Jordan bimodules over the superalgebra $M_{1|1}$

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Abstract

Let F be a field of characteristic different of 2 and let $M_{1|1}(F)^{(+)}$ denote the Jordan superalgebra of 2×2 matrices over the field F. The aim of this paper is to classify irreducible (unital and one-sided) Jordan bimodules over the Jordan superalgebra $M_{1|1}(F)^{(+)}$.

Introduction

We will assume in the paper that all algebras are algebras over a field F, $charF \neq 2$.

The theory of bimodules over simple Jordan algebras, developed by N. Jacobson in [J], was extended to Jordan superalgebras in a series of papers (see [7,8,...]).

Let's remember that a superalgebra $J = J_{\bar{0}} + J_{\bar{1}}$ is a $\mathbf{Z}_2$-graded algebra. So $J_{\bar{0}}$ is a subalgebra of J ($J_{\bar{0}}J_{\bar{0}} \subseteq J_{\bar{0}}$), $J_{\bar{1}}$ is a module over $J_{\bar{0}}$ ($J_{\bar{0}}J_{\bar{1}}, J_{\bar{1}}J_{\bar{0}} \subseteq J_{\bar{1}}$) and $J_{\bar{1}}J_{\bar{1}} \subseteq J_{\bar{1}}$. Elements lying in $J_{\bar{0}} \cup J_{\bar{0}}$ are called homogeneous elements, even if they lie in $J_{\bar{0}}$ and odd if they lie in $J_{\bar{1}}$. The parity of a homogenous element a is zero if the element a is even and one if it is odd and is represented as $|a|$.

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A Jordan superalgebra is a superalgebra $J = J_{\bar{0}} + J_{\bar{1}}$ satisfying the following two homogeneous identities:

i) $xy = (-1)^{|x||y|}yx$,

ii) $(xy)(zu) + (-1)^{|y||z|}(xz)(yu) + (-1)^{|y||u|+|z||u|}(xu)(yz) = ((xy)z)u + (-1)^{|u||z|+|u||y|+|z||y|}((xu)z)y + (-1)^{|x||y|+|x||z|+|x||u|+|z||u|}(yu)z)x$, for arbitrary homogeneous elements x, y, z, u in J.

A Jordan superalgebra is called simple if it has no nontrivial graded ideals. For more information about (simple) Jordan superalgebras we refer the reader to [K], [MZ], [RZ].

If $A = A_{\bar{0}} + A_{\bar{1}}$ is an associative superalgebra, that is, an associative algebra that has a $\mathbf{Z}_2$ grading, then we can define a new operation $\circ$ given by: $a \circ b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ for arbitrary homogeneous elements $a, b \in A$. The superalgebra obtained in this way, with the same underlying vector space and the same gradding of A and with the new product *circ* is a Jordan superalgebra, that is denoted $A^{(+)}$

In particular, if we take $A = M_{1|1}(F)$, the superalgebra of 2×2 matrices over the field F, with even part the set of diagonal matrices and its even part equal to the set of off-diagonal matrices,

$$A_{\bar{0}} = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\}, \quad A_{\bar{1}} = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\}$$

the corresponding Jordan superalgebra $J = A^{(+)} = M_{1|1}(F)^{(+)}$ is a simple Jordan superalgebra.

Definition. If V is a $\mathbf{Z}_2$-graded vector space and there exist bilinear maps $V \times J \to V$, $J \times V \to V$, we say that V is a Jordan bimodule over the Jordan superalgebra J if the split null extension $V + J$ is a Jordan superalgebra, where the multiplication in the split null extension extends the one of J, $V \cdot V = (0)$ and the multiplication of elements of J and V is given by the bilinear maps . (see [MZ2]).

In the superalgebras setting, for each bimodule we can define the opposite module. Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a Jordan bimodule over a Jordan superalgebra J. Take copies $V_{\bar{1}}^{op}$ and $V_{\bar{0}}^{op}$ of $v_{\bar{1}}$ and $V_{\bar{0}}$ with different parity. Then $V^{op} = V_{\bar{1}}^{op} + V_{\bar{0}}^{op}$ becomes a Jordan J-bimodule defining the action of J on V^{op} by:

$$av^{op} = (-1)^{|a|}(av)^{op}, \quad v^{op}a = (va)^{op}$$

.

