Martindale Quotients of Jordan Algebras

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Abstract: In this paper we introduce Martindale quotients of Jordan algebras over arbitrary rings of scalars with respect to denominator filters of ideals. For any denominatored algebra, we show the existence of maximal Martindale quotients naturally containing all Martindale quotients of the algebra with respect to the given denominator filter.

Keywords: Jordan algebra, Martindale quotient, denominator filter of ideals.

MSC2000: 17C10.

Introduction

In the last years, several authors have studied algebras of quotients of Jordan systems. The origin can be set in [8], where Martínez gives necessary and sufficient conditions for a Jordan algebra to have an algebra of fractions. She uses the Tits-Kantor-Koecher construction to move the problem into the Lie algebra setting, so that only rings of scalars containing 1/6 can be considered. In a similar fashion, taking into account Siles' work on quotients of Lie algebras [15], García and Gómez-Lozano [4] give a notion of Martindale-like quotient for linear Jordan systems over fields with respect to filters of ideals and prove the existence of maximum quotients in the nondegenerate case. In [3], the restriction on the rings of scalars is weakened to having 1/2, though only strongly prime algebras are considered. However, a full description of the maximum Martindale-like quotients of strongly prime Jordan algebras is obtained, giving a new unified approach to Zelmanovs's classification theorems [16]. In [14], Montaner gives a Jordan version of Lambek and Utumi's algebras of quotients over arbitrary rings of scalars, but only for nondegenerate algebras. His notion includes that of García and Gómez-Lozano in the case of algebras.

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In this paper we give a (quadratic) notion of Martindale quotient Q for arbitrary Jordan algebras J with respect to denominator filters of ideals. Unlike more general theories of localization in the associative case, we demand that a Martindale quotient algebra Q contains a faithful copy of the original algebra J. We impose no conditions (such as semiprimeness or nondegeneracy), only that the "denominators" are faithful to J (sturdy). This notion extends that given in the linear setting by García and Gómez-Lozano [3], and also includes the notion of Martindale-like cover [1, 2] for nondegenerate algebras. Since we do not assume any regularity condition other than the existence of a denominator filter of ideals, we cannot make use of the structure theory of nondegenerate Jordan algebras, unlike [1, 2, 3, 14]. We generalize all basic properties known in the linear case [3, 4] and even show that each Martindale quotient of a denominatored algebra is contained in a maximal one, though we leave open the problem of uniqueness of those maximal quotients.

The paper is divided into six sections, apart from a preliminary one recalling basic results and terminology. The way elements of Martindale quotients are boosted into the original algebra by denominator ideals is deeply related to annihilators, so we start in the first section with some combinatorial results concerning annihilation by powers of ideals. The second section defines Martindale quotients with respect to denominator filters of ideals and studies basic properties leading to the notion of maximum Martindale quotient. In the third section we give several examples, including a degenerate example, and prove that our notion is the quadratic generalization of that given in [4] and includes that of Martindale-like cover [1, 2]. In Section 4, we exhibit a way to build Martindale quotients out of any extension of Jordan algebras. This construction is extensively used in the next section where we show that there is a bound on the cardinality of Martindale quotients of a given denominatored algebra, implying the existence of maximal quotients; existence of a maximum quotient is equivalent to directedness of the lattice of quotients. Finally, the last section deals with the interaction between Martindale quotients and unital hulls.

0. Preliminaries

0.1 We will deal with Jordan algebras over a ring of scalars Φ. The reader is referred to [5, 7, 12] for definitions and basic properties not explicitly mentioned or proved in this section. Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $V_x y = x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively. For $y \in J$, the quadratic operator $\cap_y : J \longrightarrow J$ of inner multiplication by y is given by $\cap_y(x) = U_x y$. Each Jordan algebra is imbedded in its *free unital hull* $\widehat{J} := \Phi 1 \oplus J$. Zelmanov's structure theory shows that the proper unital hulls are those which are tight; in (6.2) we will tighten $\widehat{J}$ to get the "true" unital hull $\check{J}$.

0.2 We recall the following identities valid for arbitrary Jordan algebras which will be needed in the sequel:

(i) $x^2 \circ z = \{x, x, z\}$,
$(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$,

(ii) $\{U_b a, a, y\} = \{b, U_a b, y\}$,
$\{x, U_a b, y\} = \{\{x, a, b\}, a, y\} - \{b, U_a x, y\}$,
$\{U_b a, x, y\} = \{b, \{a, b, x\}, y\} - \{U_b x, a, y\}$,

(iii) $U_x U_y z + U_y U_x z - U_{x \circ y} z = -V_{x,y} V_{y,x} z + (U_x y^2) \circ z = \{U_x y, z, y\} - V_x U_y V_x z$,
$U_y x^2 = (x \circ y)^2 - U_x y^2 - y \circ U_x y = (x \circ y)^2 + U_x y^2 - \{x, y, x \circ y\}$,

(iv) $U_{U_x y} = U_x U_y U_x, \qquad (U_x y)^2 = U_x U_y x^2, \qquad U_{x^2} = U_x U_x$,

(v) $(U_x y) \circ z = \{x, y, x \circ z\} - U_x(y \circ z)$,

(vi) $2U_x z = (x \circ z) \circ x - x^2 \circ z$,

(vii) $[V_{x,y}, V_{z,w}] = V_{\{x,y,z\},w} - V_{z,\{y,x,w\}}$,

(viii) $U_x U_{a,b} = V_{x,b} V_{x,a} - V_{U_x b, a}$,

(ix) $U_{\{x,y,z\}} + U_{U_x y, U_z y} = U_x U_y U_z + U_z U_y U_x + U_{x,z} U_y U_{x,z}$.

Indeed, (i), (iii–vi), and the first part of (ii) follow from Macdonald's Theorem [6], the second and third identities of (ii) follow from the first one by linearization, and (vii), (viii), (ix) are respectively JP15, JP13, JP20 in [7].

0.3 A Jordan algebra J is said to be *nondegenerate* if zero is the only *absolute zero divisor*, i.e., the only $x \in J$ such that $U_x = 0$.

0.4 We recall that an *inner ideal* I of a Jordan algebra J is a Φ-submodule of J satisfying $U_I \widehat{J} \subseteq I$ [i.e., $U_I J + I^2 \subseteq I$], while an *outer ideal* of J is a Φ-submodule I of J satisfying $U_{\widehat{J}} I \subseteq I$ [i.e., $U_J I + I \circ J \subseteq I$], which implies $\{I, J, J\} \subseteq I$ by (0.2)(i). We say that I is an *ideal* of J if it is both an inner and outer ideal. The cube $I^3 = U_I I$ and the product $U_I L$ of ideals I, L of J are again ideals of J [10, p. 221].

0.5 Given elements x, y in a Jordan algebra J, the symmetric sets of three expressions

$$\beta_x(y) := \{U_x y, \cap_x y, V_x y\} = \{\cap_y x, U_y x, V_y x\} =: \beta_y(x)$$

(the three basic Jordan products of x and y) will appear frequently. For any subsets S, T, L we will call the set

$$\beta_S(T) := \bigcup_{x \in S, y \in T} \beta_x(y)$$

the *basic S-boost* of T, and

$$\begin{aligned} Z_{L;S}(T) :=& \beta_S(T) \cup \{\{x, y, z\} \mid x \in S, y \in T, z \in L\} \cup \\ & \{U_x U_y z \mid x \in S, y \in T, z \in L\} \cup \{U_x y^2 \mid x \in S, y \in T\} \end{aligned}$$

the *Zelmanov S-boost of T in L*. When any of the subsets S, T, L above consists of a single element x, we will write x instead of $\{x\}$. In the same fashion, "$\neq 0$", "$= 0$" will be abbreviations of "$\neq \{0\}$", "$= \{0\}$", respectively.

0.6 We say that a Jordan algebra J is *semiprime* if $I^3 \neq 0$, for any nonzero ideal I of J, and say that J is *prime* if $U_I L \neq 0$, for any nonzero ideals I, L of J. Every nondegenerate Jordan algebra is semiprime. An ideal I of J is said to be *essential* if $I \cap L \neq 0$ for any nonzero ideal L of J. It is obvious that the intersection of two essential ideals of J is again an essential ideal. Moreover, if I, K are essential ideals of a semiprime J, the product $U_I K$ is essential: for any nonzero ideal L of J, $L \cap I \cap K \neq 0$, hence $0 \neq (L \cap I \cap K)^3 \subseteq L \cap U_I K$.

0.7 In a Jordan algebra J, the Zelmanov annihilator $\mathrm{Zann}_J(T)$ of a subset T of J is the set of all $z \in J$ such that $Z_{J;z}(T) = Z_{J;T}(z) = 0$, i.e., for all $x \in T$, (Z1) $U_z x = 0$, (Z2) $U_x z = 0$, (Z3) $V_{z,x}\widehat{J} = 0$, (Z3)$'$ $V_{x,z}\widehat{J} = 0$, (Z4) $U_z U_x \widehat{J} = 0$, (Z4)$'$ $U_x U_z \widehat{J} = 0$. Here (Z3) $\Leftrightarrow$ (Z3)$'$ by (0.2)(i), and in its presence (Z4) $\Leftrightarrow$ (Z4)$'$ by (0.2)(iii), so $Z_{J;z}(x) = 0 \iff Z_{J;x}(z) = 0$. Avoiding the unital hull, $\mathrm{Zann}_J(x)$ is the set of z which satisfy (Z1), (Z2), (Z3a) $z \circ x = 0$, (Z3b) $\{z, x, J\} = 0$, (Z4a) $U_z U_x J = 0$, (Z4b) $U_z x^2 = 0$. Thus

$$\begin{aligned}\mathrm{Zann}_J(T) &= \{z \in J \mid U_z T = U_T z = \{z, T, \widehat{J}\} = U_z U_T \widehat{J} = 0\}\\ &= \{z \in J \mid Z_{J;z}(T) = 0\} = \{z \in J \mid Z_{J;T}(z) = 0\}\end{aligned}$$

(if $1/2 \in \Phi$ then $\{z, T, \widehat{J}\} = 0$ suffices [9, 1.4]). This is always an inner ideal, and is an ideal if T is an ideal of J [9, 1.4 p. 235]. We say T is *sturdy* if $\mathrm{Zann}_J(T) = 0$. When $T = I$ is an ideal, the condition $U_z U_I \widehat{J} = 0$ follows from $U_z I = 0$. If $I \cap \mathrm{Zann}_J(I) = 0$ then $\mathrm{Zann}_J(I)$ is the maximum ideal of J missing I: if $I \cap K = 0$ for an ideal K of J, then $Z_{J;I}(K) \subseteq I \cap K = 0 \Longrightarrow K \subseteq \mathrm{Zann}_J(I)$. This implies that sturdy ideals are always essential. Since for any ideal I of J the ideal $L := I \cap \mathrm{Zann}_J(I)$ has $L^3 = 0$, we have

(1) *essential ideals coincide with sturdy ideals in semiprime Jordan algebras.*

When I is an ideal of a nondegenerate J,

(2) $\mathrm{Zann}_J(I) = \{z \in J \mid U_z I = 0\} = \{z \in J \mid U_I z = 0\}$ (J *nondegenerate*)

(see [10, 1.2a, 1.7; 13, 1.3]).

1. Technical Lemmas Concerning the Annihilator

1.1 Lemma. *Let Q be a Jordan algebra, I and S submodules of Q, and set $S' = S \circ I + S$.*

(i) *If $q \in Q$ has $U_I q + q \circ I \subseteq S$ then*

(a) $\{q, I, I\} \subseteq S'$,

(b) $V_{q,I^3} + V_{I^3,q} \subseteq V_{S',I} + V_{I,S'}$,

and, if in addition $U_q I + U_q I^3 \subseteq S$, *then*

(c) $U_q U_{I^3} \subseteq U_{S'} U_I + U_I U_S + U_{S,I^3} U_I + U_{q,I} U_{S,I^3} + (V_{S',I} + V_{I,S'})^2 + V_{S,I^3}$.

(ii) *Annihilation of ideals and boosting them to* 0 *are closely related. Indeed, if* I *is a submodule of* Q *with* $I^3 \subseteq I$, *then*

$$q \in \mathrm{Zann}_Q(I) \Longrightarrow \beta_q(I) = 0 \ (\textit{equivalently } \beta_I(q) = 0) \Longrightarrow q \in \mathrm{Zann}_Q(I^3).$$

(iii) *If* $I \subseteq J \subseteq Q$ *where* I *is an ideal in the subalgebra* J *and* $q \in Q$ *satisfies* $\beta_q(I) \subseteq J$ (*equivalently* $\beta_I(q) \subseteq J$), *then* $V_{q,(I^3)^3} + V_{(I^3)^3,q} \subseteq V_{I,I}$.

PROOF: (i) By (0.2)(i) $\{q, I, I\} \subseteq (q \circ I) \circ I + U_{I,I} q \subseteq S \circ I + S = S'$ as in (a), so by (0.2)(ii) for $a, b \in I$ and $x = q$, $V_{q,U_a b} = V_{\{q,a,b\},a} - V_{b,U_a q} \in V_{S',I} + V_{I,S}$ and dually $V_{U_b a,q} = V_{b,\{a,b,q\}} - V_{U_b q,a} \in V_{I,S'} + V_{S,I}$ as in (b). For (c), U_{I^3} is spanned by all $U_c, U_{c,c'}$ for $c = U_a b, c' = U_{a'} b' \in I^3$ for $a, a', b, b' \in I$, and we have $U_q U_c = U_q U_a U_b U_a$ [by (0.2)(iv)] $= \big(U_{\{q,a,b\}} + U_{U_q a, U_b a} - U_b U_a U_q - U_{q,b} U_a U_{q,b}\big) U_a$ [by (0.2)(ix)] $\subseteq U_{S'} U_I + U_{S,I^3} U_I + U_I U_S + U_{q,I} U_{S,I^3}$ [by respectively (a); $U_q a \in S$; (0.2)(iv) for $y = q$ and $U_a q \in S$; $y \to q, b$ in linearized (0.2)(iv) and $U_a q \in S$], while $U_q U_{c,c'} = V_{q,c'} V_{q,c} - V_{U_q c', c}$ [by (0.2)(viii) for $x = q$, $a = c$, $b = c'$] $\subseteq \big(V_{S',I} + V_{I,S'}\big)^2 + V_{S,I^3}$ [by (b) and hypothesis $U_q I^3 \subseteq S$].

