbb{P} \equiv (m', \{Q^{ij}\}_{i=1,j=1}^q, r)$ does not satisfy the requirements considered in **Definitions 1–3**. In fact, it is possible, as we show below.

Example 4. Suppose we have two urns, each one with 10 balls. The two of them have five red, and five unpainted balls. We select one ball from the first urn and then we select a ball from the second urn, with this rule: the ball is red, if the ball selected from the first urn is red, while it is uncoloured, if the ball from the first urn is uncoloured (this rule realizes an extreme form of dependence between the selections). Once we have selected both balls, we use the following procedure to paint them in case they are uncoloured: we toss three coins, and check the number of heads:

- If the number is 3, we paint both balls with the colour red.
- If the number of heads is 2, we paint the first ball red, and the second one, white.
- If the number of heads is 1, we paint the first ball white, and the second one, red.
- Finally, if three tails are obtained, we paint white both of them.

The probability measure that models this random experiment can be written as:

$$P = m(A_1 \times B_1) P^{11} + m(A_2 \times B_2) P^{22},$$

where $A_1 = B_1 = \{r\}, A_2 = B_2 = \{r, w\}$,

$$m(A_1 \times B_1) = m(A_2 \times B_2) = 0.5 \text{ and}$$

$$P^{11} \equiv (1, 0, 0, 0) \text{ and } P^{22} \equiv (1/8, 3/8, 3/8, 1/8).$$

There do not exist m_1 and m_2 such that $m = m_1 \odot m_2$. On the other hand, each P^{ij} cannot be factorized as $P^{ij} = P_1^i \otimes P_2^j$. In other words, m and $\{P^{ij}\}_{i=1,j=1}^2$ do not satisfy the requirements from **Definitions 1 and 2**. It has no sense to check condition

Definition 3, since $P_1^{12}, P_2^{12}, P_1^{21}$ and P_2^{22} can be arbitrarily defined. Nevertheless, P coincides with the product of its marginals. In fact, $P(\{(r, r)\}) = 9/16, P(\{(r, w)\}) = P(\{(w, r)\}) = 3/16$, and $P(\{(w, w)\}) = 1/16$, and hence $P(A \times B) = P_1(A)P_2(B), \forall A, B \subseteq \{r, w\}$.

Since the probability measure that models the last experiment can be written as a product, there must exist an alternative linear convex combination,

$$P = \sum_{i=1}^2 \sum_{j=1}^2 m_1(A_i)m_2(B_j)Q^{ij}, \tag{2}$$

where $Q^{ij} = Q_1^i \otimes Q_2^j, \forall i, j$. In fact, the last experiment is equivalent to the following one: suppose we have two urns, each one with 10 balls. The two of them have five red, and five unpainted balls. We select one ball from each urn in a stochastically independent way. If some of the balls is uncoloured, we toss a coin to decide its colour (one coin for each ball). The probability measure associated to this new random experiment coincides with P and it can be written, in a natural way as in Eq. (2), where: $m_1(A_1) = m_1(A_2) = m_2(B_1) = m_2(B_2) = 0.5, Q_k^i(\{r\}) = Q_k^i(\{w\}) = 0.5, i = 1, 2, k = 1, 2$.

In the last example, we have built a product probability measure $P = P_1 \otimes P_2$ without having into account any of the requirements given in Definitions 1–3. We can also get a product probability by using some of these rules, but not all of them. In next example, we will only take into account the requirement from Definition 1, and we will get a product probability measure.

Example 5. Consider two urns with 10 balls. Both of them have the same composition: five balls are red, and the other five are unpainted. Suppose that we independently draw a ball from each one of the urns. If one ball is uncoloured, then it will be painted red or white with some procedure. This information can be represented by a mass assignment given by:

$$m(A_1 \times B_1) = m(A_1 \times B_2) = m(A_2 \times B_1) = m(A_2 \times B_2) = 0.25,$$

where $A_1 = B_1 = \{r\}, A_2 = B_2 = \{r, w\}$.

Now, we consider an specific procedure to paint the balls in case they are uncoloured (in this way we obtain a precise probability distribution on the ball colours):

- If both balls are red, we do not need to do anything.
- If the first ball is red and the second one is uncoloured, we paint it red with probability 5/8 and white, with probability 3/8.
- If the second ball is red and the first one is uncoloured, then we paint it red with probability 1/2 (and white, with the same probability).
- Finally, if both balls are unpainted, we assign them the pairs of colors (red, red), (red, white), (white, red), (white, white) with respective probabilities (1/8, 3/8, 1/4, 1/4).