If J is a unital Jordan superalgebra and V is a bimodule such that the identity of J, 1, acts as the identity on V, then we say that J is a unital Jordan bimodule over J.

A one-sided Jordan bimodule over J is a bimodule V such that $\{J, V, J\} = (0)$, where $\{x, v, y\} = (xv)y + x(vy) - (-1)^{|x||v|}v(xy)$ represents the triple Jordan product in $J + V$ and $x, y \in J, v \in V$ are homogeneous elements.

Let's denote $U(x, y)$ the operator given by $vU(x, y) = \{x, v, y\}$ and $D(x, y) = R(x)R(y) - (-1)^{|x||y|}R(y)R(x)$.

It is well known that every Jordan bimodule decomposes as a direct sum of unital and one-sided Jordan bimodules.

The aim of this paper is to give the classification of one sided Jordan bimodules (already announced by the authors some time ago) and one-sided modules over the simple Jordan superalgebra $J = M_{1|1}(F)^{(+)}$.

1 Unital bimodules

In the section J will denote the Jordan superalgebra $J = M_{1|1}^{(+)}$. We will fix the canonical basis $\{e, f, x, y\}$, where $e = e_{11},\ f = e_{22},\ x = e_{12},\ y = e_{21}$. Then $J_{\bar{0}} = Fe + Ff,\ J_{\bar{1}} = Fx + Fy,\ ef = 0,\ e^2 = 2,\ f^2 = f,\ [x, y] = e - f$.

For arbitrary elements $\alpha, \beta, \gamma \in F$, let us call $V(\alpha, \beta, \gamma)$ the 4-dimensional $\mathbf{Z}_2$-graded vector space $V = F(v, w, z, t)$ with $V_{\bar{0}} = F(v, w)$, $V_{\bar{1}} = F(z, t)$ and the action of J over V defined by:

$$\begin{aligned} ve = v,\ vf = 0,\ vx = z,\ vy = t, \\ we = 0,\ wf = w,\ wx = (\gamma - 1)z - 2\alpha t,\ wy = 2\beta z - (\gamma + 1)t, \\ ze = \frac{1}{2}z,\ zf = \frac{1}{2}z,\ zx = \alpha v,\ zy = \frac{1}{2}(\gamma + 1)v + \frac{1}{2}w, \\ te = \frac{1}{2}t,\ tf = \frac{1}{2}t,\ tx = \frac{1}{2}(\gamma - 1)v - \frac{1}{2}w,\ ty = \beta v. \end{aligned} \tag{1.1}$$

Let us notice that $R(x)^2 = \alpha I_V$, $R(y)^2 = \beta I_V$ and $R(x)R(y) + R(y)R(x) = \gamma I_V$.

It can be also checked that $vU(x, y) = w$.

To start we will prove that $V(\alpha, \beta, 0)$ is a Jordan bimodule for arbitrary $\alpha, \beta \in F$.

Lemma 1.1. *$V(\alpha, \beta, 0)$ is a (unital) Jordan bimodule over J.*

Proof

Let us define an embedding from $M_{1|1}(F)$ in $M_{2|2}(F)$ via:

$$i : M_{1|1}(F) \to M_{2|2}(F)$$

$$e \to \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$f \to \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}$$

$$x \to \begin{pmatrix} 0 & I_2 \\ A & 0 \end{pmatrix}$$

$$y \to \begin{pmatrix} 0 & B \\ I_2 & 0 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 2\alpha \end{pmatrix}, \ B = \begin{pmatrix} 2\beta & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the J-submodule of $M_{2|2}(F)$ with basis $\{\mathbf{v},\ \mathbf{w} = \mathbf{v}U(x,y),\ \mathbf{z} = \mathbf{v}x,\ \mathbf{t} = \mathbf{v}y\}$.