(ii) The first implication is obvious. For the second, $\beta_q(I) = 0 \Longrightarrow \beta_q(I^3) = 0$ and $\{q, I^3, \widehat{Q}\} = V_{q,I^3}(\widehat{Q}) = 0$ and $U_q U_{I^3} \widehat{Q} = 0$ by applying (i)(b), (i)(c) to $\widehat{Q}$ with $S = 0$.

(iii) $V_{q,(I^3)^3} + V_{(I^3)^3,q} \subseteq V_{J,I^3} + V_{I^3,J}$ [by (i)(b) for I^3 in place of I and $S = J$] $\subseteq V_{I,I}$ [by (i)(b), for $S = I$ and all $q \in J$, because $\beta_q(I) \subseteq I$ since I is an ideal in J]. ■

1.2 LEMMA. *Let* Q *be a Jordan algebra,* J *a subalgebra of* Q, $q \in Q$, *and* I *an ideal of* J *with* $\beta_I(q) = 0$. *Then* $\beta_{I^3}(\beta_J(q)) = 0$ *so* $\beta_J(q) \subseteq \mathrm{Zann}_Q((I^3)^3)$.

PROOF: Let $L := I^3$, $x \in J$. Note $V_{L,q} \widehat{Q} = V_{q,L} \widehat{Q} = 0$ by (1.1)(i)(b) with $S = 0$, and $\{q, J, L\} = q \circ (J \circ L) - \{q, L, J\}$ [by (0.2)(i)] $\subseteq V_{q,L} 1 - V_{q,L} J$ [since $J \circ L \subseteq L$ by idealness of L in J] $= 0$. Now we check $V_{z,L} \widehat{Q} = 0$ successively for $z = x \circ q, U_x q, U_q x$ in $\beta_J(q)$:

$$V_{x \circ q, L} = [V_x, V_{q,L}] + V_{q, x \circ L} \text{ [by (0.2)(vii) with } y = 1] \ = 0 \text{ since } x \circ L \subseteq L,$$
$$V_{U_x q, L} = -V_{U_x L, q} + V_{x, \{q,x,L\}} \text{ [by (0.2)(ii)] } \subseteq V_{L,q} + V_{x,\{q,J,L\}} = 0,$$
$$V_{U_q x, L} = -V_{U_q L, x} + V_{q, \{x,q,L\}} \text{ [by (0.2)(ii))] } \subseteq V_{\beta_I(q), J} + V_{Q, V_{L,q} J} = 0.$$

Now we have $\beta_J(q) \circ L = V_{\beta_J(q), L} \widehat{1} = 0$ and it remains to show $U_L \beta_J(q) =$

$U_{\beta_J(q)}L = 0$. For all $k \in L$ we have by (0.2)(iii), (iii), (v) respectively that

$$\begin{aligned}
U_k(U_x q) &= -U_x U_k q + U_{x\circ k} q - \{\{x,k,q\},x,k\} + q \circ (U_k x^2) \\
&\subseteq U_J U_I q + U_I q + \{V_{q,L}(J), J, J\} + q \circ I = 0, \\
U_k(U_q x) &= -U_q U_k x + U_{k\circ q} x - \{\{k,q,x\},k,q\} + x \circ (U_q k^2) \\
&\subseteq U_q I + U_{I\circ q} J + V_{q,L} Q + J \circ (U_q I) = 0, \\
U_k(x \circ q) &= -x \circ (U_k q) + \{k, q, k \circ x\} \subseteq J \circ U_I q + U_I q = 0,
\end{aligned}$$

and similarly we have by (0.2)(iv), (iv), (iii) respectively that

$$\begin{aligned}
U_{U_x q} k &= U_x U_q U_x k \subseteq U_J U_q I = 0, \\
U_{U_q x} k &= U_q U_x U_q k \subseteq U_q U_J U_q I = 0, \\
U_{x\circ q} k &= U_x U_q k + U_q U_x k + \{\{x,q,k\},x,q\} - k \circ (U_q x^2) \\
&\subseteq U_J U_q I + U_q I + \{V_{L,q} J, J, Q\} + L \circ U_q J = 0.
\end{aligned}$$

We have proved $\beta_{I^3}(\beta_J(q)) = 0$, so $\beta_J(q) \subseteq \mathrm{Zann}_Q((I^3)^3)$ follows from (1.1)(ii). ■

2. Denominatored Algebras and Martindale Quotients

2.1 Given a Jordan algebra J, a nonempty set $\mathcal{F}$ of ideals of J will be called a *filter* if, for any $K, L \in \mathcal{F}$, there exists $I \in \mathcal{F}$ such that $I \subseteq U_K L$ (so that $I \subseteq K \cap L$). Notice that, in particular, for any $K \in \mathcal{F}$, there exists $K' \in \mathcal{F}$ such that $K' \subseteq K^3$. We say that a filter $\mathcal{F}'$ is *finer* than $\mathcal{F}$ ($\mathcal{F}' \succ \mathcal{F}$) if for all $I \in \mathcal{F}$ there exists $I' \in \mathcal{F}'$ with $I' \subseteq I$ (for example, if $\mathcal{F}' \supseteq \mathcal{F}$), and $\mathcal{F}, \mathcal{F}'$ are *cofinal* if $\mathcal{F}' \succ \mathcal{F} \succ \mathcal{F}'$. For a filter $\mathcal{F}$, its *closure* $\overline{\mathcal{F}}$, consisting of all ideals of J which contain some ideal of $\mathcal{F}$, is a filter which contains $\mathcal{F}$ and, moreover, $\mathcal{F}$ and $\overline{\mathcal{F}}$ are cofinal. Notice that

$$\mathcal{F}' \succ \mathcal{F} \Longleftrightarrow \overline{\mathcal{F}'} \supseteq \overline{\mathcal{F}}. \tag{1}$$

A *filtered algebra* will be a pair $(J, \mathcal{F})$ where J is a Jordan algebra and $\mathcal{F}$ is a filter of ideals of J. If a filter $\mathcal{F}$ consists of sturdy ideals of J, then it will be called a *denominator filter*, and the pair $(J, \mathcal{F})$ will be called a *denominatored algebra*. Notice that the closure of a denominator filter is also a denominator filter.

For example, the set of essential ideals of a semiprime Jordan algebra is a denominator filter by (0.6) and (0.7)(1).

We also remark that, for linear Jordan algebras ($1/2 \in \Phi$), a nonempty set of ideals of J is a denominator filter if and only if it is a power filter of sturdy ideals in the sense of [3, 1.2]

2.2 A *Martindale quotient* of a denominatored algebra $(J, \mathcal{F})$ is a pair (Q, τ) where Q is a Jordan algebra, and $\tau : J \longrightarrow Q$ is an algebra monomorphism such that

for any $0 \neq q \in Q$ there exists $I \in \mathcal{F}$ which boosts q nontrivially into J, in the sense that
$$0 \neq \beta_{\tau(I)}(q) \subseteq \tau(J),$$
which implies $\{\tau(I), \tau(I), q\} \subseteq \tau(J)$ by (0.2)(i).

2.3 Remark: Notice that (1) *any Martindale quotient* (Q, τ) *of a denominatored algebra* $(J, \mathcal{F})$ *is tight over* $\tau(J)$ (or, by abuse of language, tight over J) in the sense that every nonzero ideal of Q hits $\tau(J)$: if $0 \neq L$ is an ideal of Q, we can take any $0 \neq q \in L$, and there exists $I \in \mathcal{F}$ such that $0 \neq \beta_{\tau(I)}(q) \subseteq \tau(J) \cap L$.

Also, (2) *The denominator filter* $\mathcal{F}$ *in* J *induces a denominator filter* $\widetilde{\mathcal{F}} = \{\widetilde{I} \mid \widetilde{I}$ *is an ideal of* $Q, \widetilde{I} \supseteq \tau(I)$ *for some* $I \in \mathcal{F}\}$ *on* Q, since all such $\tilde{I}$ are sturdy in Q: if $\widetilde{I} \supseteq \tau(I)$ then $0 = \tau(\mathrm{Zann}_J(I))$ [by sturdiness of I] $= \mathrm{Zann}_{\tau(J)}(\tau(I))$ [since τ is an algebra isomorphism of J with $\tau(J)$] $\supseteq \tau(J) \cap \mathrm{Zann}_Q(\widetilde{I})$ forces $\mathrm{Zann}_Q(\widetilde{I})$ (which is an ideal of Q) to vanish by (1).

2.4 Remark: *Any covering map of Martindale quotients must be injective: if* (Q, τ) *is a Martindale quotient of a denominatored algebra* $(J, \mathcal{F})$ *and* $\tau' : J \longrightarrow Q'$ *a Jordan algebra monomorphism, then any algebra homomorphism* $f : Q \longrightarrow Q'$ *which satisfies* $f\tau = \tau'$ *must be injective; in particular, this holds when* (Q', τ') *is another Martindale quotient of* $(J, \mathcal{F})$ *and* $f\tau = \tau'$. Indeed, $\mathrm{Ker}\, \tau' = 0$ implies $\mathrm{Ker}\, f \cap \tau(J) = 0$, hence $\mathrm{Ker}\, f = 0$ by tightness (2.3)(1).

2.5 Proposition. *Let* $(J, \mathcal{F})$ *be a filtered algebra. Then,* $(J, \mathcal{F})$ *is a denominatored algebra if and only if* $\beta_I(x) \neq 0$ *for all* $0 \neq x \in J$ *and* $I \in \mathcal{F}$*. As a consequence, if* $(J, \mathcal{F})$ *is a denominatored algebra, then* (J, Id_J) *is a Martindale quotient of* $(J, \mathcal{F})$.

Proof: Assume that $\mathcal{F}$ is a denominator filter. If $x \in J$ satisfies $\beta_I(x) = 0$, for some $I \in \mathcal{F}$, then $x \in \mathrm{Zann}_J(I^3)$ by (1.1)(ii). On the other hand, there is $I' \in \mathcal{F}$ such that $I' \subseteq I^3$. But this means $x \in \mathrm{Zann}_J(I')$, which forces $x = 0$ since I' is sturdy. The converse is clear and the consequence is straightforward. ∎

2.6 Proposition. *Let* (Q, τ) *be a Martindale quotient of a denominatored algebra* $(J, \mathcal{F})$.

(i) *All* $\mathcal{F}$*-boosts are nontrivial:* $\beta_{\tau(I)}(q) \neq 0$ *for all* $0 \neq q \in Q$ *and* $I \in \mathcal{F}$.

(ii) *In particular,* $\mathrm{Zann}_Q(\tau(I)) = 0$ *and* $\tau(I)$ *remains sturdy in* Q*, for all* I *in* $\mathcal{F}$.

(iii) *Cofinal denominator filters have precisely the same Martindale quotients. Moreover,* (Q, τ) *remains a Martindale quotient of* $(J, \mathcal{F}')$ *for any denominator filter* $\mathcal{F}' \succ \mathcal{F}$.

Proof: (i) Suppose on the contrary that $0 \neq q \in Q$ satisfies $\beta_{\tau(I)}(q) = 0$ for some $I \in \mathcal{F}$, hence by (1.2) applied to $\tau(J) \subseteq Q$ we have $\beta_{\tau(J)}(q) \subseteq \mathrm{Zann}_Q((\tau(I)^3)^3)$. There exists $K \in \mathcal{F}$ satisfying $0 \neq \beta_{\tau(K)}(q) \subseteq \tau(J)$. The filter $\mathcal{F}$ contains $I' \subseteq U_I I = I^3$ and $I'' \subseteq U_{I'} I' \subseteq (I^3)^3$, so $\tau(I'') \subseteq (\tau(I)^3)^3$ and $0 \neq \beta_{\tau(K)}(q) \subseteq \tau(J) \cap$

$\beta_{\tau(J)}(q) \subseteq \tau(J) \cap \mathrm{Zann}_Q(\tau(I'')) \subseteq \mathrm{Zann}_{\tau(J)}(\tau(I''))$. But since τ is an injective algebra homomorphism, $\mathrm{Zann}_{\tau(J)}(\tau(I'')) = \tau(\mathrm{Zann}_J(I'')) = 0$ by sturdiness of I'', which is a contradiction.

(ii) is a direct consequence of (i).

(iii) The fact that cofinal denominator filters have the same Martindale quotients readily follows from (i). It is clear that if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are denominator filters in J, all Martindale quotients of $(J, \mathcal{F}_1)$ are also Martindale quotients of $(J, \mathcal{F}_2)$. Since $\mathcal{F} \subseteq \overline{\mathcal{F}}$, (Q, τ) is a Martindale quotient of $(J, \overline{\mathcal{F}})$, hence of $(J, \overline{\mathcal{F}'})$ because $\overline{\mathcal{F}'} \supseteq \overline{\mathcal{F}}$ by (2.1)(1). Thus (Q, τ) is a Martindale quotient of $(J, \mathcal{F}')$ since $\mathcal{F}'$ and $\overline{\mathcal{F}'}$ are cofinal. ■

Notice that (iii) shows $(J, \mathcal{F})$ and its closure $(J, \overline{\mathcal{F}})$ have precisely the same Martindale quotients. Since always $U_I K, I \cap K \in \overline{\mathcal{F}}$ for $I, K \in \overline{\mathcal{F}}$, this shows that without loss of generality we could have required a denominator filter to be a filter in the set-theoretic sense (closed under *enlargements* $I' \supseteq I$ and *intersections* $I \cap K$), which is also *closed under products* $U_I K$.