The probability measure, P , that models the final colour of the balls can be written as

$$P = \sum_{i=1}^2 \sum_{j=1}^2 m(A_i \times B_j)P^{ij}, \text{ where}$$

$$A_1 = B_1 = \{r\}, A_2 = B_2 = \{r, w\},$$

$$m(A_1 \times B_1) = m(A_1 \times B_2) = m(A_2 \times B_1) = m(A_2 \times B_2) = 0.25 \text{ and}$$

$$P^{11} \equiv (1, 0, 0, 0) \quad P^{12} \equiv \left(\frac{5}{8}, \frac{3}{8}, 0, 0\right)$$

$$P^{21} \equiv \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right) \quad P^{22} \equiv \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}\right).$$

The probability measure describing the joint experiment (drawing the balls and then painting them) can be represented by: $\mathbb{P} \equiv (m, (P^{ij})_{i=1, j=1}^2)$. It satisfies first condition of independence (m is the product of its marginal mass assignments), but it does not satisfy the second and the third ones. On the other hand, the probability measure $P = \sum_{i=1}^2 \sum_{j=1}^2 m(A_i \times B_j)P^{ij}$ can be identified with the tuple

$$P \equiv \left(\frac{9}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16}\right),$$

so it can be factorized as

$$P = P_1 \otimes P_2 \equiv (3/4, 1/4) \otimes (3/4, 1/4).$$

We can also build some \mathbb{P} satisfying the requirements from Definitions 2 and 3, but not the property from Definition 1, and such that the probability measure P can be written as the product of its marginals. Let us show it in next example:

Example 6. Suppose that we have three urns. The first one has 3 balls: one white, one red and one uncoloured. The second urn has two balls: one red and one white. Third urn has two unpainted balls. We select one ball from the first urn. If it is coloured, we select a second ball from second urn. If, otherwise, it is uncoloured, we select the second ball from the third urn. This process defines mass assignment m . Now we determine a probability $P \in \mathcal{K}_m$ with the following procedure: once the balls have been selected, we drop two coins to decide their colour (if they are uncoloured), one coin for each ball.

The probability measure that models this experiment can be written as:

$$P = \sum_{i=1}^3 \sum_{j=1}^3 m(A_i \times B_j) P_1^i \otimes P_2^j, \text{ where}$$

$$A_1 = B_1 = \{r\}, \quad A_2 = B_2 = \{w\}, \quad A_3 = B_3 = \{r, w\},$$

the mass assignment m is determined by:

	B_1	B_2	B_3
A_1	1/6	1/6	0
A_2	1/6	1/6	0
A_3	0	0	1/3

and the marginal probability measures defined on each focal are:

$$P_1^1 \equiv (1, 0) \quad P_1^2 \equiv (0, 1) \quad P_1^3 \equiv (0.5, 0.5)$$

$$P_2^1 \equiv (1, 0) \quad P_2^2 \equiv (0, 1) \quad P_2^3 \equiv (0.5, 0.5)$$

The mass assignment m cannot be written as the product of its marginals, i.e., $m \neq m_1 \odot m_2$. So, $\mathbb{P} = \left(m, \{P^{ij}\}_{i=1}^3 \}_{j=1}^3\right)$ does not satisfy the condition described in Definition 1. But it satisfies the conditions described in Definitions 2 and 3. (There is independence inside the focal elements, but not between focals.) On the other hand, we easily check that $P(\{(r, r)\}) = P(\{(r, w)\}) = P(\{(w, r)\}) = P(\{(w, w)\}) = 0.25$. So P can be factorized as the product of its marginals. In fact:

$$P \equiv (0.25, 0.25, 0.25, 0.25) = (0.5, 0.5) \otimes (0.5, 0.5) = P_1 \otimes P_2.$$

4.3. Random set independence and independence in the selection

Let $m_1 : \wp(\Omega_1) \rightarrow [0, 1], m_2 : \wp(\Omega_2) \rightarrow [0, 1]$ two arbitrary mass assignments and let $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$ a joint mass with m_1 and m_2 as marginal masses. As we have pointed out in Section 4.1, the class of probability measures $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$, where $\mathbb{P} = \left(m, (P^{ij})_{i=1}^q \}_{j=1}^r\right)$ satisfies the three conditions considered in the last definitions, coincides with the family of product probability measures:

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\text{Pl}_{m_1}), P_2 \in \mathcal{P}(\text{Pl}_{m_2})\}.$$

On the other hand, we easily check that the class of probability measures $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$ where $\mathbb{P} = \left(m, (P^{ij})_{i=1}^q \}_{j=1}^r\right)$ satisfies the first condition, coincides with $\mathcal{P}(\text{Pl}_{m_1 \odot m_2})$. Thus, the following inclusion holds:

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\text{Pl}_{m_1}), P_2 \in \mathcal{P}(\text{Pl}_{m_2})\} \subseteq \mathcal{P}(\text{Pl}_{m_1 \odot m_2}) \tag{3}$$

The left hand side is associated to type 1 independence. The right hand side is related to random set independence. We may ask ourselves whether the inclusion in Eq. (3) is strict or not, for any pair of mass assignments m_1, m_2 , i.e., when random set independence coincides with type-1 independence. Let us notice that the probability measure $\mathbb{P} \equiv \left(m, (P^{ij})_{i=1}^q \}_{j=1}^r\right)$ in Example 5 satisfies the first condition of independence, but it does not satisfy the second and the third ones. Nevertheless, the probability measure $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \otimes B_j) P^{ij}$ can be factorized as $P = P_1 \otimes P_2$, and hence it belongs to the class $\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\text{Pl}_{m_1}), P_2 \in \mathcal{P}(\text{Pl}_{m_2})\}$. So, we ask ourselves:

Does there exist some pair m_1, m_2 such that any

$$P = \sum_{i=1}^q \sum_{j=1}^r m_1(A_i) m_2(B_j) P^{ij}$$

can be written as the product of its marginals, $P = P_1 \otimes P_2$?

The answer is “no”, except for the cases where m_1 and m_2 represent trivial situations. Let us show the following result:

Theorem 1. Let us consider two finite universes Ω_1 and Ω_2 and two arbitrary mass assignments $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$ and $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$. Let m be the “product mass assignment”, i.e., $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$ such that $m(A \times B) = m_1(A) \cdot m_2(B)$, $\forall A, B$. Let us assume that the set $\mathcal{P}(\text{Pl}_m)$ coincides with

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\text{Pl}_{m_1}), P_2 \in \mathcal{P}(\text{Pl}_{m_2})\}.$$

Then, some of the following conditions holds:

- Pl_{m_1} and Pl_{m_2} are probability measures (they are additive).
- Pl_{m_1} or Pl_{m_2} is a degenerate probability measure (i.e., at least one of the families \mathcal{F}_{m_1} or \mathcal{F}_{m_2} has only one focal with only one element.)

Proof. Let us assume that Pl_{m_2} is not a degenerate probability measure. Then there exists $B \subseteq \Omega_2$ and $Q_2 \in \mathcal{P}(Pl_{m_2})$ such that $Q_2(B) \in (0, 1)$. Let A be an arbitrary subset of Ω_1 and let $P_1, Q_1 \in \mathcal{P}(Pl_{m_1})$ such that $P_1(A) = Pl_{m_1}(A)$ and $Q_1(A) = Bel_{m_1}(A)$. (The existence of such P_1, Q_1 and Q_2 is easily checked.) Let $\vec{\alpha}, \vec{\alpha}'$ and $\vec{\beta}$ be respectively associated to each one of them. Let $\vec{\gamma} = (\gamma_{ij})_{i=1, j=1}^r$ be defined as $\gamma_{ij}(x, y) = \alpha_i(x)\beta_j(y)I_B(y) + \alpha'_i(x)\beta_j(y)I_{B^c}(y)$. We can check that $\vec{\gamma}$ represents a probability measure, R , on $\Omega_1 \times \Omega_2$ such that (a) $R \in \mathcal{P}(Pl_m)$, (b) $R_2 = Q_2, R_2(A \times B) = P_1(A)Q_2(B)$ and (c) $R_2(A \times B^c) = Q_1(A)Q_2(B)$. We easily derive that $Pl_{m_1}(A) = P_1(A) = Q_1(A) = Bel_{m_1}(A)$. Since A is an arbitrary set, we conclude that Pl_{m_1} is additive. \square

