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Idempotents and Peirce gradings of Jordan algebras ☆

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ABSTRACT

In this paper we find a necessary and sufficient condition for a Peirce grading of a Jordan algebra J to come from a Peirce decomposition with respect to an idempotent of a Jordan algebra $\tilde{J}$ containing J as a subalgebra. We also show that the above condition holds automatically when J is nondegenerate.

Introduction

Peirce gradings $V = V_2 \oplus V_1 \oplus V_0$ of a Jordan system V are introduced in [4,11] as a generalization of Peirce decompositions with respect to idempotents. As shown in [11], Peirce gradings are directly linked to involutive automorphisms and gradings.

In [4, 4.1], it is shown that the components V_α, $\alpha = 0, 1, 2$, inherit nondegeneracy and Von Neumann regularity from V, among other properties. In [1], it is shown that V_0 and V_2 inherit strong primeness, primitivity, semiprimitivity and simplicity from V. These results extend those obtained by McCrimmon [6,7] about inheritance of simplicity by the diagonal components of a Peirce decomposition with respect to an idempotent.

Neher [11] gives examples of Peirce gradings not coming from idempotents. Indeed, if we consider a Jordan system $\tilde{J}$ with an idempotent e and take any nonzero ideal J of $\tilde{J}$ such that $e \notin J$, then the usual Peirce decomposition $\tilde{J} = \tilde{J}_2(e) \oplus \tilde{J}_1(e) \oplus \tilde{J}_0(e)$ gives rise to a Peirce grading in J (with

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$J_\alpha = \tilde{J}_\alpha(e) \cap J$ for $\alpha \in \{0, 1, 2\}$) which does not come, in general, from an idempotent of J. We will study, in the case of Jordan algebras, if every Peirce grading of J comes from a Peirce decomposition with respect of an idempotent in an algebra $\tilde{J}$ containing J as a subalgebra.

The paper is divided into three parts, apart from a preliminary section in which we outline some known results and definitions. In the first section we study Peirce gradings of associative algebras and show that they always come from idempotents in bigger algebras. We also show that not all Peirce gradings in special Jordan algebras are induced by associative Peirce gradings. In the second section we give a necessary condition on Peirce gradings of a Jordan algebra J for them to come from idempotents in a bigger algebra, and build the natural extension $\tilde{J}$ of J where the idempotent element should be found. Finally, in Section 3, we prove that the necessary condition of the previous section is also sufficient for Jordan algebras without 2-torsion, by showing that then $\tilde{J}$ is indeed a Jordan algebra. We also prove that this additional condition holds automatically in sufficiently regular (for example nondegenerate) Jordan algebras.

0. Preliminaries

0.1. We will work with associative and Jordan algebras over an arbitrary ring of scalars Φ. The reader is referred to [3,9,10] for basic results, notation, and terminology, though we will stress here some definitions.

0.2. A *quadratic algebra* J in the sense of [5, Section 1] is given on a Φ-module by its squares x^2 and products $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y, so that we have their linearizations, given by $x \circ y = V_x y := (x + y)^2 - x^2 - y^2$, and $U_{x,z} y = V_{x,y} z = \{x, y, z\} := U_{x+z} y - U_x y - U_z y$. A quadratic algebra is a *Jordan algebra* if it strictly satisfies

QJ1: $V_{x,x} = V_{x^2}$,
QJ2: $U_x V_x = V_x U_x$,
QJ3: $U_x(x^2) = (x^2)^2$,
QJ4: $U_x U_y(x^2) = (U_x y)^2$,
QJ5: $U_{x^2} = U_x^2$,
QJ6: $U_{U_x y} = U_x U_y U_x$,

for any $x, y \in J$, i.e., the polynomials $j_1(x, y) := V_{x,x} y - V_{x^2} y$, $j_2(x, y) := U_x V_x y - V_x U_x y$, $j_3(x) := U_x(x^2) - (x^2)^2$, $j_4(x, y) := U_x U_y(x^2) - (U_x y)^2$, $j_5(x, y) := U_{x^2} y - U_x^2 y$, $j_6(x, y, z) := U_{U_x y} z - U_x U_y U_x z$ vanish strictly on J, i.e., vanish on any scalar extension of J.