2.7 PROPOSITION. *Let (Q', τ') be a Martindale quotient of a denominatored algebra $(J, \mathcal{F})$, and let $\tau : J \longrightarrow Q$, $f, g : Q \longrightarrow Q'$ be algebra homomorphisms with $f\tau = g\tau = \tau'$. Then f and g agree on any $q \in Q$ boostable into $\tau(J)$:*

$$\beta_{\tau(I)}(q) \subseteq \tau(J) \textit{ for some } I \in \mathcal{F} \Longrightarrow f(q) = g(q).$$

In particular, if Q, Q' are both Martindale quotients then any $f : Q \longrightarrow Q'$ with $f\tau = \tau'$ is unique and injective, so that if $(Q, \tau) = (Q', \tau')$ then $f = Id_Q$.

PROOF: Our goal is to prove that $q' = f(q) - g(q) \in Q'$ vanishes. By (2.6)(i) it will suffice to prove

$$\beta_{\tau'(L)}(q') = 0 \text{ for } L \in \mathcal{F}, L \subseteq I^3.$$

As usual, we show q' is killed by all three pieces of $\beta_{\tau'(L)}$. Using $f\tau = g\tau = \tau'$ on J, for any $k \in I$ we have

$$\begin{aligned} U_{\tau'(k)} q' &= U_{\tau'(k)} f(q) - U_{\tau'(k)} g(q) = U_{f\tau(k)} f(q) - U_{g\tau(k)} g(q) \\ &= f(U_{\tau(k)} q) - g(U_{\tau(k)} q) \in (f - g)(\tau(J)) = 0 \\ \tau'(k) \circ q' &= \tau'(k) \circ f(q) - \tau'(k) \circ g(q) = f\tau(k) \circ f(q) - g\tau(k) \circ g(q) \\ &= f(\tau(k) \circ q) - g(\tau(k) \circ q) \in (f - g)(\tau(J)) = 0, \end{aligned}$$

and for any $k \in L$ we have

$$\begin{aligned} U_{q'} \tau'(k) &= U_{f(q) - g(q)} \tau'(k) = (-U_{f(q)} + U_{g(q)} + U_{f(q), f(q) - g(q)}) \tau'(k) \\ &= -U_{f(q)} f\tau(k) + U_{g(q)} g\tau(k) - \{f(q), \tau'(k), q'\} \\ &= -f(U_q \tau(k)) + g(U_q \tau(k)) - \{q', \tau'(k), f(q)\} \\ &\in (g - f)(\tau(J)) - \{q', \tau'(L), Q'\} = 0 \end{aligned}$$

[using the above to replace S, I, Q, q in (1.1)(i)(b) by 0, $\tau'(I)$, Q', q', respectively, noticing $\tau'(L) \subseteq \tau'(I^3) = (\tau'(I))^3$].

When Q, Q' are Martindale quotients, then all q are boostable, so $f = g$ is unique, and it is injective by (2.4). ■

2.8 A Martindale quotient (Q,τ) of a denominatored algebra $(J,\mathcal{F})$ will be called a *maximum* if for any other Martindale quotient (Q',τ') of $(J,\mathcal{F})$, there exists an algebra homomorphism $f : Q' \longrightarrow Q$ such that $f\tau' = \tau$.

The following result is a consequence of (2.7).

2.9 Theorem (Universal Property for Maximum Martindale Quotients). *Let (Q,τ) be a maximum Martindale quotient of a denominatored algebra $(J,\mathcal{F})$. If (Q',τ') is a Martindale quotient of $(J,\mathcal{F})$, then there exists a unique algebra homomorphism $f : Q' \longrightarrow Q$ such that $f\tau' = \tau$. Moreover, f is necessarily injective. Thus maximum Martindale quotients of a given denominatored algebra $(J,\mathcal{F})$ are unique up to isomorphism.*

Proof: If (Q,τ), (Q',τ') are both maximum Martindale quotients of $(J,\mathcal{F})$, then the unique algebra homomorphisms $f : Q' \longrightarrow Q$, $f' : Q \longrightarrow Q'$, such that $f\tau' = \tau$, $f'\tau = \tau'$ are mutually inverse isomorphisms since by (2.7) $f \circ f' = Id_Q$ and $f' \circ f = Id_{Q'}$. ■

3. Examples

3.1 Subexample. *If (Q,τ) is a Martindale quotient of a denominatored algebra $(J,\mathcal{F})$ then so is (Q',τ') for every subalgebra Q' of Q with $\tau(J) \subseteq Q'$, where τ' denotes the restriction of τ.*

This shows that in general there will be lots of "smaller" quotients (think of rational numbers with denominators restricted to a multiplicatively closed subset of the subset of the integers). The more interesting question is whether there are larger Martindale quotients (see Section 5).

3.2 Sturdy Ideal Example. *If $(J,\mathcal{F})$ is a denominatored Jordan algebra where J is an ideal of Q and all $I \in \mathcal{F}$ remain sturdy in Q ($\mathrm{Zann}_Q(I) = 0$ for all $I \in \mathcal{F}$), then (Q,τ), for τ the inclusion map, is a Martindale quotient of $(J,\mathcal{F})$.*

Indeed, always $\beta_I(q) \subseteq J$, and by (1.1)(ii) $\beta_I(q) = 0 \Longrightarrow q \in \mathrm{Zann}_Q(I^3) = 0$ because I^3 contains an ideal L of $\mathcal{F}$ which remains sturdy in Q.

3.3 Lemma. *If $(J,\mathcal{F})$ is a unital denominatored Jordan algebra, then any Martindale quotient (Q,τ) of $(J,\mathcal{F})$ is unital with the same unit as J: $1_Q = \tau(1_J)$.*

Proof: We shall use Peirce decompositions (see [7, Section I.5]). Indeed $e = \tau(1_J)$ is an idempotent of Q such that $\tau(J) \subseteq Q_2(e)$, and for any $q \in Q_1(e) \cup Q_0(e)$, $\beta_{\tau(J)}(q) \subseteq U_q\tau(J) + U_{\tau(J)}q + q \circ \tau(J) \subseteq U_q Q_2(e) + U_{Q_2(e)}q + q \circ Q_2(e) \subseteq Q_0(e) + 0 + Q_1(e)$. Thus, for any $I \in \mathcal{F}$ such that $\beta_{\tau(I)}(q) \subseteq \tau(J)$, we have that $\beta_{\tau(I)}(q) \subseteq$

$Q_2(e) \cap (Q_0(e) + Q_1(e)) = 0$, which implies $q = 0$ by (2.6)(i). This shows that $Q = Q_2(e)$, i.e., e is the unit element of Q. ■

3.4 Simple Examples. *If J is a unital Jordan algebra and $\mathcal{F} = \{J\}$ (when J is simple this is the only possible filter), then the only Martindale quotient is, up to isomorphism, $(Q, \tau) = (J, Id)$. If J is simple but not necessarily unital, $\mathcal{F} = \{J\}$ is the unique denominator filter of J, but now there can be many quotients: if $J = A^{(+)}$ for $A \subseteq \mathrm{End}(V_\Delta)$ the ideal of finite-rank endomorphisms of an infinite-dimensional right vector space over a division ring Δ, then any Jordan subalgebra Q with $A^{(+)} \subseteq Q \subseteq \mathrm{End}(V_\Delta)^{(+)}$ is a Martindale quotient of $(J, \mathcal{F})$.*

For a unital J, $\mathcal{F} = \{J\}$, and (Q, τ) a Martindale quotient of $(J, \mathcal{F})$, we have by (3.3) $q = U_{\tau(1)}q \in U_{\tau(J)}q \subseteq \tau(J)$, so $Q = \tau(J)$.

When $J = A^{(+)}$, as above, since J is an ideal of Q, we can use (3.2) as soon as we check that the lone ideal $I = J$ in $\mathcal{F}$ is sturdy in Q: if $q \in Q$ is nonzero, then there exists $v \in V$ such that $q(v) = w \neq 0$ and $U_q a(v) = qaq(v) = w$ for any finite rank transformation a with $a(w) = v$, so $U_q a \neq 0$ and $q \notin \mathrm{Zann}_Q(J)$.

3.5 Nondegenerate Examples. *Let J be a nondegenerate Jordan algebra and $\mathcal{F}$ be the set of all essential ideals of J (which is a denominator filter of J (2.1)), and let $\tau : J \longrightarrow Q$ be a Jordan algebra monomorphism. Then (Q, τ) is a Martindale quotient of $(J, \mathcal{F})$ iff Q is a Martindale-like cover of $\tau(J)$ in the sense of* [2, 2.1, 2.4] *(i.e. for any $0 \neq q \in Q$ there exists an essential I such that $\beta_{\tau(I)}(q) \subseteq \tau(J)$ and $U_{\tau(I)}q \neq 0$).*

Sufficiency is obvious and necessity follows from the following general observation (improving on (2.6)(i)).

3.6 Lemma. *If (Q, τ) is a Martindale quotient for a nondegenerate denominatored Jordan algebra $(J, \mathcal{F})$ then $U_{\tau(I)}q \neq 0$ for all $0 \neq q \in Q$ and $I \in \mathcal{F}$.*

Proof: Replacing J by $\tau(J)$, we may assume $J \subseteq Q$ and τ is the inclusion map. Since Q is tight over J (2.3), Q is also nondegenerate [11, 2.9(iii)]. Given $0 \neq q \in Q$ and $I \in \mathcal{F}$ there is $L \in \mathcal{F}$ satisfying $\beta_L(q) \subseteq J$, and by the definition of filter we may choose such an L with $L \subseteq I$. We claim that $U_L q \neq 0$ (hence $U_I q \neq 0$ too). Otherwise, $U_L q = 0$, so $U_L U_q L = 0$ by nondegeneracy and [1, 3.4], which implies $U_q L \subseteq \mathrm{Zann}_J(L)$ by (0.7)(2), but $\mathrm{Zann}_J(L) = 0$ by sturdiness of L, hence $U_q L = 0$. Also $U_L(L \circ q) \subseteq L \circ U_L q + \{L \circ L, q, L\} \subseteq L \circ U_L q + U_L q = 0$ by (0.2)(v) implies $L \circ q \subseteq \mathrm{Zann}_J(L) = 0$ by (0.7)(2) again. Thus $\beta_L(q) = 0$, which contradicts (2.6)(i). ■

3.7 Linear Examples. *For a denominatored linear Jordan algebra $(J, \mathcal{F})$ $(1/2 \in \Phi)$ and a Jordan algebra monomorphism $\tau : J \longrightarrow Q$, (Q, τ) is a Martindale quotient of $(J, \mathcal{F})$ iff it is an algebra of Martindale-like quotients of J with respect to $\mathcal{F}$ in the sense of* [3, 1.3].

Proof: As above, we may assume $J \subseteq Q$. We must show that the Martindale quotient condition $0 \neq \beta_I(q) \subseteq J$ and the Martindale-like quotient condition $0 \neq$

$I \circ q \subseteq J$ for a given $0 \neq q \in Q$ are equivalent. If Q is a Martindale quotient and $0 \neq \beta_I(q) \subseteq J$ we claim $q \circ I \subseteq J$ is nonzero, since otherwise $U_I q = 2U_I q \subseteq \big(I \circ (I \circ q) - I^2 \circ q\big)$ [by (0.2)(vi)] $= 0$, and any $I' \in \mathcal{F}$ such that $I' \subseteq I^3$ would have $\beta_{I'}(q) \subseteq U_I q + 2U_q I^3 + q \circ I \subseteq 0 + V_{q,I^3} q + 0 = 0$ [by (1.1)(i)(b) with $S = 0$], which contradicts (2.6)(i).

Conversely, if Q is a Jordan algebra of Martindale-like quotients of J with respect to $\mathcal{F}$, for a given $0 \neq q \in Q$, there are $K, L \in \mathcal{F}$ satisfying $0 \neq K \circ q \subseteq J$, $L \circ q^2 \subseteq J$, and we claim $0 \neq \beta_I(q) \subseteq J$ for any $I \subseteq L \cap K^3$ in $\mathcal{F}$: $0 \neq I \circ q$ [by [3, 1.5]] $\subseteq K \circ q \subseteq J$ and by (0.2)(vi) again both $U_I q \subseteq K \circ (K \circ q) - K^2 \circ q \subseteq J$ and $U_q I \subseteq (q \circ I) \circ q - q^2 \circ I \subseteq (q \circ U_K K) \circ q - q^2 \circ L \subseteq K \circ q - q^2 \circ L \subseteq J$ [since by (0.2)(v) $q \circ U_K K \subseteq \{K, K, q \circ K\} - U_K(q \circ K) \subseteq \{K, K, J\} - U_K J \subseteq K$]. ∎

3.8 Admonitory Example. We give an example to show that in characteristic 2 there can be unexpectedly large "quotients" involving weird quadratic forms. Consider a Jordan algebra $J = \Phi e \oplus M$ for $\Phi e \cong \Phi$ whose Peirce 1-space $J_1 = M$ relative to the idempotent e is a trivial bimodule, $M^2 = U_M M = 0$. Then $\mathcal{F} = \{J\}$ is a denominator filter, and J imbeds naturally as an ideal in a unital special Martindale quotient algebra

$$\mathcal{E} := J \oplus \mathcal{E}_0 = \Phi e \oplus M \oplus \mathcal{E}_0 = \begin{pmatrix} \Phi & 0 \\ M & \mathcal{E}_0 \end{pmatrix}^{(+)} \hookrightarrow \begin{pmatrix} \Phi & M^* \\ M & \mathcal{E}_0 \end{pmatrix}^{(+)} \cong \mathrm{End}_\Phi(J)^{(+)}$$

for $\mathcal{E}_0 := \mathrm{End}_\Phi(M)$ under

$$\begin{aligned} U_{\alpha e \oplus m \oplus T_0}(\beta e \oplus n \oplus S_0) &:= \alpha^2 \beta e \oplus \big(\alpha\beta m + \alpha T_0(n) + T_0 S_0(m)\big) \oplus T_0 S_0 T_0, \\ (\alpha e \oplus m \oplus T_0)^2 &:= \alpha^2 e \oplus \big(\alpha m + T_0(m)\big) \oplus T_0^2. \end{aligned}$$

But in the presence of 2-torsion there can be larger unnatural quotients. Denote by $\mathcal{RS}_\Phi(M) := \{\lambda \in \Phi \mid \lambda M = 0, \quad \lambda^2 = 2\lambda = 0\}$, the ideal in Φ of scalars in the radical of the squaring quadratic form $\lambda \to \lambda^2$ which kill M, and let $\mathcal{WQ}_\Phi(M) := \{\text{weird quadratic maps } \omega : M \to \mathcal{RS}_\Phi(M) \mid \omega(M, M) = 0\}$. For convenience we will assume our quotients contain J and the imbedding τ is inclusion.