This bimodule is isomorphic to $V(\alpha, \beta, 0)$.

Now let us consider arbitrary elements $\alpha, \beta, \gamma \in F$. We can take elements $\alpha', \beta' \in F$ such that $\gamma^2 - 4\alpha\beta - 1 = -4\alpha'\beta' - 1$, that is, $\gamma^2 = 4(\alpha\beta - \alpha'\beta')$.

Lemma 1.2. *There is an isomorphism $\varphi : M_{1|1}(F)^{(+)} \longrightarrow M_{1|1}(F)^{(+)}$ such that for every $v \in V(\alpha, \beta, \gamma)$ we have $vR(\varphi(x))^2 = \alpha' v$, $vR(\varphi(y))^2 = \beta' v$ and $v(R(\varphi(x))R(\varphi(y)) + R(\varphi(y))R(\varphi(x))) = 0$.*

Proof.
From $\gamma^2 - 4\alpha\beta = -4\alpha'\beta'$ it follow that matrices

$$A' = \begin{pmatrix} 0 & 2\alpha' \\ -2\beta' & 0 \end{pmatrix}, \ A = \begin{pmatrix} \gamma & -2\alpha \\ 2\beta & -\gamma \end{pmatrix}$$

have the same determinant and both of them have zero trace.

Consequently both matrices are similar, that is, there is an invertible matrix P (without loss of generality we can assume that $|P| = 1$) suchat that $A' = PAP^{-1}$.

If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we only need to consider φ the automorphism of $M_{1|1}(F)^{(+)}$ given by $\varphi(e) = e$, $\varphi(f) = f$, $\varphi(x) = ax + by$, $\varphi(y) = cx + dy$.

Notice that this lemma says that $V(\alpha, \beta, \gamma)$ is a unital module over $M_{1|1}(F)^{(+)}$ and that there is a semiisomorphism between $V(\alpha, \beta, \gamma)$ and $V(\alpha', \beta', 0)$.

That is, we have proved the following result.

Theorem 1.3. *a) For arbitrary elements* $\alpha, \beta, \gamma \in F$*, the action of* $J = M_{1|1}(F)^{(+)}$ *over* $V = V(\alpha, \beta, \gamma)$ *over the graded vector space* $V = F(v, w, z, t)$ *given by (1.1) defines a structure of unital* J*-bimodule* $V(\alpha, \beta, \gamma)$*.*

b) Given $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in F$*, the* J *bimodules* $V(\alpha, \beta, \gamma)$ *and* $V(\alpha', \beta', \gamma')$ *are isomorphic if and only if* $\alpha = \alpha'$*,* $\beta = \beta'$ *and* $\gamma = \gamma'$

c) The J *bimodules* $V(\alpha, \beta, \gamma)$ *and* $V(\alpha', \beta', \gamma')$ *are semi- isomorphic if and only if* $\gamma^2 - 4\alpha\beta = \gamma'^2 - 4\alpha'\beta'$*.*

Lemma 1.4. *If* $\gamma^2 - 4\alpha\beta - 1 \neq 0$ *then the bimodule* $V = V(\alpha, \beta, \gamma)$ *is irreducible. If* $\gamma = 1$ *and* $\alpha = 0$ *then* $Fw + Fwy$ *is the only proper submodule of* $V = V(\alpha, \beta, \gamma)$*. In all other cases,* $Fw + Fwx$ *is the only proper submodule of* $V = V(\alpha, \beta, \gamma)$*.*

Proof.

Let $(0) \neq V'$ a nonzero submodule of $V = V(\alpha, \beta, \gamma)$. Then $V' \cap V_{\bar{0}} \neq (0)$, since otherwise $V'x = V'y = (0)$.

Applying to an arbitrary element $\tilde{v}$ in V' the following Jordan identity: $R(x)R(e)R(y) - R(y)R(e)R(x) - R([x, y]e) - R(xe)R(y) + R(ye)R(x) - R([x, y])R(e) = 0$, we get that $\tilde{v}(R(e) - R(e - f)R(e)) = 0$.