0.3. One can get Jordan algebras from associative algebras by *symmetrization*: Given an associative algebra R with products xy, for any $x, y \in R$, one can build a Jordan algebra, denoted $R^{(+)}$, on the same Φ-module and with the same squares as R ($x^2 = xx$) and Jordan products $U_x y = xyx$. A Jordan algebra is said to be *special* when it is a subalgebra of $R^{(+)}$, for some associative algebra R.

0.4. A Jordan algebra J is said to be *unital* if there exists a *unit element* $1 \in J$ such that, $U_1 x = x$, $U_x 1 = x^2$, for any $x \in J$. The unit element is unique (when it exists) and idempotent ($1^2 = 1$). Any Jordan algebra J has a *unitization* $\hat{J} = J \oplus \Phi 1$, which is a Jordan algebra built on the direct sum of J and a free Φ-module with basis $\{1\}$, given by

$$(x + \lambda 1)^2 = x^2 + 2\lambda x + \lambda^2 1, \qquad U_{x+\lambda 1}(y + \mu 1) = U_x y + \mu x^2 + \lambda^2 y + \lambda x \circ y + 2\lambda\mu x + \lambda^2 \mu 1,$$

for any $x, y \in J$, $\lambda, \mu \in \Phi$. The algebra J is an ideal of $\hat{J}$, and 1 is the unit element of $\hat{J}$ [5].

0.5. We will need the following identities of Jordan algebras, which are direct consequences of Macdonald's Theorem [2]:

(i) $\{x, y, y\} = x \circ y^2$, $\{x, y^2, y\} = \{x, y, y^2\} = x \circ y^3$,
(ii) $(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$,
(iii) $2x^3 = x \circ x^2$,
(iv) $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$,
(v) $(x \circ y)^2 = x \circ U_y x + U_x y^2 + U_y x^2$,
(vi) $x^{2m+n} = U_{x^m} x^n$, $(x^m)^n = x^{mn}$,
(vii) $U_x U_y x = U_{x \circ y} x - x \circ U_y x^2 - U_y x^3$,
(viii) $U_z U_x U_z y = U_z U_{x \circ z} y - \{z^2 \circ x, y, U_z x\} + \{x, z^2, y\} \circ U_z x - \{y, z, U_z U_x z\} - U_{z^2} U_x y$,
(ix) $2U_x y = x \circ (x \circ y) - x^2 \circ y$.

0.6. Following [1,4,9,11], given a Jordan algebra J, a *Peirce grading* of J is a decomposition $J = J_2 \oplus J_1 \oplus J_0$ (direct sum of Φ-modules) such that, for any $\alpha, \beta, \gamma \in \{0, 1, 2\}$,

(i) $U_{J_\alpha} J_\beta \subseteq J_{2\alpha-\beta}$,
(ii) $\{J_\alpha, J_\beta, J_\gamma\} \subseteq J_{\alpha-\beta+\gamma}$,
(iii) $\{J_0, J_2, J\} = \{J_2, J_0, J\} = 0$,
(iv) $J_2^2 \subseteq J_2$, $J_0^2 \subseteq J_0$, $J_1^2 \subseteq J_0 + J_2$,
(v) $J_2 \circ J_1 + J_0 \circ J_1 \subseteq J_1$, $J_2 \circ J_0 = 0$,

where $J_\lambda = 0$ if $\lambda \notin \{0, 1, 2\}$. A Jordan algebra equipped with a Peirce grading will be said *Peirce graded*. We will consider the natural projections $\pi_\alpha : J \to J_\alpha$, $\pi_\alpha(x) = x_\alpha$, $\alpha \in \{0, 1, 2\}$. Notice that (i)–(v) involve at most degree two multiplications, so that any scalar extension of a Peirce graded Jordan algebra is naturally Peirce graded.

0.7. Notice that J_0 y J_2 can be exchanged, i.e., taking $J'_\alpha = J_{2-\alpha}$ for $\alpha \in \{0, 2\}$, we obtain a new Peirce grading $J = J'_2 \oplus J'_1 \oplus J'_0$.