3.9 Proposition. (1) *The Martindale quotients Q for $(J, \mathcal{F})$ as above ($J = \Phi e \oplus M, M = J_1(e), M^2 = U_M M = 0, \mathcal{F} = \{J\}$) are precisely all $Q = J \oplus Q_0$, where Q_0 is a Jordan algebra, with multiplication given by the Product Formula*

$$\begin{aligned} U_{x \oplus q_0}(y \oplus p_0) &= \big(U_x^J y + \omega_{p_0}(m) e\big) + \big(\alpha \nu_{q_0}(n) + \nu_{q_0}\nu_{p_0}(m)\big) \oplus U_{q_0}^{Q_0}(p_0), \\ (x \oplus q_0)^2 &= x^2 + \nu_{q_0}(m) \oplus q_0^2 \end{aligned}$$

for $x = \alpha e \oplus m$, $y = \beta e \oplus n \in J$, $q_0, p_0 \in Q_0$, where the Peirce 0-component Q_0 relative to e is a Jordan algebra with a linear specialization $\nu : Q_0 \to \mathrm{End}(M)$ and a linear map $\omega : Q_0 \to \mathcal{WQ}_\Phi(M)$ satisfying

(Axiom 1) $\nu_{U_{q_0}p_0} = \nu_{q_0}\nu_{p_0}\nu_{q_0}, \quad \nu_{q_0^2} = \nu_{q_0}\nu_{q_0},$

(Axiom 2) $\omega_{U_{q_0}p_0}(m) = \omega_{p_0}(\nu_{q_0}(m)), \quad \omega_{q_0^2} = 0$

(hence $\omega_{\{q_0,p_0,s_0\}}(m) = \omega_{p_0}(\nu_{q_0}(m), \nu_{s_0}(m)) = 0, \quad \omega_{q_0\circ p_0} = 0$),

(Axiom 3) $\omega_{q_0} = 0$ *and* $\nu_{q_0} = 0 \Longrightarrow q_0 = 0$.

The algebra Q *is unital iff* Q_0 *is unital,* $\nu_{1_{Q_0}} = Id_M$*, and* $\omega_{1_{Q_0}}(M) = 0$*. In this case* $1_Q = e_2 + e_0$ *with* $e_2 := e$ *and* $e_0 := 1_{Q_0}$*.*

(2) *There is a maximum algebra of quotients* $Q^{max} := J \oplus Q_0^{max}$ *for* $Q_0^{max} := \mathrm{End}(M)^{(+)} \oplus \mathcal{WQ}_\Phi(M)$ *under*

$$U^{max}_{T_0\oplus\tau}(S_0 \oplus \sigma) = T_0S_0T_0 \oplus \sigma T_0, \quad (T_0 \oplus \tau)^{(2,max)} = T_0^2 \oplus 0$$

with ν^{max}, ω^{max} *defined by*

$$\nu^{max}_{T_0\oplus\tau} := T_0, \quad \omega^{max}_{T_0\oplus\tau} := \tau$$

and having unit $e_0 := Id_M \oplus 0$*. Any quotient* Q *imbeds in this* Q^{max} *via* $\varphi = Id_J \oplus \nu \oplus \omega$ *:*

$$\varphi(x \oplus q_0) = x \oplus (\nu_{q_0} \oplus \omega_{q_0}).$$

(3) *In particular, if* $\Phi = \mathbb{Z}[\varepsilon]$ *for a 2-dual number* $\varepsilon\,[2\varepsilon = \varepsilon^2 = 0]$*, and* $M = \mathbb{Z}m\,(\varepsilon m = 0)$ *then* $\mathrm{End}_\Phi(M) = \mathbb{Z}e_0$*,* $e_0 = Id_M$*,* $\mathcal{RS}(\Phi) = \mathbb{Z}\varepsilon = \mathbb{Z}_2\varepsilon$*,* $\mathcal{WQ}_\Phi(M) = \mathbb{Z}\varepsilon\omega_0 = \mathbb{Z}_2\varepsilon\omega_0$ *for* $\omega_0(\alpha m) = \alpha^2$ *the natural quadratic form on* M*, and* $J = \Phi e_2 \oplus M$ *has maximum quotient* $Q^{max} = J \oplus Q_0^{max}$ *for* $Q_0^{max} = (\mathbb{Z}e_0)^{(+)} \oplus \mathbb{Z}\varepsilon\omega_0 = (\mathbb{Z}e_0)^{(+)} \oplus \mathbb{Z}_2\varepsilon\omega_0$*. In terms of direct sums of* $\mathbb{Z}$*-modules* $J = \mathbb{Z}e_2 \oplus (\mathbb{Z}\varepsilon)e_2 \oplus \mathbb{Z}m = \mathbb{Z}e_2 \oplus (\mathbb{Z}_2\varepsilon)e_2 \oplus \mathbb{Z}m$*,* $Q^{max} \cong \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \oplus \mathbb{Z}_2\varepsilon e_2 \oplus \mathbb{Z}_2\varepsilon\omega_0 e_0$ *and* $Q_0^{max} = \mathbb{Z}[\varepsilon\omega_0]e_0 = \mathbb{Z}e_0 \oplus (\mathbb{Z}\varepsilon\omega_0)e_0$*.*

Proof: (1) (Necessity) Assume Q is a Martindale quotient of $(J, \mathcal{F})$. It can be readily seen that Q results only from addition of a Peirce 0-component relative to $e = e_2$: if $q = q_0 + q_1 + q_2 \in Q$ (using subscripts to indicate the Peirce components) then $q_2 = U_e q \in U_J q \subseteq J$, $q_1 = V_e(q) - 2q_2 \in V_J q + J \subseteq J$. Set $\nu_{q_0} := V_{q_0}|_M$ and $\cap_{q_0}(m) = U_m q_0 = \omega_{q_0}(m)e_2$ and let E_i $(i = 0, 1, 2)$ denote the Peirce projections with respect to e_2. Then Peirce orthogonality and triviality of M shows that the product in Q is given by the above Product Formula (use (0.2)(i) and notice that $U_{M,M}Q_0 = E_2(U_{M,M}Q_0) = E_2((M \circ Q_0) \circ M) \subseteq E_2(M \circ M) = 0$). To see that ω_{q_0} maps to $\mathcal{RS}_\Phi(M)$, observe that the scalar $\lambda := \omega_{q_0}(m)$ satisfies $\lambda e_2 = U_m q_0$ by definition, hence $2\lambda e_2 = 2U_m q_0 = U_{m,m}q_0 = 0$; $\lambda^2 e_2 = (U_m q_0)^2 = U_m U_{q_0} m^2 = 0$ [by (0.2)(iv) and $M^2 = 0$], and $\lambda n = n \circ U_m q_0 = -U_m(n \circ q_0) = 0$, for all $n \in M$ [by (0.2)(v) and $M \circ M = U_M M = 0$]. To see that $\omega_{q_0} \in \mathcal{WQ}_\Phi(M)$, note that $U_{M,M}q_0 = 0$ implies $\omega_{q_0}(M, M) = 0$.

Then ν is the usual linear Peirce specialization of the Peirce 0-space on the Peirce 1-space, so Axiom 1 holds. For Axiom 2, $\omega_{q_0^2}(m)e_2 = U_m(q_0^2) = 0$ by (0.2)(iii), and then $\omega_{U_{q_0}p_0}(m)e_2 = U_{m\circ q_0}p_0 = \omega_{p_0}(V_{q_0}m)e_2$. Axiom 3 is the necessary and sufficient condition for Q to be a Martindale quotient: always $\beta_J(q) \subseteq J$, and $\beta_J(q) = 0 \iff q = q_0, V_{q_0}|_M = 0, \cap_{q_0}|_M = 0 \iff q = q_0, \nu_{q_0} = 0, \omega_{q_0} = 0$. Thus the Axioms are necessary.

If Q has a unit 1 (which means $U_1 = Id_Q$ and $U_q 1 = q^2$ for all $q \in Q$), it is readily seen that $1 = e_2 + e_0$ for some $e_0 \in Q_0$. A direct computation shows that $U_1 = Id_Q$ is equivalent to $U_{e_0} = E_0$, $U_{e_2,e_0} = E_1$, while $q^2 = U_q 1$ is equivalent to $q_0^2 = U_{q_0} e_0$, $\omega_{e_0} = 0$, and $V_{q_0,e_0}|_M = V_{q_0}|_M$. From this, the above criterion of unitality of Q readily follows, taking into account that always $U_{e_0,e_2} = V_{e_0} E_1$.

(Sufficiency) It is more tedious to prove that Axioms 1,2 and the Product Formula are sufficient to produce a quadratic Jordan algebra. For convenience we pass to the free unital hull $\widehat{Q_0} = \Phi e_0 \oplus Q_0$ (where Axiom 3 might not hold any longer) with linear specialization $\widehat{\nu}$ of $\widehat{Q_0}$ via $\widehat{\nu}_{\alpha e_0 + q_0} := \alpha Id_M + \nu_{q_0}$ satisfying obviously Axiom 1, and quadratic $\widehat{\omega}_{\alpha e_0 \oplus q_0} := \omega_{q_0}$ satisfying Axiom 2 since $\widehat{\omega}_{U_{\alpha e_0 + q_0}\beta e_0 + p_0} = \alpha^2 \omega_{p_0} + \omega_{U_{q_0} p_0} = \omega_{p_0} \cdot (\alpha Id_M + \nu_{q_0}) = \widehat{\omega}_{\beta e_0 + p_0} \widehat{\nu}_{\alpha e_0 + q_0}$ (using that ω_{q_0} maps to $\mathcal{RS}_\Phi(M)$) and $\widehat{\omega}_{(\alpha e_0 + q_0)^2} = \omega_{2\alpha q_0 + q_0^2} = 0$. We will assume from the start that Q_0 is unital (with unit e_0 satisfying $\nu_{e_0} = Id_M$, $\omega_{e_0}(M) = 0$), and verify the quadratic axioms (QJA1-3).

By definition of $\mathcal{WQ}_\Phi(M)$, the Product Formula, and Axioms 1,2 we have

$$
\begin{aligned}
(\star) \quad & V_M U_M E_0 = 2U_M E_0 = U_{M,M} E_0 = U_M V_{Q_0,Q_0} E_0 = \omega_{Q_0}(M) E_1 = 0, \\
(\star\star) \quad & V_{q_0,m_1} E_2 = V_{q_0 \circ m_1} E_2 = V_{q_0} V_{m_1} E_2, \quad V_{m_1,q_0} E_0 = V_{q_0 \circ m_1} E_0.
\end{aligned}
$$

Unitality (QJA1) $U_1 = Id_Q$ for $1 = e_2 + e_0$ follows from the Product Formula. To establish (QJA2-3) we must look carefully at the operators involving general elements $q := \alpha e_2 \oplus m_1 \oplus q_0$, $p := \beta e_2 \oplus n_1 \oplus p_0$: by the Product Formula $U_q = \alpha^2 U_{e_2} + U_{m_1} + U_{q_0} + \alpha U_{e_2,m_1} + \alpha U_{e_2,q_0} + U_{m_1,q_0} = \sum_{i,j} X_{ij} =: X$ for Peirce components $X_{ij} := E_i U_q E_j$, and similarly $U_p = \sum_{i,j} Y_{ij} =: Y, V_{q,p} = \sum_{i,j} S_{ij} =: S, V_{p,q} = \sum_{i,j} T_{ij} =: T$ where the Peirce components are given by

$$
\begin{aligned}
& X_{21} = X_{02} = X_{01} = 0, \quad X_{22} = \alpha^2 E_2, \quad X_{00} = U_{q_0} = U_{q_0} E_0, \quad X_{11} = \alpha V_{q_0} E_1, \\
& X_{20} = U_{m_1} E_0, \quad X_{12} = \alpha V_{m_1} E_2, \quad X_{10} = V_{q_0} V_{m_1} E_0, \\
& Y_{21} = Y_{02} = Y_{01} = 0, \quad Y_{22} = \beta^2 E_2, \quad Y_{00} = U_{p_0} = U_{p_0} E_0, \quad Y_{11} = \beta V_{p_0} E_1, \\
& Y_{20} = U_{n_1} E_0, \quad Y_{12} = \beta V_{n_1} E_2, \quad Y_{10} = V_{p_0} V_{n_1} E_0, \\
& S_{01} = S_{21} = S_{02} = S_{20} = 0, \quad S_{22} = 2\alpha\beta E_2, \quad S_{11} = (\alpha\beta + V_{q_0} V_{p_0}) E_1, \\
& S_{00} = V_{q_0,p_0} E_0, \quad S_{10} = (\alpha V_{n_1} + V_{p_0 \circ m_1}) E_0, \quad S_{12} = (\beta V_{m_1} + V_{q_0} V_{n_1}) E_2, \\
& T_{01} = T_{21} = T_{02} = T_{20} = 0, \quad T_{22} = 2\beta\alpha E_2, \quad T_{11} = (\beta\alpha + V_{p_0} V_{q_0}) E_1, \\
& T_{00} = V_{p_0,q_0} E_0, \quad T_{10} = (\beta V_{m_1} + V_{q_0 \circ n_1}) E_0, \quad T_{12} = (\alpha V_{n_1} + V_{p_0} V_{m_1}) E_2
\end{aligned}
$$