But the odd part of $V = V(\alpha, \beta, \gamma)$ lies in the $\frac{1}{2}$- Peirce component of e and f, so $\tilde{v} = 0$.

That is, if $V' \cup V_{\bar{0}} = (0)$, then $V' = (0)$.

If $\{e, V', e\} \neq (0)$, then $v \in V'$ and so $V' = V$.

If $\{e, V', e\} = (0)$, then $V' \cup V_{\bar{0}} = Fw$. But $wU(x, y) = (\gamma^2 - 4\alpha\beta - 1)v \in V'$. So $v \in V'$, that is, $V = V'$ as soon as $\gamma^2 - 4\alpha\beta - 1 \neq 0$. This proves irreducibility of $V = V(\alpha, \beta, \gamma)$ when $\gamma^2 - 4\alpha\beta - 1 \neq 0$.

So from now on we assume that $\gamma^2 - 4\alpha\beta - 1 = 0$

Now let's consider the case $\gamma = 1$ and $\alpha = 0$. Then $wU(x, y) = wx = 0$ and $wy = 2\beta z - 2t$ and $V' = F(w, wx)$.

Otherwise, $wx = (\gamma - 1)z - 2\alpha t \neq 0$ and $wy = 2\beta z - (\gamma + 1)t$ implies that $(\gamma + 1)wx - 2\alpha wy = (\gamma^2 - 4\alpha\beta - 1)z - 0t = 0$, that is, $F(w, wx) = V'$.

Notation. If $\gamma^2 - 4\alpha\beta - 1 = 0$, let's denote $V'(\alpha, \beta, \gamma)$ the only proper nonzero submodule of $V = V(\alpha, \beta, \gamma)$ (that can be expressed as $F(w, wx)$ except when $\alpha = 0$, $\gamma = 1$ that can be expressed as $F(w, wy)$) and $\tilde{V}(\alpha, \beta, \gamma) = V(\alpha, \beta, \gamma)/V'(\alpha, \beta, \gamma)$.

Now we can prove the classification result.

Theorem 1.5. *Every irreducible finite dimensional unital Jordan bimodule over* $J = M_{1|1}(F)^{(+)}$ *is isomorphic to one of the bimodules* $V = V(\alpha, \beta, \gamma)$*, if* $\gamma^2 - 4\alpha\beta - 1 \neq 0$*, or to* $V'(\alpha, \beta, \gamma)$ *or* $\tilde{V}(\alpha, \beta, \gamma)$ *if* $\gamma^2 - 4\alpha\beta - 1 = 0$ *or their opposite modules.*

Proof.

Let V be an irreducible unital finite dimensional J-bimodule. Up to opposite, we can assume that $V_{\bar{0}} = \{e, V_{\bar{0}}, e\} + \{f, V_{\bar{0}}, f\}$ and $V_{\bar{1}} = \{e, V_{\bar{1}}, f\}$.

The operators $R(x)^2$, $R(y)^2$, $R(x)R(y) + R(y)R(x)$ commute with the action of J. By Schur's Lemma they act as scalars α, β, γ respectively.

We claim that for every subspace W of $\{e, V_{\bar{0}}, e\}$ the vector space $U = W + WU(J_{\bar{1}}, J_{\bar{1}}) + WJ_{\bar{1}}$ is a J-bimodule. Indeed, since $W \subseteq \{e, V_{\bar{0}}, e\}$, we have that $WJ_{\bar{1}} \subseteq \{e, V_{\bar{1}}, f\}$ and $WU(J_{\bar{1}}, J_{\bar{1}}) \subseteq \{f, V_{\bar{0}}, f\}$. Hence each summand W, $WU(J_{\bar{1}}, J_{\bar{1}})$ and $WJ_{\bar{1}}$ is invariant under multiplication by e and f, so under multiplication by $J_{\bar{0}}$.