0.8. Given an idempotent e in a Jordan algebra J, the usual *Peirce decomposition* $J = J_2(e) \oplus J_1(e) \oplus J_0(e)$ of J with respect to e, where

$$J_2(e) = U_e J, \qquad J_1(e) = (V_e - 2U_e) J, \qquad J_0(e) = (\mathrm{Id} - V_e + U_e) J \tag{1}$$

[3, Section I.5], is an example of Peirce grading of J which, following [11], will be called an *idempotent Peirce grading*. Notice the lack of "(0, 2)-symmetry" in idempotent Peirce gradings. On the other hand, we recall the basic fact that, for any $x_\alpha \in J_\alpha(e)$,

$$e \circ x_\alpha = \alpha x_\alpha. \tag{2}$$

1. Peirce gradings in associative algebras

1.1. Given an associative algebra R, a *Peirce grading* of R is a decomposition $R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$ (direct sum of Φ-modules) such that, for any $\alpha, \beta, \gamma, \delta \in \{0, 1\}$,

(i) $R_{\alpha\beta} R_{\gamma\delta} \subseteq R_{\alpha\delta}$, if $\beta = \gamma$,
(ii) $R_{\alpha\beta} R_{\gamma\delta} = 0$, if $\beta \neq \gamma$.

We will consider the natural projections $\pi_{\alpha\beta} : R \to R_{\alpha\beta}$, $\pi_{\alpha\beta}(x) = x_{\alpha\beta}$, $\alpha, \beta \in \{0, 1\}$.

Given an idempotent e in an associative algebra R, the usual *Peirce decomposition* $R = R_{11}(e) \oplus R_{10}(e) \oplus R_{01}(e) \oplus R_{00}(e)$ of R with respect to e is an example of Peirce grading of R.

1.2. Given a Peirce grading of an associative algebra $R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$, let $\tilde{R} = R \oplus \Phi e$ be the Φ-module obtained as a direct sum of R and a free Φ-module with basis $\{e\}$. Define a Φ-algebra structure on $\tilde{R}$ by the fact that R is a subalgebra of $\tilde{R}$ and

$$e^2 = e, \qquad xe = x_{11} + x_{01}, \qquad ex = x_{11} + x_{10},$$

for any $x \in R$, i.e., for any $x, y \in R$ and $\lambda, \mu \in \Phi$,

$$(\lambda e + x)(\mu e + y) = \lambda\mu e + \lambda(y_{11} + y_{10}) + \mu(x_{11} + x_{01}) + xy.$$

The proof of the following result is straightforward.

1.3. Theorem. *Under the conditions of* (1.2), $\tilde{R}$ *is an associative algebra such that* R *is an ideal of* $\tilde{R}$, *and* $R_{\alpha\beta} = \tilde{R}_{\alpha\beta}(e) \cap R$, *for any* $\alpha, \beta \in \{0, 1\}$.

1.4. Any Peirce grading $R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$ of an associative algebra R obviously induces a Peirce grading of $J = R^{(+)}$ by taking $J_2 = R_{11}$, $J_1 = R_{01} + R_{10}$, $J_0 = R_{00}$. The converse is false: Let $R = G(e_1, e_2)$ be the nonunital Grassmann or exterior algebra in two generators over Φ (Φ is an arbitrary ring of scalars), and let $J = R^{(+)}$, which is a trivial algebra. Then, $J_0 = \Phi e_1$, $J_2 = \Phi e_2$, $J_1 = \Phi e_1 e_2$ is a Peirce grading of J which does not come from any (associative) Peirce grading of R in the above sense, since the associative product $J_0 J_2$ is nonzero.

2. Peirce gradings of Jordan algebras

2.1. We will remark some basic multiplication properties of a Peirce graded Jordan algebra $J = J_2 \oplus J_1 \oplus J_0$, which are direct consequences of (0.5)(i), (ii) together with the definition (0.6). If $x_\alpha, y_\alpha, z_\alpha \in J_\alpha$, $\alpha = 0, 1, 2$, then