[using $(\star)$ $(\star\star)$ for S_{ij}, T_{ij}]. To establish (QJA2), $SX - XT = 0$, we check that $E_i(SX - XT)E_j = 0$ directly, using $(\star)$ $(\star\star)$, for all (i,j) except $(i,j) = (0,0)$, where we use (QJA2) for Q_0, and $(1,0)$, where we use $V_{q_0}V_{m_1}V_{p_0,q_0}s_0 = V_{q_0}V_{\{p_0,q_0,s_0\}}m_1 = V_{q_0}(V_{p_0}V_{q_0}V_{s_0}+V_{s_0}V_{q_0}V_{p_0})m_1$ [linearizing Axiom 1] $= \big(V_{q_0}V_{p_0}V_{q_0}V_{m_1}+V_{p_0\circ m_1}U_{q_0}\big)s_0$ and $V_{q_0}V_{q_0\circ n_1}s_0 = V_{q_0}V_{s_0}V_{q_0}n_1 = V_{U_{q_0}s_0}n_1$ [by Axiom 1] $= V_{n_1}U_{q_0}s_0$. This completes the proof of (QJA2).

Finally, for the Fundamental Formula (QJA3), $U_{U_qp} = U_qU_pU_q = XYX$, we have by the Product Formula

$$s := U_qp = \gamma e_2 + r_1 + s_0 \quad \mathit{for} \quad \gamma := \alpha^2\beta + \omega_{p_0}(m_1),\ \gamma^2 = \alpha^4\beta^2,$$
$$s_0 := U_{q_0}p_0, \quad r_1 := \alpha\beta m_1 + \alpha q_0\circ n_1 + V_{q_0}V_{p_0}m_1$$

so $U_s = \sum_{i,j} Z_{ij} =: Z$ has Peirce components

$$\begin{aligned}
&Z_{21} = Z_{02} = Z_{01} = 0, \quad Z_{22} = \alpha^4\beta^2E_2,\\
&Z_{00} = U_{s_0} = U_{q_0}U_{p_0}U_{q_0}, \quad Z_{11} = \gamma V_{s_0}E_1 = \alpha^2\beta V_{q_0}V_{p_0}V_{q_0}E_1,\ \text{[by (QJA3), Ax 1,}\star]\\
&Z_{12} = \gamma V_{r_1}E_2 = \alpha^3\beta^2V_{m_1}E_2+\alpha^3\beta V_{q_0}V_{n_1}E_2+\alpha^2\beta V_{q_0}V_{p_0}V_{m_1}E_2, \qquad [\text{by } \star, \star\star]\\
&Z_{20} = U_{r_1}E_0 = \alpha^2\beta^2U_{m_1}E_0+\alpha^2U_{n_1}U_{q_0}E_0+U_{m_1}U_{p_0}U_{q_0}E_0, \qquad [\text{by } \star,\ \text{Ax } 2]\\
&Z_{10} = V_{s_0}V_{r_1}E_0 = V_{q_0}V_{p_0}V_{q_0}\big(\alpha\beta V_{m_1} + \alpha V_{q_0\circ n_1} + V_{V_{q_0}V_{p_0}m_1}\big)E_0. \qquad [\text{by Ax } 1]
\end{aligned}$$

By direct calculation the Z_{ij} are the same as the components E_iXYXE_j of $U_qU_pU_q$: $E_2XYXE_1 = E_0XYXE_2 = E_0XYXE_1 = 0,\ E_2XYXE_2 = Z_{22},\ E_0XYXE_0 = Z_{00},\ E_1XYXE_1 = Z_{11},\ E_1XYXE_2 = Z_{12},\ E_2XYXE_0 = Z_{20},\ E_1XYXE_0 = Z_{00}$ [using Ax 1, $\star$]. Thus (QJA3) holds.

Axioms 1-2 hold strictly (on all scalar extensions), so the identities (QJA1-3) hold strictly. Once we have proved that Q is a Jordan algebra we return to the general case where Q_0 might not be unital, but Axiom 3 holds. From this axiom and the Product Formula it is readily checked that (Q,τ) is a Martindale quotient of $(J,\mathcal{F})$, which finishes the verification (1).

(2) It is easy to verify that Q_0^{max} is a Jordan algebra which satisfies the 3 axioms: (QJA1-3) are straightforward; since any product $\{x,y,z\}$ in Q_0^{max} equals the corresponding product of the first components of x,y,z in $\mathrm{End}(M)^{(+)}$, the linearizations of (QJA1-3) also hold. For $q_0 = T_0\oplus\tau$, $p_0 = S_0\oplus\sigma$ we have Axiom 1 since $\nu_{U_{q_0}p_0} = \nu_{T_0S_0T_0\oplus\sigma\circ T_0} = T_0S_0T_0 = \nu_{q_0}\nu_{p_0}\nu_{q_0},\ \nu_{q_0^2} = \nu_{T_0^2\oplus 0} = T_0T_0 = \nu_{q_0}\nu_{q_0}$, while Axiom 2 holds because $\omega_{U_{q_0}p_0} = \omega_{T_0S_0T_0\oplus\sigma T_0} = \sigma T_0 = \omega_{p_0}\nu_{q_0}$ and $\omega_{q_0^2} = \omega_{T_0^2\oplus 0} = 0$, and Axiom 3 follows from $\omega_{q_0} = 0 \Rightarrow \tau = 0$ and $\nu_{q_0} = 0 \Rightarrow T_0 = 0$. Q_0^{max} has unit $e_0 = Id_M\oplus 0$ and $\nu_{e_0} = Id_M$ and $\omega_{e_0} = 0$ as in the unitality criterion, hence Q^{max} is unital and a Martindale quotient of $(J,\mathcal{F})$ with multiplication given by the Product Formula by (1).

For the universal imbedding property of Q in Q^{max}, the linear map φ is injective since $\mathrm{Ker}(\varphi) = \{q_0\in Q_0 \mid \nu_{q_0} = 0,\ \omega_{q_0} = 0\}$ vanishes by Axiom 3, and it is a homomorphism of Jordan algebras since both have Product Formulas, maps ν,ω, and algebras Q_0, Q_0^{max} which correspond under φ:

$$\begin{aligned}
\nu_{q_0} &= \nu^{max}_{\nu_{q_0}\oplus\omega_{q_0}} = \nu^{max}_{\varphi(q_0)}, \qquad \omega_{q_0} = \omega^{max}_{\nu_{q_0}\oplus\omega_{q_0}} = \omega^{max}_{\varphi(q_0)},\\
\varphi(U_{q_0}p_0) &= \nu_{U_{q_0}p_0} \oplus \omega_{U_{q_0}p_0} = \nu_{q_0}\nu_{p_0}\nu_{q_0} \oplus \omega_{p_0}\nu_{q_0} \qquad \text{[by Ax 1,2]}\\
&= U^{max}_{\nu_{q_0}\oplus\omega_{q_0}}(\nu_{p_0}\oplus\omega_{p_0}) = U^{max}_{\varphi(q_0)}(\varphi(p_0)),\\
\varphi(q_0^2) &= \nu_{q_0^2} \oplus \omega_{q_0^2} = \nu_{q_0}\nu_{q_0} \oplus 0\text{[by Ax 1,2]} = (\nu_{q_0}\oplus\omega_{q_0})^{(2,max)} = \varphi(q_0)^{(2,max)}.
\end{aligned}$$

This establishes that Q^{max} is a maximum quotient.

(3) This follows immediately from (2) since $\lambda = \alpha + \beta\varepsilon \in \mathbb{Z}[\varepsilon]$ has $\lambda M = 0$ and $\lambda^2 = 2\lambda = 0$ if and only if $\alpha = 0$, and $\omega(m) = \beta\varepsilon \Rightarrow \omega(\alpha m) = \alpha^2\beta\varepsilon = \beta\varepsilon\omega_0(\alpha m) \Rightarrow \omega = \beta\varepsilon\omega_0$. ■

When imbedded in Q^{max}, the quotient Q need not split into a direct sum of three components, but it does contain an ideal $W_0 := \{q_0 \in Q_0 \mid \nu_{q_0} = 0\} = Ker(\nu)$ of Q_0 (by Axiom 1) and an ample outer ideal $\mathcal{E}_0 := \{q_0 \in Q_0 \mid \omega_{q_0} = 0\}$ of Q_0 with $\mathcal{E}_0 \oplus W_0 \subseteq Q_0$ which is a direct sum of subspaces (though not of algebras) by Axiom 3. [For outerness, $\mathcal{E}_0$ is invariant under all U_{p_0} and all V_{p_0} since $\omega_{U_{p_0}q_0} = \omega_{q_0}\nu_{p_0} = 0$ and $\omega_{p_0\circ q_0} = 0$ by Axiom 2, and for ampleness, we use again Axiom 2 together with $\omega_{Q_0}(M) \subseteq \mathcal{RS}\Phi(M)$].

4. Building Martindale Quotients out of Extensions

It is important that once we can boost an element q into J, we can boost it into any ideal K of J we wish.

4.1 Lemma. *Let J be a subalgebra of a Jordan algebra Q, and $q \in Q$ an element boosted into J by an ideal I of J. Then for any other ideal K of J, the cube of the ideal $I' := (I \cap K)^3$ boosts q into K: if $\beta_I(q) \subseteq J$ then*

(i) $\beta_{I'^3}(q) \subseteq U_{I'}q + U_q {I'}^3 + q \circ I' \subseteq K,$

(ii) $V_{I'^3,q} + V_{q,I'^3} \subseteq V_{K,K}.$

Proof: To establish (i), $U_{I'}q + q \circ I' \subseteq K$ since for $k = U_a b \in I'$ $(a, b \in I \cap K)$, $k' \in I'$ we have $U_k q = U_a U_b U_a q$ [by (0.2)(iv)] $\in U_K U_K(U_I q) \subseteq U_K U_K J \subseteq K$ and $U_{k,k'}q, k \circ q \in V_{I',q}\widehat{J} \subseteq K$ since

$$\text{(iii)} \qquad V_{I',q} + V_{q,I'} \subseteq V_{I\cap K,J} + V_{J,I\cap K} \subseteq V_{K,J} + V_{J,K}$$

by (1.1)(i) for $I \cap K$ in place of I, and $S = J$. Finally, $U_q {I'}^3 \subseteq K$ since for $a, b \in I' \subseteq I^3 \cap K$ (0.2)(iii) implies $U_q U_a b = (U_{q\circ a} - U_a U_q - V_{q,a}V_{a,q} + V_{U_q a^2})b \in U_{q\circ I}K + U_K(U_q I) + V_{q,I'}V_{I',q}K + V_{U_q I}K \subseteq U_J K + U_K J + (V_{K,J} + V_{J,K})^2 K + V_J K$ [by (iii)] $\subseteq K$.

(ii) follows from (i) since $U_{I'}q+q\circ I' \subseteq K$ implies $V_{I'^3,q}+V_{q,I'^3} \subseteq V_{I',K}+V_{K,I'} \subseteq V_{K,K}$ by (1.1)(i) [I' in place of I, $S=K$]. ■

Note that if $I,K \in \mathcal{F}$ for a filter $\mathcal{F}$, then we can choose $L \in \mathcal{F}$ such that $L \subseteq I'^3 := ((I\cap K)^3)^3$, so that L boosts q into K.

4.2 THEOREM. *Let $f : J \longrightarrow Q$ be a homomorphism of Jordan algebras and $\mathcal{F}$ be a filter of ideals of J.*

(i) *The boostable elements*

$$Q(f,\mathcal{F}) := \{q \in Q \mid \exists I \in \mathcal{F} \text{ with } \beta_{f(I)}(q) \subseteq f(J)\}$$

of Q form a Jordan subalgebra of Q containing $f(J)$ (and the unit of Q if Q is unital).

(ii) *The filter annihilator*

$$\begin{aligned}\operatorname{Ann}(f,\mathcal{F}) :&= \{q \in Q(f,\mathcal{F}) \mid \exists I \in \mathcal{F} \text{ with } \beta_{f(I)}(q) = 0\}\\ &= \{q \in Q(f,\mathcal{F}) \mid \exists I \in \mathcal{F} \text{ with } q \in \operatorname{Zann}_Q(f(I))\}\end{aligned}$$

(the elements boosted to 0 *by some ideal in the filter) is an ideal of $Q(f,\mathcal{F})$.*

(iii) *If all the ideals $f(I)$, for $I\in\mathcal{F}$, are sturdy in $f(J)$ (for example, if $\mathcal{F}$ is a denominator filter of J and f is injective) then*

(a) $\operatorname{Ann}(f,\mathcal{F}) \cap f(J) = 0$,

(b) *if $q \in Q(f,\mathcal{F})$ then $\beta_{f(I)}(q) \subseteq \operatorname{Ann}(f,\mathcal{F})$, for some $I \in \mathcal{F}$, implies $q \in \operatorname{Ann}(f,\mathcal{F})$.*

If, in addition, f is injective, then it induces the algebra monomorphism

(c) $\tilde{f} : J \longrightarrow \tilde{Q}(f,\mathcal{F}) := Q(f,\mathcal{F})/\operatorname{Ann}(f,\mathcal{F})$

given by $\tilde{f}(x) = f(x) + \operatorname{Ann}(f,\mathcal{F})$. Moreover, $(J,\mathcal{F})$ is a denominatored algebra and $(\tilde{Q}(f,\mathcal{F}),\tilde{f})$ is a Martindale quotient of $(J,\mathcal{F})$.