There is a reason for this: the second member in the inclusion of Eq. (3) is always convex, while the first member is not (type-1 independence usually implies that the joint credal set is not convex). We could relax the problem, by asking whether the two sets are equivalent (they have the same set of extreme points). The answer continues being “no” with the exception of some particular cases as the following theorem states:

Theorem 2. Let us consider two finite universes Ω_1 and Ω_2 and two arbitrary mass assignments $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$ and $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$. Let m be the “product mass assignment”, i.e., $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$ such that $m(A \times B) = m_1(A) \cdot m_2(B), \forall A, B$. Let us assume that the set of extreme probabilities in $\mathcal{P}(Pl_m)$ coincides with the extreme probabilities in

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(Pl_{m_1}), P_2 \in \mathcal{P}(Pl_{m_2})\}.$$

Then, for at least one of the masses m_i all its focal elements are pairwise disjoint.

Proof. To simplify the notation in the proof, let us denote Pl_m as Pl and Pl_{m_i} as $Pl_i (i = 1, 2)$.

Assume that m_1 has two focal elements such that $A_1 \cap A_2 \neq \emptyset$. As they are different, at least one of them is not included into the other. Without loss of generality, assume that $A_1 - A_2 \neq \emptyset$. Assume also that B_1 and B_2 are also two focal elements for m_2 such that $B_1 \cap B_2 \neq \emptyset$ and $B_1 - B_2 \neq \emptyset$.

In these conditions, we are going to prove that there is an extreme point in $\mathcal{P}(Pl)$ which does not belong to $\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(Pl_1), P_2 \in \mathcal{P}(Pl_2)\}$.

As Pl is an order-2 capacity, there is an extreme probability $P \in \mathcal{P}(Pl)$ such that it maximizes $P((A_1 - A_2) \times (B_1 - B_2))$ and $P((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2))$, at the same time, i.e.,

$$P((A_1 - A_2) \times (B_1 - B_2)) = Pl((A_1 - A_2) \times (B_1 - B_2)) = Pl_1(A_1 - A_2) \cdot Pl_2(B_1 - B_2)$$

and

$$P((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)) = Pl((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2))$$

We are going to prove that there is no probability P' in $\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(Pl_1), P_2 \in \mathcal{P}(Pl_2)\}$, fulfilling the two equalities satisfied by P . Assume that $P' = P'_1 \otimes P'_2$ also satisfies

$$P'((A_1 - A_2) \times (B_1 - B_2)) = Pl_1(A_1 - A_2) \cdot Pl_2(B_1 - B_2)$$

and

$$P'((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)) = Pl((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)).$$

We are going to obtain a contradiction. Let us consider the following values:

$$\begin{aligned} a_1 &= Pl_1(A_1) - Pl_1(A_1 \cap A_2), & a_2 &= Pl_1(A_1) - Pl_1(A_1 - A_2), & a_3 &= Pl_1(A_1) - a_1 - a_2 \\ b_1 &= Pl_2(B_1) - Pl_2(B_1 \cap B_2), & b_2 &= Pl_2(B_1) - Pl_2(B_1 - B_2), & b_3 &= Pl_2(B_1) - b_1 - b_2 \end{aligned}$$

The following facts can be easily obtained:

- $Pl_1(A_1) = a_1 + a_2 + a_3, \quad Pl_2(B_1) = b_1 + b_2 + b_3.$
- $Pl_1(A_1 - A_2) = a_1 + a_3, \quad Pl_2(B_1 - B_2) = b_1 + b_3.$
- $a_2 \geq m_1(A_2) > 0, \quad a_3 \geq m_1(A_1) > 0, \quad b_2 \geq m_2(B_2) > 0, \quad b_3 \geq m_2(B_1) > 0.$

The contradiction is a consequence of the following sequence of facts:

- $Pl((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)) = a_1 \cdot b_1 + a_1 \cdot b_3 + a_2 \cdot b_2 + a_2 \cdot b_3 + a_3 \cdot b_1 + a_3 \cdot b_2 + a_3 \cdot b_3.$ This equality can be obtained from the fact that this plausibility is equal to the plausibility of $Pl(A_1 \times B_1) = (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$ minus the mass of all the focal elements $A \times B$ such that

$$A \cap (A_1 - A_2) \neq \emptyset, \quad A \cap (A_1 \cap A_2) = \emptyset, \quad B \cap (B_1 - B_2) = \emptyset, \quad B \cap (B_1 \cap B_2) \neq \emptyset$$

and those verifying

$$A \cap (A_1 - A_2) = \emptyset, \quad A \cap (A_1 \cap A_2) \neq \emptyset, \quad B \cap (B_1 - B_2) \neq \emptyset, \quad B \cap (B_1 \cap B_2) = \emptyset,$$

and taking into account that the masses of these focal elements add $a_1 \cdot b_2 + b_1 \cdot a_2$.