(i) $x_2^2 \circ x_1 = (x_1 \circ x_2) \circ x_2$, $x_0^2 \circ x_1 = (x_1 \circ x_0) \circ x_0$,
$(x_2 \circ y_2) \circ x_1 = (x_1 \circ x_2) \circ y_2 + (x_1 \circ y_2) \circ x_2$,
$(x_0 \circ y_0) \circ x_1 = (x_1 \circ x_0) \circ y_0 + (x_1 \circ y_0) \circ x_0$,
(ii) $(x_1 \circ x_2) \circ x_0 = (x_1 \circ x_0) \circ x_2 = \{x_0, x_1, x_2\}$,
(iii) $\pi_2((x_1 \circ x_2) \circ y_1) = \{x_2, x_1, y_1\}$, $\pi_0((x_1 \circ x_2) \circ y_1) = \{x_1, x_2, y_1\}$,
$\pi_2((x_1 \circ x_0) \circ y_1) = \{x_1, x_0, y_1\}$, $\pi_0((x_1 \circ x_0) \circ y_1) = \{x_0, x_1, y_1\}$,
(iv) $\pi_2(\{x_1^2, y_1, z_1\}) = \{\pi_2(x_1^2), y_1, z_1\}$, $\pi_0(\{x_1^2, y_1, z_1\}) = \{\pi_0(x_1^2), y_1, z_1\}$,
$\pi_2(\{x_1, y_1^2, z_1\}) = \{x_1, \pi_0(y_1^2), z_1\}$, $\pi_0(\{x_1, y_1^2, z_1\}) = \{x_1, \pi_2(y_1^2), z_1\}$,
(v) $\pi_2(U_{x_1} y_1^2) = U_{x_1}(\pi_0(y_1^2))$, $\pi_0(U_{x_1} y_1^2) = U_{x_1}(\pi_2(y_1^2))$,
(vi) $\pi_2(x_1^4) = (\pi_2(x_1^2))^2$, $\pi_0(x_1^4) = (\pi_0(x_1^2))^2$.

2.2. Let J be a subalgebra of a Jordan algebra $\tilde{J}$, and assume that J is equipped with a Peirce grading $J = J_2 \oplus J_1 \oplus J_0$ induced by an idempotent $e \in \tilde{J}$, i.e., $J_\alpha = \tilde{J}_\alpha(e) \cap J$. Then

$$x_1^3 = \pi_0\big(x_1^2\big) \circ x_1 = \pi_2\big(x_1^2\big) \circ x_1, \quad \text{for any } x_1 \in J_1. \tag{1}$$

Moreover, (1) holds strictly on J in the sense that any linearization of (1) holds in J, i.e., (1) holds in any (naturally Peirce graded) scalar extension of J. This is due to the fact that the hypotheses extend naturally to scalar extensions of J and $\tilde{J}$.

Indeed,

$$\begin{aligned} x_1^3 &= e \circ x_1^3 \quad \big(\text{by (0.8)(2), since } x_1^3 \in J_1 \text{ by (0.6)(i)}\big) \\ &= \big\{e, x_1^2, x_1\big\} \quad \big(\text{by (0.5)(i)}\big) \\ &= \big(e \circ x_1^2\big) \circ x_1 - \big\{x_1^2, e, x_1\big\} \quad \big(\text{by (0.5)(ii)}\big) \end{aligned}$$

$$= 2\pi_2(x_1^2) \circ x_1 - \{\pi_0(x_1^2), e, x_1\} - \{\pi_2(x_1^2), e, x_1\} \quad \text{(by (0.6)(iv) and (0.8)(2))}$$
$$= 2\pi_2(x_1^2) \circ x_1 - \{\pi_2(x_1^2), e, x_1\} \quad \text{(by (0.6)(iii) since } e \in \tilde{J}_2(e))$$
$$= 2\pi_2(x_1^2) \circ x_1 - \pi_2(x_1^2) \circ (e \circ x_1) + \{\pi_2(x_1^2), x_1, e\} \quad \text{(by (0.5)(ii))}$$
$$= 2\pi_2(x_1^2) \circ x_1 - \pi_2(x_1^2) \circ x_1 \quad \text{(by (0.8)(2) and (0.6)(ii))}$$
$$= \pi_2(x_1^2) \circ x_1.$$

The second equality follows from the first together with the following general fact:

2.3. If $J = J_2 \oplus J_1 \oplus J_0$ is a Peirce graded Jordan algebra and $x_1 \in J_1$, then

$$x_1^3 = \pi_0(x_1^2) \circ x_1 \quad \Leftrightarrow \quad x_1^3 = \pi_2(x_1^2) \circ x_1.$$

Just notice that $2x_1^3 = x_1 \circ x_1^2 = x_1 \circ \pi_0(x_1^2) + x_1 \circ \pi_2(x_1^2)$ using (0.5)(iii) and (0.6)(iv).