PROOF: Notice that $\mathcal{F}' = \{f(I) \mid I \in \mathcal{F}\}$ is a filter of ideals of the subalgebra $f(J)$ of Q, so $Q(f,\mathcal{F}) = Q(\tau,\mathcal{F}')$, $\operatorname{Ann}(f,\mathcal{F}) = \operatorname{Ann}(\tau,\mathcal{F}')$, where $\tau : f(J) \longrightarrow Q$ is the inclusion map. We will henceforth assume that J is a subalgebra of Q and f is the inclusion (we are not assuming our new J is the same as the old since f need not be a monomorphism).

(i) Imbedding Q in a unital hull $\widehat{Q}$, it suffices to prove the unital version (this guarantees $Q(f,\mathcal{F})$ is closed under squares $q^2 = U_q1$ as soon as it is closed under U-products U_qq' - note that always $1 \in \widehat{Q}(f,\mathcal{F})$ since $\beta_I(1) \subseteq I \subseteq J$ for all $I \in \mathcal{F}$).

Throughout we fix $q_1, q_2 \in Q(f,\mathcal{F})$ boosted by the ideals $I_1, I_2 \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, we can find $I \in \mathcal{F}$ with $I \subseteq I_1 \cap I_2$ a common booster for q_1 and q_2. We also fix $L, M \in \mathcal{F}$ with $L \subseteq (I^3)^3$ and $M \subseteq (L^3)^3$.

Let $q = q_1$ or q_2. From (1.1)(i) for $S = J$, we obtain

$$\{I, I, q\} \subseteq J. \tag{1}$$

If, in addition, we use (4.1) for $K = I$ (so $I' = I^3$) or $K = L$ (so $I' = (I \cap L)^3 = L^3$), we obtain respectively,

$$U_{I^3}q + q \circ I^3 + U_q L \subseteq I, \quad V_{L,q} + V_{q,L} \subseteq V_{I,I} \tag{2}$$

or

$$U_{L^3}q + q \circ L^3 + U_q M \subseteq L, \quad V_{M,q} + V_{q,M} \subseteq V_{L,L}. \tag{3}$$

First, $\alpha q \in Q(f, \mathcal{F})$ for any $\alpha \in \Phi$ since $\beta_I(\alpha q) \subseteq \alpha U_I q + \alpha^2 \cap_I q + \alpha V_I q \subseteq J$. Next, $q_1 + q_2 \in Q(f, \mathcal{F})$ since $\beta_L(q_1 + q_2) \subseteq J$: $(U_L + V_L)(q_1 + q_2) \subseteq (U_I + V_I)q_1 + (U_I + V_I)q_2 \subseteq J$ and $\cap_L(q_1 + q_2) \subseteq \cap_L q_1 + \cap_L q_2 + \{q_1, L, q_2\} \subseteq J + J + V_{q_1,L}q_2 \subseteq J + V_{I,I}q_2$ (by (2)) $\subseteq J$ by (1). Finally $U_{q_1}q_2 \in Q(f, \mathcal{F})$ since $\beta_M(U_{q_1}q_2) \subseteq J$: $V_M(U_{q_1}q_2) \subseteq \{q_1 \circ M, q_2, q_1\} + U_{q_1}(M \circ q_2)$ (by (0.2)(v)) $\subseteq V_{L,q_2}q_1 + U_{q_1}L$ (by (3)) $\subseteq V_{I,I}q_1 + I$ (by (2)) $\subseteq J$ by (1); $U_{U_{q_1}q_2}M = U_{q_1}U_{q_2}U_{q_1}M$ (by (0.2)(iv)) $\subseteq U_{q_1}U_{q_2}L$ (by (3)) $\subseteq U_{q_1}I$ (by (2)) $\subseteq J$; and $U_L(U_{q_1}q_2) \subseteq U_{L \circ q_1}q_2 + U_{q_1}(U_L q_2) + V_{q_1,L}V_{L,q_1}q_2 + V_{U_{q_1}L^2}q_2$ (by (0.2)(iii)) $\subseteq U_I q_2 + U_{q_1}I + V_{I,I}V_{I,I}q_2 + V_I q_2$ (by (2)) $\subseteq J$ by (1).

(ii) Note that the two conditions for $q \in A := \mathrm{Ann}(f, \mathcal{F})$ are equivalent using (1.1)(ii): if $\beta_I(q) = 0$ then $q \in \mathrm{Zann}_Q(I^3)$, and $q \in \mathrm{Zann}_Q(I')$ for any $I' \in \mathcal{F}$ with $I' \subseteq I^3$. As in (i), we may assume that Q is unital. To see that A is an ideal, we again consider $q_1, q_2 \in A$, $p \in Q(f, \mathcal{F})$ which we may assume have a common booster $I \in \mathcal{F}$ such that $\beta_I(q_i) = 0$ $(i = 1, 2)$, $\beta_I(p) \subseteq J$, where again by (1.1)(i) with $S = 0$ or J,

$$\{I, I, q_i\} = 0, \qquad \text{for } i = 1, 2, \qquad \text{or} \qquad \{I, I, p\} \subseteq J, \tag{4}$$

and the relations (1–3) still hold for $q = q_1, q_2, p$ for $L, M \in \mathcal{F}$ as above. Also, by (1.1)(i) with $S = 0$,

$$V_{I^3,q_i} = V_{q_i,I^3} = 0 \qquad \text{for } i = 1, 2. \tag{5}$$

Clearly A is closed under scaling, and it is closed under sums since as in (i) above $\beta_{I^3}(q_1 + q_2) = 0$: $(U_I + V_I)(q_1 + q_2) \subseteq (U_I + V_I)q_1 + (U_I + V_I)q_2 = 0$, and $\cap_{I^3}(q_1 + q_2) \subseteq \cap_{I^3}q_1 + \cap_{I^3}q_2 + V_{q_1,I^3}q_2 = 0$ by (5).

By unitality, A will be an ideal as soon as all $U_p q_1$ and $U_{q_1}p$ lie in A. Put $q = q_1$. For $U_p q$, we have $\beta_M(U_p q) = 0$: $V_M(U_p q) = \{p \circ M, q, p\} + U_p(M \circ q)$ (by (0.2)(v)) $\subseteq V_{L,q}p + U_p(I \circ q)$ (by (3)) $= 0$ by (5); $\cap_L(U_p q) = U_p U_q U_p L$ (by (0.2)(iv)) $\subseteq U_p U_q I$ (by (2)) $= 0$; and $U_L(U_p q) \subseteq U_{L \circ p}q + U_p U_L q + V_{p,L}V_{L,p}q + V_{U_p L^2}q$ (by (0.2)(iii)) $\subseteq U_I q + U_p(U_I q) + V_{I,I}V_{I,I}q + V_I q$ (by (2)) $= 0$ by (4). For $U_q p$ we have $\beta_{I^3}(U_q p) = 0$: $V_{I^3}(U_q p) \subseteq \{I^3 \circ q, p, q\} + U_q(I^3 \circ p)$ (by (0.2)(v)) $\subseteq \{I \circ q, p, q\} + U_q I$ (by (2)) $= 0$; $\cap_I(U_q p) = U_q U_p U_q I$ (by (0.2)(iv)) $= 0$; and $U_{I^3}(U_q p) = U_{I^3 \circ q}p + U_q U_{I^3}p + V_{q,I^3}V_{I^3,q}p + V_{U_q(I^3)^2}p$ (by (0.2)(iii)) $\subseteq U_{I \circ q}p + U_q I + 0 + V_{U_q I}p$ (by (2), (5)) $= 0$.

(iii) Now assume that all $I \in \mathcal{F}$ are sturdy in J.

(a) If $x \in J \cap \mathrm{Ann}(f, \mathcal{F})$, then there exists $L \in \mathcal{F}$ such that $x \in \mathrm{Zann}_Q(L) \cap J \subseteq \mathrm{Zann}_J(L) = 0$ by sturdiness.

(b) Let $q \in Q(f, \mathcal{F})$, so that there exists $L \in \mathcal{F}$ such that $\beta_L(q) \subseteq J$ and suppose that there is $I \in \mathcal{F}$, such that $\beta_I(q) \subseteq \mathrm{Ann}(f, \mathcal{F})$. Let $K \in \mathcal{F}$ satisfy $K \subseteq I \cap L$. Hence, (a) yields $\beta_K(q) \subseteq \mathrm{Ann}(f, \mathcal{F}) \cap J = 0$, which implies $q \in \mathrm{Ann}(f, \mathcal{F})$.

The rest of (iii) is straightforward. ∎

5. Existence of Maximal Martindale Quotients

5.1 We let $\mathcal{M}q(J, \mathcal{F})$ denote the class of Martindale quotients of a denominatored algebra $(J, \mathcal{F})$. If (Q_1, τ_1), $(Q_2, \tau_2) \in \mathcal{M}q(J, \mathcal{F})$ we will say that (Q_1, τ_1) is *less than or equal to* (Q_2, τ_2), and write $(Q_1, \tau_1) \leq (Q_2, \tau_2)$, if there exists an algebra *homomorphism* $f : Q_1 \longrightarrow Q_2$ such that $\tau_2 = f\tau_1$. By (2.4), any such covering f is actually a *monomorphism*. We say that (Q_1, τ_1) is *isomorphic* to (Q_2, τ_2), and write $(Q_1, \tau_1) \cong (Q_2, \tau_2)$ if there exists an algebra isomorphism $f : Q_1 \longrightarrow Q_2$ such that $f\tau_1 = \tau_2$. $[Q, \tau]$ will denote the class of all Martindale quotients of $(J, \mathcal{F})$ isomorphic to (Q, τ). A Martindale quotient $(Q, \tau) \in \mathcal{M}q(J, \mathcal{F})$ will be said to be *maximal* if any other $(Q', \tau') \in \mathcal{M}q(J, \mathcal{F})$ bigger than or equal to (Q, τ) is necessarily isomorphic to it: $(Q, \tau) \leq (Q', \tau') \Longrightarrow (Q, \tau) \cong (Q', \tau')$.

A priori, the collection of isomorphism classes of quotients of $(J, \mathcal{F})$ form a *class*; we wish to show they can be fully represented by a *set*, indeed a partially ordered set.

5.2 Proposition. *Given a denominatored algebra $(J, \mathcal{F})$, there is a bound on the cardinalities of all Martindale quotients of $(J, \mathcal{F})$. Moreover, every Martindale quotient is isomorphic to an algebra based on a subset of the fixed set*

$$X(J, \mathcal{F}) = \uplus_{I \in \mathcal{F}} X_I, \quad X_I := \mathrm{Quad}_\Phi(I, J) \times \mathrm{Hom}_\Phi(I, J) \times \mathrm{Hom}_\Phi(I, J),$$

where $\uplus$ denotes the disjoint union, $\mathrm{Quad}_\Phi(I, J)$ *denotes the set of quadratic maps from I to J, and* $\mathrm{Hom}_\Phi(I, J)$ *denotes the set of Φ-linear maps from I to J.*

Proof: Given a Martindale quotient (Q, τ) of $(J, \mathcal{F})$ we can, as usual, assume that $J \subseteq Q$ and τ is inclusion. We define a set-theoretic map $\varphi : Q \longrightarrow X := X(J, \mathcal{F})$, by choosing for each $q \in Q$ an ideal $I_q \in \mathcal{F}$ such that $\beta_{I_q}(q) \subseteq J$, and then define $\varphi(q) := (\cap_q, U_q, V_q) \in X_{I_q} \subseteq X(J, \mathcal{F})$. We claim that φ is injective. Indeed, if $\varphi(q) = \varphi(q')$, then $I_q = I_{q'} =: I \in \mathcal{F}$ has

$$\cap_q = \cap_{q'}, V_q = V_{q'} \text{ on } I \quad \Longrightarrow \quad U_I(q - q') = V_I(q - q') = 0,$$

hence $V_{q-q', I^3} = 0$ by (1.1)(i) for $S = 0$. Thus, for any $x \in I^3$, $\cap_x(q - q') = U_{q-q'}x = U_q x + U_{q'}x - \{q, x, q'\} = 2U_q x - \{q, x, q'\}$ [since $U_q = U_{q'}$ on I] $= \{q, x, q\} - \{q, x, q'\} =$

$\{q, x, q-q'\} \in V_{q-q', I^3} Q = 0$, so $\cap_{I^3}(q-q') = 0$. Therefore $\beta_K(q-q') = 0$ for any $K \in \mathcal{F}$ such that $K \subseteq I^3$, hence by (2.6)(i) $q - q' = 0$ and $q = q'$. By set-theoretic transfer, the bijection $Q \longrightarrow \varphi(Q) \subseteq X$ becomes an isomorphism of Q with an algebra Q' based on a subset of X. ■