Now using that $R(J_{\bar{1}})R(J_{\bar{1}}) \subseteq U(J_{\bar{1}}, J_{\bar{1}}) + D(J_{\bar{1}}, J_{\bar{1}}) + R(J_{\bar{0}})$, we get that $WR(J_{\bar{1}})R(J_{\bar{1}}) \subseteq WU(J_{\bar{1}}, J_{\bar{1}}) + WD(J_{\bar{1}}, J_{\bar{1}}) + WR(J_{\bar{0}}) \subseteq U$. That implies that $WR(J)R(J) \subseteq U$.

So, we only need to prove that $WU(J_{\bar{1}}, J_{\bar{1}})R(J_{\bar{1}}) \subseteq WJ_{\bar{1}}$. But $U(J_{\bar{1}}, J_{\bar{1}}) \subseteq R(J_{\bar{1}})R(J_{\bar{1}}) + R(J_{\bar{0}})$ and $R(J_{\bar{1}})R(J_{\bar{1}})R(J_{\bar{1}}) \subseteq R(J)R(J) + D(J_{\bar{1}}), J_{\bar{1}})R(J_{\bar{1}})$. Now using that $D(J_{\bar{1}}, J_{\bar{1}})$ acts as an scalar multiplication we gets what we wanted.

In the same way we can prove that for every $W \subseteq \{f, V_{\bar{0}}, f\}$, the subspace $W + WU(J_{\bar{1}}, J_{\bar{1}}) + WJ_{\bar{1}}$ is a J-bimodule.

Since we assume V to be irreducible, it follows that $dim_F\{e, V_{\bar{0}}, e\} \leq 1$ and $dim_F\{f, V_{\bar{0}}, f\} \leq 1$, $dimV_{\bar{1}} \leq 2$.

If $\gamma^2 - 4\alpha\beta - 1 \neq 0$, let us show that $v \simeq V(\alpha, \beta, \gamma)$, where $R(x)^2$ acts on V as αI_V, $R(y)^2$ acts as βI_V and $R(x)R(y) + R(y)R(x)$ acts as γI_V. We have already seen that $V_{\bar{0}} \neq (0)$. The operator $U(x, y)^2$ acts on $V_{\bar{0}}$ as the multiplication by $\gamma^2 - 2\alpha\beta - 1$. This implies that both $\{e, V_{\bar{0}}, e\}$ and $\{f, V_{\bar{0}}, f\}$ are different of zero (multiplication by $U(x, y)$ exchange them both).

Choose $0 \neq v \in \{e, V_{\bar{0}}, e\}$. We know that $w = vU(x, y) \in \{f, V_{\bar{0}}, f\}$. Let us prove that $vx, vy \in V_{\bar{1}}$ are linearly independent. Suppose that $vy = \lambda vx$, $\lambda \in F$. Then $vR(y)R(x) = (vy)x = \lambda(vx)x = \lambda\alpha v$ and $vU(x, y) = vR(x)R(y) - vR(y)R(x) - vR([x, y]) = v(R(x)R(y) + R(y)R(x)) - 2vR(y)R(x) - vR(e - f) = (\gamma - 2\lambda\alpha - 1)v$, that is, $vU(x, y) \in Fv$, which is a contradiction.

Hence $F(v, w = vU(x, y), vx, vy)$ is a J-bimodule and the multiplication table coincides with the one of $V(\alpha, \beta, \gamma)$.

Now let's consider the case $\gamma^2 - 4\alpha\beta - 1 = 0$. In this case $V_{\bar{0}}U(x, y)^2 = (0)$. If $\{e, V_{\bar{0}}, e\} \neq (0)$ and $0 \neq v \in \{e, V_{\bar{0}}, e\}$, then $w = vU(x, y) = 0$. Indeed, if $w = vU(x, y) \neq 0$, then V is generated by w, $wU(x, y)$, wx, wy. But $wU(x, y) = vU(x, y)^2 = 0$. So, $dim_F V_{\bar{0}} \leq 1$, which contradicts $v, w \in V_{\bar{0}}$. Hence $w = vU(x, y) = 0$. This says that $V \simeq V'(\alpha, \beta, gamma)$.