2.4. There are examples of Peirce gradings not fulfilling (2.2)(1): Let $S = \Phi[X]$ be the free nonunital associative algebra in one generator X and let Φ be an arbitrary ring of scalars. Let L be the (associative) ideal of S generated by X^4, and $A = S/L$. Clearly, A is an associative algebra and a free Φ-module with basis

$$\{a = X + L,\ b = X^2 + L,\ c = X^3 + L\}.$$

In the special Jordan algebra $J = A^{(+)}$ one can readily check that $J_2 = \Phi b$, $J_1 = \Phi a + \Phi c$, $J_0 = 0$ is a Peirce grading, but $a^3 = c \neq 0$ and $\pi_0(a^2) \circ a = 0$ since $\pi_0(a^2) = 0$ because $a^2 = b \in J_2$.

2.5. Let $J = J_2 \oplus J_1 \oplus J_0$ be a Peirce graded Jordan algebra. Let $\tilde{J} = J \oplus \Phi e$ be the Φ-module obtained as a direct sum of J and a free Φ-module with basis $\{e\}$. We can define a quadratic algebra structure (cf. [5, Section 1]) on $\tilde{J}$ by establishing, for any $x, y \in J$:

(i) J is a subalgebra of $\tilde{J}$,
(ii) $e^2 = e$, which implies
$e \circ e = 2e$,
(iii) $e \circ x = 2x_2 + x_1$,
(iv) $U_e e = e$,
(v) $U_e x = x_2$, which implies
$\{e, x, e\} = 2x_2$,
(vi) $U_x e = x_2^2 + x_2 \circ x_1 + \pi_0(x_1^2)$, which implies
$\{x, e, y\} = x_2 \circ y_2 + x_2 \circ y_1 + y_2 \circ x_1 + \pi_0(x_1 \circ y_1)$,
(vii) $\{e, x, y\} = x_2 \circ y_2 + \pi_2(x_1 \circ y_1) + x_2 \circ y_1 + x_1 \circ y_0$,
(viii) $\{e, e, x\} = 2x_2 + x_1$,

i.e., for any $x, y \in J$, $\alpha, \beta \in \Phi$,

(a) $(x + \alpha e)^2 = x^2 + \alpha(2x_2 + x_1) + \alpha^2 e$, and
(b) $U_{x+\alpha e}(y + \beta e) = U_x y + \beta(x_2^2 + x_2 \circ x_1 + \pi_0(x_1^2)) + \alpha^2 y_2 + \alpha\beta(2x_2 + x_1) + \alpha(x_2 \circ y_2 + \pi_2(x_1 \circ y_1) + x_1 \circ y_2 + x_0 \circ y_1) + \alpha^2 \beta e$.

Under these conditions J is automatically an ideal of $\tilde{J}$ and the quotient $\tilde{J}/J$ is isomorphic to the subalgebra Φe of $\tilde{J}$. Notice that Φe is a Jordan algebra isomorphic to $\Phi^{(+)}$. We have the natural projections $\mu : \tilde{J} \to J$ and $\tau : \tilde{J} \to \Phi e$, given by $\mu(x + \alpha e) = x$, $\tau(x + \alpha e) = \alpha e$, for any $x \in J, \alpha \in \Phi$, and we remark that τ is a quadratic algebra epimorphism, whose kernel is J.

2.6. The "(0–2)-symmetry" of the Peirce grading in J disappears in $\tilde{J}$ (2.5). However, we will be able to make use of that symmetry by further extending $\tilde{J}$. Let $\widehat{\tilde{J}}$ be the unitization of $\tilde{J}$. Thus $\widehat{\tilde{J}} = J \oplus \Phi e \oplus \Phi 1$. Notice that the subalgebra $J \oplus \Phi 1$ of $\widehat{\tilde{J}}$ is simply the unitization $\hat{J}$ of J, hence a Jordan algebra. Let $J' = J$ Peirce graded by $J'_2 = J_0$, $J'_1 = J_1$, $J'_0 = J_2$, build $\tilde{J}'$ as in (2.5), and let $\widehat{\tilde{J}'}$ be its unitization.

We can define a linear map $\varphi : \widehat{\tilde{J}} \to \widehat{\tilde{J}'}$ by

$$\varphi|_{\hat{J}} = \mathrm{Id}_{\hat{J}}, \qquad \varphi(e) = 1 - e, \tag{1}$$

which satisfies

$$\varphi\big(\pi_\alpha(x)\big) = \pi_\alpha(x) = \pi'_{2-\alpha}(x) = \pi'_{2-\alpha}\big(\varphi(x)\big), \quad \alpha = 1, 2, \tag{2}$$

for any $x \in J$.