- $P'_1(A_1 - A_2) = Pl_1(A_1 - A_2) = a_1 + a_3, \quad P'_2(B_1 - B_2) = Pl_2(B_1 - B_2) = b_1 + b_3.$

These equalities are a consequence of the fact that

$$\begin{aligned} Pl((A_1 - A_2) \times (B_1 - B_2)) &= P'((A_1 - A_2) \times (B_1 - B_2)) = \\ P'_1(A_1 - A_2) \cdot P'_2(B_1 - B_2) &\leq Pl_1(A_1 - A_2) \cdot Pl_2(B_1 - B_2) = \\ Pl((A_1 - A_2) \times (B_1 - B_2)). & \end{aligned}$$

As the first and the last member of the sequence are the same, the inequality is in fact an equality: $P'_1(A_1 - A_2) = Pl_1(A_1 - A_2)$ and $P'_2(B_1 - B_2) = Pl_2(B_1 - B_2)$.

- $P'_1(A_1 \cap A_2) \leq a_2, \quad P'_2(B_1 \cap B_2) \leq b_2.$

These inequalities are obtained taking into account that

$$P'_1(A_1 \cap A_2) = P'_1(A_1) - P'_1(A_1 - A_2) \leq Pl_1(A_1) - P'_1(A_1 - A_2) = Pl_1(A_1) - Pl_1(A_1 - A_2) = (a_1 + a_2 + a_3) - (a_1 + a_3) = a_2,$$

And analogously for the other inequality $P'_2(B_1 \cap B_2) \leq b_2$.

- $P'(((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2))) \leq (a_1 + a_3)(b_1 + b_3) + a_2 \cdot b_2$ This inequality is obtained from:
 $P'((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)) = P'((A_1 - A_2) \times (B_1 - B_2)) +$
 $P'((A_1 \cap A_2) \times (B_1 \cap B_2)) = Pl_1(A_1 - A_2) \cdot Pl_2(B_1 - B_2) + P'_1(A_1 - A_2) \cdot P'_2(B_1 - B_2) \leq (a_1 + a_3)(b_1 + b_3) + a_2 \cdot b_2.$

Finally, taking into account the first and last elements of this list, we obtain that:

$Pl((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)) - P'((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2)) \geq a_2 \cdot b_3 + b_2 \cdot a_3 > 0,$ and $P'((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2))$ cannot be equal to $Pl((A_1 - A_2) \times (B_1 - B_2) \cup (A_1 \cap A_2) \times (B_1 \cap B_2))$, which is in contradiction with what it was assumed. \square

If $m = m_1 \odot m_2$, then it satisfies the inclusion in Eq. (3) and we have shown, that except in some particular cases, this inclusion is not an equality. However, we can prove some results showing the m is a *reasonable* mass function, showing that it is a maximal element under order $m_1 \preceq m_2$ (m_1 is less informative than m_2), where $m_1 \preceq m_2$ if and only if $\mathcal{P}(Pl_{m_2}) \subseteq \mathcal{P}(Pl_{m_1})$. We will also give a characterization in terms of mass assignments with rectangles as focal elements.

Theorem 3. Let us consider two finite universes Ω_1 and Ω_2 and two arbitrary mass assignments $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$ and $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$. Let m be the “product mass assignment”, i.e., $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$ such that $m(A \times B) = m_1(A) \cdot m_2(B)$, $\forall A, B$. Assume that m' is another mass assignment satisfying the inclusion $\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(Pl_{m_1}), P_2 \in \mathcal{P}(Pl_{m_2})\} \subseteq \mathcal{P}(Pl_{m'})$ and such that $m \preceq m'$, then $m' = m$.