If $\{f, V_{\bar{0}}, f\} \neq (0)$, then $V \simeq \bar{V}(\alpha, \beta, \gamma)$, what proves the theorem.

2 One sided modules

Let $S = S(J)$ be the unital universal associative enveloping algebra of the Jordan algebra $J = M_{1|1}^{(+)}$. Denote $x = e_{12},\ y = e_{21},\ e = e_{11},\ f = e_{22},\ v = e - f$, then $J = alg_{Jord}\langle x, y\rangle$ and $S = alg_{As}\langle x, y\rangle$.

We have $x \circ e = x,\ y \circ e = y, [x, y] = v$. Observe that x^2, y^2 lie in the center $Z(S)$ of S. Moreover, we have

$$\begin{aligned}[x \circ y, x] &= [y, x^2] = 0,\\ [x \circ y, y] &= [x, y^2] = 0,\end{aligned}$$

hence $x \circ y \in Z(S)$.

Lemma 2.1. *Let* $A = F[x^2, y^2],\ B = F[x^2, y^2, x \circ y]$.
1) The algebra S *is a free* B*-module with free generators* $1, x, y, xy$.
2) The center $Z(S) = B$.
3) $B = A[x \circ y]$, *where* $(x \circ y)^2 = 1 + 4x^2y^2$.

Proof. We have $yx = x \circ y - xy,\ xyx = (x \circ y)x - x^2y,\ yxy = (x \circ y)y - y^2x,\ (xy)^2 = (x \circ y)xy - x^2y^2$, which proves that S is spanned over B by elements $1, x, y, xy$. Let $z = \alpha + \beta x + \gamma y + \delta xy \in Z(S)$ with $\alpha, \beta, \gamma, \delta \in B$, then $0 = [x, z] = \gamma[x, y] + \delta x[x, y] = \gamma v + \delta xv$. Multiplying by v, we get $\gamma + \delta x = 0$, which gives $\gamma = \delta = 0$. Similarly, we get $\beta = 0$, hence $Z(S) = B$. The similar argument shows that if $\alpha + \beta x + \gamma y + \delta xy = 0$ then $\alpha = \beta = \gamma = \delta = 0$, which proves 1). Finally,

$$(x \circ y)^2 = (xy)^2 + (yx)^2 + 2x^2y^2 = [x, y]xy + [y, x]yx + 4x^2y^2 = v^2 + 4x^2y^2 = 1 + 4x^2y^2,$$

proving 3).

□

The algebra S has a natural $\mathbf{Z}_2$-grading induced by the grading of J:

$$S_{\bar{0}} = B + Bxy,\ S_{\bar{1}} = Bx + By.$$

The category of one-sided Jordan J-superbimodules is isomorphic to the category of right associative $\mathbf{Z}_2$-graded S-modules. In particular, irreducible superbimodules over J correspond to irreducible $\mathbf{Z}_2$-graded S-modules.

Let $M = M_{\bar{0}} + M_{\bar{1}}$ be an irreducible $\mathbf{Z}_2$-graded S-module and $\varphi : S \to End_F M$ be the corresponding representation. Then $\varphi(B)$ lies in the even part of the centralizer D of S-module M, which is a graded division algebra (see, for example, [1]). Denote $\alpha = \varphi(x^2)$, $\beta = \varphi(y^2)$, $\gamma = \varphi(x \circ y)$, $K = F(\alpha, \beta, \gamma)$, then K is a field, $K = F(\alpha, \beta) + F(\alpha, \beta)\gamma$ where $\gamma^2 = 4\alpha\beta + 1$. Moreover, the graded algebra $\bar{S} = \varphi(S)$ has dimension at most 4 over K.

The algebra $\bar{S}$ and the module M may be considered over the field K, then M is a faithful irreducible graded module over the K-algebra $\bar{S}$. By [2, Lemma 4.2], M up to opposing grading is isomorphic to a minimal graded right ideal of $\bar{S}$. Since $\dim_K \bar{S} \leq 4$, we have $\dim_K M \leq 2$. Moreover, the case $\dim_K M = 1$ can appear only when $\bar{S} = K$ which is impossible since $[\varphi(x), \varphi(y)] \neq 0$. Therefore, $\dim_K \bar{S} = 4$ and $\dim_K M = 2$.