The proof of the following result is straightforward.

2.7. Proposition. *Under the conditions of* (2.6), *φ is a quadratic algebra isomorphism.*

3. Main results

3.1. In this section, when J is a Peirce graded Jordan algebra, $\tilde{J}$ will denote the quadratic algebra built in (2.5).

3.2. As in [5], we also consider the notion of *commutative Jordan algebra*: a linear algebra $(C, \circ)$ over Φ, such that

$$x \circ y = y \circ x \quad \text{and} \quad \big(x^2 \circ y\big) \circ x = x^2 \circ (y \circ x)$$

hold strictly on J. Notice that, when $1/2 \in \Phi$, these are the usual linear Jordan algebras.

3.3. Proposition. *Let $(J, (\ )^2, U)$ be a quadratic Φ-algebra such that*

(i) $2U_x y = x \circ (x \circ y) - x^2 \circ y$, *for any $x, y \in J$ and*
(ii) *the identity $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$ holds strictly on J.*

Then, the polynomials $4j_1$, $8j_2$, $8j_3$, $32j_4$, $16j_5$, and $64j_6$ vanish strictly on J.

Proof. Let us consider the linear algebra $(J, \circ)$, whose squares will be denoted

$$x^{\tilde{2}} := x \circ x = 2x^2. \tag{1}$$

By (ii), $(J, \circ)$ is a commutative Jordan algebra, so that $(J, (\ )^{\tilde{2}}, \tilde{U})$ is a Jordan algebra [5, Cor. on p. 277], where

$$\begin{aligned} \tilde{U}_x y &= 2x \circ (x \circ y) - (x \circ x) \circ y = 2\big(x \circ (x \circ y) - x^2 \circ y\big) = 4U_x y, \\ \tilde{V}_x y &= x \,\tilde{\circ}\, y = (x + y)^{\tilde{2}} - x^{\tilde{2}} - y^{\tilde{2}} = 2x \circ y = 2V_x y, \end{aligned} \tag{2}$$

hence it strictly satisfies

QJ1: $\tilde{V}_{x,x} = \tilde{V}_{x^{\tilde{2}}}$, i.e., $4V_{x,x} = 4V_{x^2}$,
QJ2: $\tilde{U}_x \tilde{V}_x = \tilde{V}_x \tilde{U}_x$, i.e., $8U_x V_x = 8V_x U_x$,

QJ3: $\tilde{U}_x(x^{\tilde{2}}) = (x^{\tilde{2}})^{\tilde{2}}$, i.e., $8U_x(x^2) = 8(x^2)^2$,
QJ4: $\tilde{U}_x\tilde{U}_y(x^{\tilde{2}}) = (\tilde{U}_x y)^{\tilde{2}}$, i.e., $32U_xU_y(x^2) = 32(U_x y)^2$
QJ5: $\tilde{U}_{x^{\tilde{2}}} = \tilde{U}_x^2$, i.e., $16U_{x^2} = 16U_x^2$
QJ6: $\tilde{U}_{\tilde{U}_x y} = \tilde{U}_x\tilde{U}_y\tilde{U}_x$, i.e., $64U_{U_x y} = 64U_xU_yU_x$.

We have shown that $4j_1$, $8j_2$, $8j_3$, $32j_4$, $16j_5$, and $64j_6$ vanish strictly on J. □

3.4. Lemma. *Let $J = J_2 \oplus J_1 \oplus J_0$ be a Peirce graded Jordan algebra. For any $x, y \in J$, we have the following equalities in $\tilde{J}$:*

(i) $(e \circ e) \circ x + ((x \circ e) \circ e) \circ e = e \circ (e \circ x) + (x \circ e) \circ (e \circ e)$,
(ii) $(x^2 \circ e) \circ e + ((x \circ e) \circ e) \circ x = x^2 \circ (e \circ e) + (x \circ e) \circ (e \circ x)$,
(iii) $(e \circ y) \circ x + ((x \circ e) \circ y) \circ e = e \circ (y \circ x) + (x \circ e) \circ (y \circ e)$.

If, in addition, the Peirce grading in J satisfies (2.2)(1)*, then also*

(iv) $(x^2 \circ e) \circ x = x^2 \circ (e \circ x)$,
(v) $(x^2 \circ y) \circ e + ((x \circ e) \circ y) \circ x = x^2 \circ (y \circ e) + (x \circ e) \circ (y \circ x)$.