5.3 Let $\mathcal{M}q(J,\mathcal{F})_X$ denote the **set** of all Martindale quotients (Q,τ) based on subsets of $X = X(J,\mathcal{F})$, and $\mathrm{Iso}_X(J,\mathcal{F})$ be the quotient of $\mathcal{M}q(J,\mathcal{F})_X$ by the restriction of the isomorphism relation, i.e., the set of isomorphism classes $[Q,\tau]_X$ of the quotients $(Q,\tau) \in \mathcal{M}q(J,\mathcal{F})_X$. (The X-class $[Q,\tau]_X$ is not all of $[Q,\tau]$, it contains only those isomorphic algebras based on subsets of X.) By (5.2) every Martindale quotient (Q',τ') is isomorphic to some $(Q,\tau) \in \mathcal{M}q(J,\mathcal{F})_X$, and we can define the X-*representative* $[Q',\tau']_X := [Q,\tau]_X \in \mathrm{Iso}_X(J,\mathcal{F})$ which is independent of our particular choice of the isomorphic Q. By abuse of language we say the set $\mathrm{Iso}_X(J,\mathcal{F})$ *represents* all isomorphism classes of Martindale quotients. Notice that, for $(Q,\tau), (Q',\tau') \in \mathcal{M}q(J,\mathcal{F})$,

$$(Q,\tau) \cong (Q',\tau') \Longleftrightarrow [Q,\tau] = [Q',\tau'] \Longleftrightarrow [Q,\tau]_X = [Q',\tau']_X.$$

5.4 PROPOSITION. *For any denominatored algebra $(J,\mathcal{F})$, we can define a binary relation $\leq$ on $\mathrm{Iso}_X(J,\mathcal{F})$ given by $[Q_1,\tau_1]_X \leq [Q_2,\tau_2]_X$ if $(Q_1,\tau_1) \leq (Q_2,\tau_2)$ so that $(\mathrm{Iso}_X(J,\mathcal{F}),\leq)$ is a nonempty inductive partially ordered set and hence contains maximal elements. Therefore $\mathcal{M}q(J,\mathcal{F})$ contains maximal elements, and any element (Q,τ) of $\mathcal{M}q(J,\mathcal{F})$ is less than or equal to a maximal element of $\mathcal{M}q(J,\mathcal{F})$.*

PROOF: $\mathrm{Iso}_X(J,\mathcal{F})$ is nonempty because $(J, Id_J) \in \mathcal{M}q(J,\mathcal{F})$ by (2.5). The relation $\leq$ is well-defined on isomorphism classes since it doesn't depend on class representatives: $f : Q_1 \longrightarrow Q_2$ covering τ_1,τ_2 induces $f' : Q_1' \longrightarrow Q_2'$ covering τ_1',τ_2' for any quotients $(Q_i',\tau_i') \cong (Q_i,\tau_i)$. Moreover we have

$$(1) \qquad (Q_1,\tau_1) \leq (Q_2,\tau_2) \text{ in } \mathcal{M}q(J,\mathcal{F}) \iff [Q_1,\tau_1]_X \leq [Q_2,\tau_2]_X \text{ in } \mathrm{Iso}_X(J,\mathcal{F}).$$

To see $\leq$ is a partial ordering, it is clearly reflexive and transitive, and it is antisymmetric on classes (not on individual algebras) since if $[Q,\tau]_X \leq [Q',\tau']_X \leq [Q,\tau]_X$ then there exist algebra homomorphisms $f' : Q \longrightarrow Q'$ and $f : Q' \longrightarrow Q$ such that $\tau' = f'\tau$ and $\tau = f\tau'$. Hence, as in the proof of (2.9), $f'f\tau' = \tau' = Id_{Q'}\tau'$ and $ff'\tau = \tau = Id_Q\tau$ imply $f'f = Id_{Q'}$ and $ff' = Id_Q$ by uniqueness (2.7). This shows that $(Q,\tau) \cong (Q',\tau')$, i.e., $[Q,\tau]_X = [Q',\tau']_X$ by (5.3).

To check inductiveness, let $\{[Q_\iota,\tau_\iota]_X \mid \iota \in S\}$ be a chain in $(\mathrm{Iso}_X(J,\mathcal{F})_X,\leq)$: for any $\iota,\kappa \in S$, either $[Q_\iota,\tau_\iota]_X \leq [Q_\kappa,\tau_\kappa]_X$ or $[Q_\kappa,\tau_\kappa]_X \leq [Q_\iota,\tau_\iota]_X$. When $[Q_\iota,\tau_\iota]_X \leq [Q_\kappa,\tau_\kappa]_X$, let $f_{\iota\kappa} : Q_\iota \longrightarrow Q_\kappa$ denote the unique (2.7) algebra monomorphism satisfying $\tau_\kappa = f_{\iota\kappa}\tau_\iota$, so by uniqueness $f_{\iota\lambda} = f_{\kappa\lambda} \circ f_{\iota\kappa}$. The direct limit of $(Q_\iota, f_{\iota\kappa})$ will play the role of the "union" of the sets Q_ι's to get a suitable upper bound. Indeed, $Q = \varinjlim Q_\iota$ is a Jordan algebra, we have monomorphisms

$f_\iota : Q_\iota \longrightarrow Q$ synthesizing the $f_{\kappa\iota} : Q_\iota \longrightarrow Q_\kappa$, and they induce an algebra monomorphism $\tau : J \longrightarrow Q$ given by $\tau = f_\iota \tau_\iota$, for all $\iota \in S$ (recall that Q can be built as $Q = \uplus_{\iota\in S} Q_\iota/\mathcal{R}$, where $\mathcal{R}$ denotes the binary relation in the disjoint union $\uplus_{\iota\in S} Q_\iota$ given by $x\mathcal{R}f_{\iota\kappa}(x)$ when $x \in Q_\iota$ and $(Q_\iota, \tau_\iota) \le (Q_\kappa, \tau_\kappa)$, which can be readily seen to be an equivalence relation). It is easy to see that (Q, τ) is a Martindale quotient of $(J, \mathcal{F})$, and the very definition of τ shows that $[Q, \tau]_X$ is an upper bound for the chain $\{[Q_\iota, \tau_\iota]_X\}_{\iota\in S}$ in $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$. This guarantees the existence of maximal elements in $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$ by Zorn's Lemma, and, indeed, the fact that any element of $\mathrm{Iso}_X(J, \mathcal{F})$ is less than or equal to a maximal element. The assertions on maximal elements of $\mathcal{M}q(J, \mathcal{F})$ follow from (1) and its elementary consequence for any $(Q, \tau) \in \mathcal{M}q(J, \mathcal{F})$:

(2) $\quad (Q, \tau)$ is a maximal quotient $\iff [Q, \tau]_X$ is maximal in $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$. ■

5.5 REMARK: Notice that for a Martindale quotient (Q, τ) of $(J, \mathcal{F})$, being maximum in the sense of (2.8) is just $[Q, \tau]_X$ being the maximum of $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$. Note that the above direct limit Q need not be based on X even though the Q_ι are, but, by (5.3), $[Q, \tau]_X = [Q', \tau']_X$ for an isomorphic quotient $Q' \in \mathcal{M}q(J, \mathcal{F})_X$.

The construction of the previous section can be used to characterize the existence of maximum Martindale quotients.

5.6 THEOREM. *Let $(J, \mathcal{F})$ be a denominatored algebra. The following assertions are equivalent:*

(i) $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$ *has a maximum.*

(ii) *There exist a Jordan algebra Q and an algebra monomorphism $f : J \longrightarrow Q$ such that for any $(Q_1, \tau_1) \in \mathcal{M}q(J, \mathcal{F})$, there exists an algebra homomorphism $f_1 : Q_1 \longrightarrow Q$, such that $f_1\tau_1 = f$.*

(iii) *For any $(Q_1, \tau_1), (Q_2, \tau_2) \in \mathcal{M}q(J, \mathcal{F})$, there exist a Jordan algebra Q, an algebra monomorphism $f : J \longrightarrow Q$, and algebra homomorphisms $f_i : Q_i \longrightarrow Q$, such that $f_i\tau_i = f$ for $i = 1, 2$.*

(iv) $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$ *is directed: for any $(Q_1, \tau_1), (Q_2, \tau_2) \in \mathcal{M}q(J, \mathcal{F})$, there exists $(Q, \tau) \in \mathcal{M}q(J, \mathcal{F})$, such that $(Q_i, \tau_i) \le (Q, \tau)$, for $i = 1, 2$, equivalently, there exists $[Q, \tau]_X \in \mathrm{Iso}_X(J, \mathcal{F})$ such that $[Q_i, \tau_i]_X \le [Q, \tau]_X$.*

PROOF: (i)$\Longrightarrow$(ii): If $(\mathrm{Iso}_X(J, \mathcal{F}), \le)$ has a maximum $[Q, \tau]_X$, one just needs to take $f = \tau$ and the existence of f_1 in (ii) will follow from equivalence (5.4)(1).

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(iv): We can consider $\widetilde{Q}(f, \mathcal{F})$ of (4.2)(iii), so that f induces $\tau := \tilde{f} : J \longrightarrow \widetilde{Q}(f, \mathcal{F})$ and $(\widetilde{Q}(f, \mathcal{F}), \tau) \in \mathcal{M}q(J, \mathcal{F})$. Moreover, it can be readily seen that, for $i = 1, 2$, $f_i(Q_i) \subseteq Q(f, \mathcal{F})$, so that f_i induces an algebra homomorphism $g_i : Q_i \longrightarrow \widetilde{Q}(f, \mathcal{F})$ such that $g_i\tau_i = \tau$, which shows $(Q_i, \tau_i) \le (\widetilde{Q}(f, \mathcal{F}), \tau)$.

(iv)$\Longrightarrow$(i): Being directed implies that there is at most one maximal element. This, together with (5.4), yields the existence of a maximum. ∎

It is known [14] that each nondegenerate Jordan algebra has a maximum Martindale quotient algebra, but it is not known if this holds in general. However, it always has a minimum quotient (itself), and any two quotients have an infimum.

5.7 To define the infimum of two Martindale quotients (Q_1,τ_1), (Q_2,τ_2) of a denominatored algebra $(J,\mathcal{F})$, we begin with their direct sum $Q_1 \boxplus Q_2$ with $f : J \longrightarrow Q_1 \boxplus Q_2$ defined by $f(x) = (\tau_1(x), \tau_2(x))$ for any $x \in J$. The map f is clearly an algebra monomorphism, so that we can take the Martindale quotient $(\widetilde{Q}(f,\mathcal{F}), \tilde{f})$ as in (4.2)(iii) (for $Q = Q_1 \boxplus Q_2$). Moreover, (2.6)(i) implies that $\mathrm{Ann}(f,\mathcal{F}) = 0$, so that by (4.2) $(\widetilde{Q}(f,\mathcal{F}), \tilde{f}) = (Q(f,\mathcal{F}), \tilde{f})$ and we obtain a Martindale quotient

(1) $(Q_1 \wedge Q_2, \tau_1 \wedge \tau_2) := (Q(f,\mathcal{F}), \tilde{f})$ ($\tilde{f}$ the co-restriction of f),

where $Q_1 \wedge Q_2$ consists of all $(q_1, q_2) \in Q_1 \boxplus Q_2$ satisfying

(2) there exists $I \in \mathcal{F}$ such that for any $a \in I$ there are $x = x(a), y = y(a), z = z(a) \in J$ with

$$U_{q_i}\tau_i(a) = \tau_i(x), \quad U_{\tau_i(a)}q_i = \tau_i(y), \quad q_i \circ \tau_i(a) = \tau_i(z) \quad \text{for } i = 1,2.$$

5.8 Theorem. *Martindale quotients $\mathcal{M}q(J,\mathcal{F})$ are directed downwards: $(Q_1 \wedge Q_2, \tau_1 \wedge \tau_2)$ is the infimum of $(Q_1,\tau_1), (Q_2,\tau_2)$ in $\mathcal{M}q(J,\mathcal{F})$, equivalently, $[Q_1 \wedge Q_2, \tau_1 \wedge \tau_2]_X$ is the infimum of $[Q_1,\tau_1]_X, [Q_2,\tau_2]_X$ in the poset* $(\mathrm{Iso}_X(J,\mathcal{F}), \leq)$.

Proof: Clearly, the restriction of the natural projection $\pi_i : Q(f,\mathcal{F}) \longrightarrow Q_i$ ($\pi_i((q_1,q_2)) = q_i$) is an algebra homomorphism such that $\pi_i\tilde{f} = \tau_i$, for $i = 1,2$, hence $(Q(f,\mathcal{F}), \tilde{f}) \leq (Q_i,\tau_i)$ for $i = 1,2$. If another $(Q,\tau) \in \mathcal{M}q(J,\mathcal{F})$ satisfies $(Q,\tau) \leq (Q_i,\tau_i)$ for $i = 1,2$, then there exist algebra homomorphisms $g_i : Q \longrightarrow Q_i$ such that $g_i\tau = \tau_i$, $i = 1,2$. Again we can define the algebra homomorphism $g : Q \longrightarrow Q_1 \boxplus Q_2$ given by $g(q) = (g_1(q), g_2(q))$, and it can be readily seen from the construction that $g(Q) \subseteq Q(f,\mathcal{F})$: $g(q)$ satisfies (5.7)(2) for any $I \in \mathcal{F}$ with $I \subseteq I_1 \cap I_2$, where $I_i \in \mathcal{F}$ $(i = 1,2)$ satisfies $\beta_{\tau_i(I_i)}g_i(q) \subseteq \tau_i(J)$; this is due to the fact that $g_i\tau = \tau_i$, and injectivity of g_i (2.4) and τ. Thus we can restrict g in the image to the algebra homomorphism $\tilde{g} : Q \longrightarrow Q(f,\mathcal{F})$ clearly satisfying $\tilde{g}\tau = \tilde{f}$, which proves $(Q,\tau) \leq (Q(f,\mathcal{F}), \tilde{f})$. ∎

The infimum of $(Q_1,\tau_1) \wedge (Q_2,\tau_2) := (Q_1 \wedge Q_2, \tau_1 \wedge \tau_2) \in \mathcal{M}q(J,\mathcal{F})$ found in (5.8), together with its explicit construction (5.7), and (2.7), gives us another way to describe the order relation $\leq$.