Observe also that by the density theorem for graded modules (see, for example, [1]), $\bar{S}$ is a dense graded subalgebra of the algebra $End_D M \subseteq End_K^{gr} M = M_{1|1}(K)$. Clearly, this implies that $\bar{S} = M_{1|1}(K)$.

Consider the elements $a = \frac{\gamma+1}{2} - xy,\ b = xy - \frac{\gamma-1}{2} \in B$. We have $a^2 = a$, $b^2 = b$, $a+b = 1$, hence up to change of indices $\varphi(a) = e_{11},\ \varphi(b) = e_{22}$.

We will separate the two cases:

1. Let first $\gamma \neq 1$. Chose an element $m \in M_{\bar{0}} \cup M_{\bar{1}}$ such that $ma = m$, then we have $m = \frac{\gamma+1}{2} m - mxy$, which gives

$$mxy = \tfrac{\gamma-1}{2} m,\ \ \beta mx = \tfrac{\gamma-1}{2} my. \tag{2.1}$$

In particular, $mxy \neq 0$, $m' = mx \neq 0$, and $M = Km + Km'$. We have by (1)

$$\begin{aligned} m'x &= \alpha m; \\ my &= \tfrac{2\beta}{\gamma-1} mx = \tfrac{2\beta}{\gamma-1} m'; \\ m'y &= mxy = \tfrac{\gamma-1}{2} m. \end{aligned}$$

2. Let now $\gamma = 1$, then $a = 1 - xy$, $b = xy$. Choose an element $m \in M_{\bar{0}} \cup M_{\bar{1}}$ such that $m = mb \neq 0$, then $m' = mx \neq 0$ and again $M = Km + Km'$. We have

$$\begin{aligned} m'x &= \alpha m; \\ my &= mby = mxyy = \beta mx = \beta m'; \\ m'y &= mxy = m. \end{aligned}$$

Observe that for $\gamma = -1$ in case 1 we obtain the formulas of case 2. The condition $\gamma^2 = 1$ is equivalent to $\alpha\beta = 0$, therefore we will distinguish four non-isomorphic cases: $\gamma \neq \pm 1$; $\alpha = 0,\ \beta \neq 0$; $\alpha \neq 0,\ \beta = 0$; $\alpha = \beta = 0$.

Resuming, we have

Theorem 2.2. *Let M be an irreducible one-sided Jordan bimodule over $J = M_{1|1}(F)^{(+)}$. Then there exist an extension field $K = F(\alpha, \beta, \gamma)$ with $\gamma^2 = 4\alpha\beta + 1$ such that* $\dim_K M = 2$, *$M = Km + Km'$, and up to opposite*

grading the action of J on M is given as follows:
1. $\gamma \neq \pm 1$ *(or* $\alpha\beta \neq 0$*).*

$$\begin{aligned} m \cdot x &= \tfrac{1}{2} m'; \\ m' \cdot x &= \tfrac{1}{2} \alpha m; \\ m \cdot y &= \tfrac{\beta}{\gamma - 1} m'; \\ m' \cdot y &= mxy = \tfrac{\gamma - 1}{4} m. \end{aligned}$$

2. $\gamma = \pm 1$ *(or* $\alpha\beta = 0$*).*

$$\begin{aligned} m \cdot x &= \tfrac{1}{2} m'; \\ m' \cdot x &= \tfrac{1}{2} \alpha m; \\ m \cdot y &= \tfrac{1}{2} \beta m'; \\ m' \cdot y &= \tfrac{1}{2} m. \end{aligned}$$

In the second case we have 3 non-isomorphic subclasses: $\alpha = 0$, $\beta \neq 0$; $\alpha \neq 0$, $\beta = 0$; $\alpha = \beta = 0$.

The module M is finite dimensional if and only if the elements α, β are algebraic over F. In particular, if the field F is algebraically closed and M is finite dimensional, then $K = F$.

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