Proof. The proof is very simple. If m' satisfies the last inclusion, then for any $A \times B$, we have that

$$Pl_{m'}(A \times B) \geq \sup_{P_1, P_2} P_1(A) \cdot P_2(B) = Pl_{m_1}(A) \cdot Pl_{m_2}(B).$$

On the other hand, as $m' \preceq m$

$$Pl_{m'}(A \times B) \leq Pl_m(A \times B) = Pl_{m_1}(A) \cdot Pl_{m_2}(B)$$

Therefore, $Pl_{m'}(A \times B) = Pl_{m_1}(A) \cdot Pl_{m_2}(B)$. In an analogous way, it can be proved that $Bel_{m'}(A \times B) = Bel_{m_1}(A) \cdot Bel_{m_2}(B)$.

These product properties are stated in [16] as characterizing random set independence, and therefore $m = m'$. \square

The following theorem proves that random set independence can be characterized by the product property of belief functions, if we assume that all the focal elements are rectangles (i.e., product sets $A \times B$).

Theorem 4. Let us consider two finite universes Ω_1 and Ω_2 and two arbitrary mass assignments $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$ and $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$. Let m be the “product mass assignment”, i.e., $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$ such that $m(A \times B) = m_1(A) \cdot m_2(B)$, $\forall A, B$. If m' is a mass function on $\Omega_1 \times \Omega_2$ with rectangles as focal sets and such that $Bel_{m'}(A \times B) = Bel_{m_1}(A) \cdot Bel_{m_2}(B)$, then $m = m'$.

Proof. The proof is a consequence of Proposition 3 in [12] in which it is shown that we can express a belief function for any set in terms of the values of the beliefs in a family of sets including the focal elements. As Bel_m and $Bel_{m'}$ have the same values in the rectangles and this family of sets includes the focal elements of them, they must have the same value of belief for any set. \square

As a consequence of this theorem, we can say that the product random set m is characterized by having rectangles as focal elements and with belief in the rectangles defined from the set $\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(Pl_{m_1}), P_2 \in \mathcal{P}(Pl_{m_2})\}$, through equation:

$$Bel_m(A_1 \times A_2) = \inf\{P_1(A_1) \otimes P_2(A_2) : P_1 \in \mathcal{P}(Pl_{m_1}), P_2 \in \mathcal{P}(Pl_{m_2})\}$$

5. Conclusion and open problems

We have considered three rules to build probability measures on product spaces in evidence theory framework. Each one of them reflects a particular aspect of independence, as we illustrate in Examples 1–3. They are simple examples about draw-

ing pairs of balls from urns. As we show there, first condition reflects that the selections of both balls are independent. The second condition means that there is independence between the procedures of painting the balls, for a particular selection of a pair of balls. Finally, third condition reflects independence between the selection of a ball and the procedure used to choose the colour to paint the other ball.

In a more general and applied context, first condition is related to the idea of independence among mechanisms of observation of variables. If we add second and third conditions, independence between the actual variables holds. But, as we have checked in Examples 4–6, none of these conditions is strictly necessary to guarantee this independence. When there is no imprecision in the observations, second and third conditions do not apply (they are trivially satisfied when the focals are singletons). In that case, independence between the variables and between their observations are the same (perception and reality do coincide). But when imprecision appears, there is no an implication relationship between independence of the observations and independence of the variables.

We hope, that these results can help to understand the hypothesis under which random set independence and independence in the selection can be applied. We consider that it is interesting to continue investigating conditions under which random set independence can be characterized by some simple properties, among those belief functions such that the associated credal set contains the set of product marginal probabilities.

In a future, we plan to extend these ideas to non finite universes. In the general setting, upper probabilities induced by multi-valued mappings [6] would play the role of plausibility measures. Furthermore, random set independence would be generalized as the stochastic independence between two multi-valued mappings defined on the same probability space. On the other hand, it is well known that the upper probability induced by a multi-valued mapping dominates all the probability measures induced by its measurable selections. Thus, the notion of type 1 independence considered in the paper would imply the stochastic independence between the pairs of random variables, each component being a selection of each multi-valued mapping. The way that the notions captured in Definitions 2 and 3 can be translated into this general setting would be a matter of study.

A new combination rule has appeared recently in the literature, the *cautious combination rule* [7,11,10], which can be applied to situations in which the items of evidence to be combined come from non distinct or overlapping sources. Every combination rule defines an independence concept through the decomposition of the joint as combination of the marginals. In the future, we plan to study the independence and conditional independence concepts associated to this combination.

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