Proof. Using (0.6), for any $x, y \in J$,

$$x^2 = \underbrace{x_2^2 + \pi_2\big(x_1^2\big)}_{=\pi_2(x^2)} + \underbrace{x_1 \circ x_2 + x_1 \circ x_0}_{=\pi_1(x^2)} + \underbrace{x_0^2 + \pi_0(x_1^2)}_{=\pi_0(x^2)}, \tag{1}$$

$$x \circ y = \underbrace{x_2 \circ y_2 + \pi_2(x_1 \circ y_1)}_{=\pi_2(x\circ y)} + \underbrace{x_1 \circ y_2 + x_1 \circ y_0 + x_2 \circ y_1 + x_0 \circ y_1}_{=\pi_1(x\circ y)} + \underbrace{x_0 \circ y_0 + \pi_0(x_1 \circ y_1)}_{=\pi_0(x\circ y)}. \tag{2}$$

Using (1), (2), (0.6), and (2.5), one can readily check:

(i) $(e \circ e) \circ x + ((x \circ e) \circ e) \circ e = e \circ (e \circ x) + (x \circ e) \circ (e \circ e) = 12x_2 + 3x_1$,
(ii) $(x^2 \circ e) \circ e + ((x \circ e) \circ e) \circ x = x^2 \circ (e \circ e) + (x \circ e) \circ (e \circ x) = 12x_2^2 + 4\pi_2(x_1^2) + 2x_1^2 + 6x_1 \circ x_2 + 2x_1 \circ x_0$,
(iii) $(e \circ y) \circ x + ((x \circ e) \circ y) \circ e = e \circ (y \circ x) + (x \circ e) \circ (y \circ e) = 6x_2 \circ y_2 + 3y_2 \circ x_1 + 3y_1 \circ x_2 + x_1 \circ y_1 + 2\pi_2(x_1 \circ y_1) + x_0 \circ y_1 + x_1 \circ y_0$.

Now, let us assume that the Peirce grading in J satisfies (2.2)(1). Using (1), (2), (0.6), and (2.5), we obtain:

$$\begin{aligned}\big(x^2 \circ e\big) \circ x = 4x_2^3 &+ 2\pi_2\big(x_1^2\big) \circ x_2 + 2x_2^2 \circ x_1 + 2\pi_2\big(x_1^2\big) \circ x_1 \\ &+ (x_1 \circ x_2) \circ x_2 + (x_1 \circ x_2) \circ x_1 + (x_1 \circ x_2) \circ x_0 \\ &+ (x_1 \circ x_0) \circ x_2 + (x_1 \circ x_0) \circ x_1 + (x_1 \circ x_0) \circ x_0\end{aligned} \tag{3}$$

and

$$\begin{aligned}x^2 \circ (e \circ x) = 4x_2^3 &+ 2\pi_2\big(x_1^2\big) \circ x_2 + 2(x_1 \circ x_2) \circ x_2 + 2(x_1 \circ x_0) \circ x_2 \\ &+ x_2^2 \circ x_1 + \pi_2\big(x_1^2\big) \circ x_1 + (x_1 \circ x_2) \circ x_1 \\ &+ (x_1 \circ x_0) \circ x_1 + x_0^2 \circ x_1 + \pi_0\big(x_1^2\big) \circ x_1.\end{aligned} \tag{4}$$

By (2.1)(i)(ii), and (2.2)(1), (3) and (4) coincide, i.e., (iv) holds.

Let $a=(x^2\circ y)\circ e+((x\circ e)\circ y)\circ x$, $b=x^2\circ(y\circ e)+(x\circ e)\circ(y\circ x)$.