5.9 Corollary. *Given $(Q_1,\tau_1), (Q_2,\tau_2) \in \mathcal{M}q(J,\mathcal{F})$, the following are equivalent:*

(i) $(Q_1,\tau_1) \leq (Q_2,\tau_2)$,

(ii) $(Q_1,\tau_1) \cong (Q_1,\tau_1) \wedge (Q_2,\tau_2)$,

(iii) *for any* $q_1 \in Q_1$*, there exists* $q_2 \in Q_2$ *such that* (q_1, q_2) *satisfies* (5.7)(2).

Proof: (i) $\Longleftrightarrow$ (ii): Apply (5.3), (5.4)(1) and (5.8).

(i) $\Longrightarrow$ (iii): If $(Q_1, \tau_1) \le (Q_2, \tau_2)$, (Q_1, τ_1) can play the role of (Q, τ) in the proof of (5.8). Hence $g_1 = Id_{Q_1}$ by (2.7), and, for any $q_1 \in Q_1$, we can take $q_2 := g_2(q_1)$.

(iii) $\Longrightarrow$ (ii): As shown in the proof of (5.8), the restriction of the projection $\pi_1 : Q_1 \wedge Q_2 \longrightarrow Q_1$ is an algebra homomorphism satisfying $\pi_1(\tau_1 \wedge \tau_2) = \tau_1$, hence it is injective by (2.4). Given any $q_1 \in Q_1$, the element $q_2 \in Q_2$ as in (iii) satisfies $(q_1, q_2) \in Q_1 \wedge Q_2$, and, obviously, $\pi_1((q_1, q_2)) = q_1$, which shows that $\pi_1 : Q_1 \wedge Q_2 \longrightarrow Q_1$ is surjective. ∎

6. Unital Hulls and Martindale Quotients

6.1 Given a Jordan algebra J, a unital hull J_1 of J is usually understood to be any unital Jordan algebra such that J is a subalgebra of J_1 and $J_1 = \Phi 1 + J$ is generated (as an algebra, equivalently, as a Φ-module) by J and the unit element 1.

More generally, we define a *unital hull* of a Jordan algebra J to be a pair (J_1, μ_1), where J_1 is a unital algebra, and $\mu_1 : J \longrightarrow J_1$ is an algebra monomorphism such that J_1 is generated by $\mu_1(J)$ and 1_{J_1}. A unital hull (J_1, μ_1) of J will be said *tight over* J if every nonzero ideal of J_1 hits $\mu_1(J)$, i.e., $I \cap \mu_1(J) \neq 0$ for any nonzero ideal I of J_1.

6.2 Proposition. *Given a Jordan algebra* J *with* $\mathrm{Zann}_J(J) = 0$*, all tight unital hulls are isomorphic to* $(\check{J}, \check{\mu}) := (\pi(\widehat{J}), \pi\iota)$*, for* $\widehat{J}$ *the free unital hull of* J*,* $\iota : J \longrightarrow \widehat{J}$ *the natural inclusion, and* $\pi : \widehat{J} \longrightarrow \widehat{J}/\mathrm{Zann}_{\widehat{J}}(\iota(J))$ *the natural projection. If* (J_1, μ_1) *is a tight unital hull of* J*, then there is a unique isomorphism* $f : \check{J} \longrightarrow J_1$ *such that* $f\check{\mu} = \mu_1$.

Proof: Given a tight unital hull (J_1, μ_1) of J, the free unital hull $\widehat{J} = \Phi 1 \oplus J$ has a canonical unital algebra epimorphism $g : \widehat{J} \longrightarrow J_1$ such that $g\iota = \mu_1$. By (0.7) the hypothesis $\mathrm{Zann}_J(J) = 0$ guarantees that $\mathrm{Zann}_{\widehat{J}}(\iota(J))$ is the maximum ideal of $\widehat{J}$ not hitting $\iota(J)$. But $\mathrm{Ker}\, g$ is also an ideal of $\widehat{J}$ not hitting $\iota(J)$ [since $g\iota = \mu_1$ is a monomorphism], and is maximal with respect to this property. Indeed, if $I \supset \mathrm{Ker}\, g$ were a bigger ideal then $g(I)$ would be a nonzero ideal of J_1, so by tightness of J_1 over J we would have $\mu_1(J) \cap g(I) \neq 0$: there are nonzero $z \in I$, $a \in J$ with $0 \neq g(z) = \mu_1(a) = g(\iota(a))$. Then $z - \iota(a) \in \mathrm{Ker}\, g \subseteq I$ and $\iota(a) = z - (z - \iota(a)) \in I$, so $0 \neq \iota(a) \in I \cap \iota(J)$, and the bigger ideal I would hit $\iota(J)$. Thus we must have

$$\mathrm{Ker}\, g = \mathrm{Zann}_{\widehat{J}}(\iota(J))$$

and the isomorphism $f : \widehat{J}/\mathrm{Ker}\, g = \check{J} \longrightarrow J_1$ induced by g satisfies $f\check{\mu} = f\pi\iota = g\iota = \mu_1$. The uniqueness of f comes from the fact that f is determined on $\check{\mu}(J)$ and $\check{1}$. ∎

6.3 Remark: In the above proof, assume, to simplify notation, that $J \subseteq \widehat{J}$. Since $J \cap \mathrm{Zann}_{\widehat{J}}(J) = \mathrm{Zann}_J(J)$, $\mathrm{Zann}_{\widehat{J}}(J)$ avoids hitting J if and only if $\mathrm{Zann}_J(J) = 0$. At the opposite extreme, if $J^2 = U_J J = 0$ then $\mathrm{Zann}_{\widehat{J}}(J) = \Phi_0 1 + J \supseteq J$ for $\Phi_0 = \{\alpha \in \Phi \mid 2\alpha J = \alpha^2 J = 0\}$ which is an ideal of Φ. Any 2-dual number ε ($2\varepsilon = \varepsilon^2 = 0$) belongs to Φ_0.

Notice that the above elementary approach to tight unital hulls resembles formally that to Martindale quotients of the previous sections. This interaction goes further than mere formal similarity, as shown in the next result.

6.4 Proposition. *Let $(J, \mathcal{F})$ be a denominatored algebra, and (J_1, μ_1) be a tight unital hull of J. Then:*

(i) *(J_1, μ_1) is a Martindale quotient of $(J, \mathcal{F})$.*

(ii) *$\mu_1(\mathcal{F}) := \{\mu_1(I) \mid I \in \mathcal{F}\}$ is a denominator filter for J_1, and, for any Martindale quotient (Q, τ) of $(J_1, \mu_1(\mathcal{F}))$, $(Q, \tau\mu_1)$ is a Martindale quotient of $(J, \mathcal{F})$.*

(iii) *Given any Martindale quotient (Q, τ) of $(J, \mathcal{F})$, and tight unital hull (Q_1, ν_1) of Q, then (Q_1, τ_1) for $\tau_1 := \nu_1\tau$ is also a Martindale quotient of $(J, \mathcal{F})$ which can be built from a quotient of a tight unital hull of J as in* (ii). *In particular, maximal Martindale quotients of $(J, \mathcal{F})$ are always unital and can be viewed as Martindale quotients of tight unital hulls of J.*

Proof: We may assume, without loss of generality, that J is contained in J_1 and μ_1 is the inclusion map. It is clear that any $I \in \mathcal{F}$ is an ideal of J_1. Moreover, I is sturdy in J_1 since $\mathrm{Zann}_{J_1}(I) \cap J = \mathrm{Zann}_J(I) = 0$, which implies $\mathrm{Zann}_{J_1}(I) = 0$ by tightness of J_1 over J. This shows that $(J_1, \mathcal{F})$ is a denominatored algebra.

(i) This follows from the Sturdy Ideal Example (3.2).

(ii) The fact that any Martindale quotient (Q, τ) of $(J_1, \mathcal{F})$ is a Martindale quotient of $(J, \mathcal{F})$ is straightforward [by (4.1)(i) $\beta_{\tau(I)}(q) \subseteq \tau(J_1) \Longrightarrow \beta_{(\tau(I)^3)^3}(q) \subseteq \tau(I) \subseteq \tau(J)$].

(iii) Notice that $Q_1 = Q(\tau_1, \mathcal{F})$ as in (4.2), since the latter clearly contains $\nu_1(Q)$ and 1_{Q_1}. On the other hand $\mathrm{Ann}(\tau_1, \mathcal{F}) \cap \nu_1(Q) = \nu_1(\mathrm{Ann}(\tau, \mathcal{F})) = \nu_1(0) = 0$, hence $\mathrm{Ann}(\tau_1, \mathcal{F}) = 0$ by tightness, so that (Q_1, τ_1) is also a Martindale quotient of $(J, \mathcal{F})$ by (4.2)(iii). Let J_1 be the subalgebra of Q_1 generated by $\tau_1(J)$ and 1_{Q_1}, and let $\sigma : J_1 \longrightarrow Q_1$ be the inclusion map. Considering the correstriction $\tau_1 : J \longrightarrow J_1$, we claim that J_1 is tight over J. Indeed, an ideal L of J_1 not hitting $\tau_1(J)$ consists necessarily of elements $q \in J_1 \subseteq Q_1$ such that $\beta_{\tau_1(J)}(q) \subseteq L \cap \tau_1(J) = 0$, which are zero by (2.6)(i) since (Q_1, τ_1) is a Martindale quotient of $(J, \mathcal{F})$. Finally, it is obvious that (Q_1, σ) is a Martindale quotient of $(J_1, \mathcal{F}_1)$, where $\mathcal{F}_1 = \{\tau_1(I) \mid I \in \mathcal{F}\}$ is a denominator filter of J_1 by tightness of J_1 over J. ■

6.5 Remark: We can obtain an alternative proof of (3.3) by using (6.4): Let (Q_1, ν_1) be a tight unital hull of Q. By (6.4)(iii), (Q_1, τ_1) for $\tau_1 = \nu_1\tau$ is another Martindale quotient of $(J, \mathcal{F})$. But it is easy to show that $q = \tau_1(1_J) - 1_{Q_1}$ satisfies

$\beta_{\tau_1(J)}(q) \subseteq U_q\tau_1(J) + U_{\tau_1(J)}q + q \circ \tau_1(J) = 0$, hence $q = 0$ by (2.6)(i), i.e., $1_{Q_1} = \tau_1(1_J) \in \nu_1(Q)$. Thus $Q_1 = \nu_1(Q)$, ν_1 is an isomorphism, and Q is unital with unit $\nu_1^{-1}(1_{Q_1}) = \tau(1_J)$.

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REFERENCES

[1] J. A. Anquela, T. Cortés, E. García, M. Gómez-Lozano, "Polynomial Identities and Speciality of Martindale-like Covers of Jordan algebras", *J. Pure Appl. Algebra* **202** (2005) 1-10.

[2] J. A. Anquela, T. Cortés, E. García, M. Gómez-Lozano, "Jordan Centers and Martindale-like Covers", *J. Algebra* **305** (2006) 615-628.

[3] J. A. Anquela, E. García, M. Gómez-Lozano, "Maximal Algebras of Martindale-like Quotients of Strongly Prime Linear Jordan Algebras", *J. Algebra* **280** (2004) 367-383.

[4] E. García, M. Gómez-Lozano, "Jordan Systems of Martindale-like Quotients" *J. Pure Appl. Algebra* **194** (2004) 127-145.

[5] N. Jacobson, *Structure Theory of Jordan Algebras.* The University of Arkansas Lecture Notes in Mathematics, University of Arkansas, Fayetteville 1981.

[6] R. Lewand, K. McCrimmon, "Macdonald's Theorem for Quadratic Jordan Algebras", *Pacific J. Math.* **35** (1970) 681-707.

[7] O. Loos, *Jordan Pairs.* Lecture Notes in Math., Vol. 460, Springer-Verlag, Berlin, 1975.

[8] C. Martínez, "The Ring of Fractions of a Jordan Algebra", *J. Algebra* **237** (2001) 798-812.

[9] K. McCrimmon, "The Zelmanov Annihilator and Nilpotence of the Nil Radical in Quadratic Jordan Algebras with Chain Conditions", *J. Algebra* **67** (1980) 230-253.

[10] K. McCrimmon, "Strong Prime Inheritance in Jordan Systems", *Algebras, Groups, Geom.* **1** (1984) 217-234.

[11] K. McCrimmon, "A Characterization of the Nondegenerate Radical in Quadratic Jordan Triple Systems", *Algebras, Groups, Geom.* **4** (1987) 145-164.

[12] K. McCrimmon, E. Zelmanov, "The Structure of Strongly Prime Quadratic Jordan Algebras", *Adv. Math.* **69** (1988) 133-222.

[13] F. Montaner, “Local PI-Theory of Jordan Systems II”, *J. Algebra* **241** (2001) 473-514.

[14] F. Montaner, “Maximal Algebras of Quotients of Jordan Algebras” (preprint) Posted on the *Jordan Theory Preprint Archives*, http://homepage.uibk.ac.at/∼c70202/jordan/archive/maxquot/index.html

[15] M. Siles-Molina, “Algebras of Quotients of Lie Algebras”, *J. Pure Appl. Algebra* **188** (2004) 175-188.

[16] E. Zelmanov, “Prime Jordan Algebras II”, *Siberian Math. J.* **24** (1983) 73-85.