Using (1), (2), (0.6), and (2.5), it can be checked that

$$\begin{aligned}\pi_2(a)=2x_2^2\circ y_2+2x_1^2\circ y_2+2(x_2\circ y_2)\circ x_2+(x_1\circ y_1)\circ x_2\\ +\pi_2\big(2(x_1\circ x_2)\circ y_1+2(x_1\circ x_0)\circ y_1+2(x_2\circ y_1)\circ x_1\\ +(x_1\circ y_2)\circ x_1+(x_1\circ y_0)\circ x_1\big),\end{aligned}$$

$$\begin{aligned}\pi_2(b)=2x_2^2\circ y_2+2x_1^2\circ y_2+2x_2\circ(x_2\circ y_2)+2x_2\circ(x_1\circ y_1)\\ +\pi_2\big((x_1\circ x_2)\circ y_1+(x_1\circ x_0)\circ y_1+x_1\circ(x_1\circ y_2)\\ +x_1\circ(x_1\circ y_0)+x_1\circ(x_2\circ y_1)+x_1\circ(x_0\circ y_1)\big),\end{aligned}$$

and $\pi_2(a)=\pi_2(b)$ because

$$\pi_2\big((x_1\circ x_2)\circ y_1+(x_1\circ x_0)\circ y_1+(x_2\circ y_1)\circ x_1\big)=x_2\circ(x_1\circ y_1)+\pi_2\big(x_1\circ(x_0\circ y_1)\big)$$

by (2.1)(iii) and (0.5)(ii).

We also have, by (1), (2), (0.6) and (2.5)

$$\begin{aligned}\pi_1(a)=(x_1\circ x_2)\circ y_2+(x_1\circ x_0)\circ y_2+(x_1\circ x_2)\circ y_0+(x_1\circ x_0)\circ y_0\\ +x_2^2\circ y_1+x_1^2\circ y_1+x_0^2\circ y_1+2(x_2\circ y_2)\circ x_1+(x_1\circ y_1)\circ x_1\\ +2(x_2\circ y_1)\circ x_2+2(x_2\circ y_1)\circ x_0+(x_1\circ y_2)\circ x_2+(x_1\circ y_2)\circ x_0\\ +(x_1\circ y_0)\circ x_2+(x_1\circ y_0)\circ x_0,\end{aligned}$$

$$\begin{aligned}\pi_1(b)=2(x_1\circ x_2)\circ y_2+2(x_1\circ x_0)\circ y_2+x_2^2\circ y_1+x_1^2\circ y_1+x_0^2\circ y_1\\ +2x_2\circ(x_1\circ y_2)+2x_2\circ(x_1\circ y_0)+2x_2\circ(x_2\circ y_1)+2x_2\circ(x_0\circ y_1)\\ +x_1\circ(x_2\circ y_2)+x_1\circ(x_1\circ y_1)+x_1\circ(x_0\circ y_0),\end{aligned}$$

and $\pi_1(a)=\pi_1(b)$ because

$$\begin{aligned}(x_1\circ x_2)\circ y_0+(x_1\circ x_0)\circ y_0+(x_2\circ y_2)\circ x_1+2(x_2\circ y_1)\circ x_0+(x_1\circ y_2)\circ x_0+(x_1\circ y_0)\circ x_0\\ =(x_1\circ x_2)\circ y_2+(x_1\circ x_0)\circ y_2+x_2\circ(x_1\circ y_2)+x_2\circ(x_1\circ y_0)+2x_2\circ(x_0\circ y_1)+x_1\circ(x_0\circ y_0)\end{aligned}$$

using (2.1)(i), (ii).

Let φ be the map defined in (2.6). By (2.6)(1)(2), and the fact that φ is a quadratic algebra homomorphism (2.7),

$$\begin{aligned}\varphi\big(\pi_0(a-b)\big)&=\pi_2'\big(\varphi(a-b)\big)=\pi_2'\Big(\big(x^{2'}\circ' y\big)\circ'(1-e)+\big(\big(x\circ'(1-e)\big)\circ' y\big)\circ' x-\Big(x^{2'}\circ'\Big(y\circ'(1-e)\Big)\\ &\quad+\Big(x\circ'(1-e)\Big)\circ'(y\circ' x)\Big)\Big)\\ &=\pi_2'\Big(\big(x^{2'}\circ' y\big)\circ' 1+\big((x\circ' 1)\circ' y\big)\circ' x-\Big(x^{2'}\circ'(y\circ' 1)+(x\circ' 1)\circ'(y\circ' x)\Big)\Big)\\ &\quad-\pi_2'\Big(\big(x^{2'}\circ' y\big)\circ' e+\big((x\circ' e)\circ' y\big)\circ' x-\Big(x^{2'}\circ'(y\circ' e)+(x\circ' e)\circ'(y\circ' x)\Big)\Big)\\ &=\pi_2'\Big(2x^{2'}\circ' y+(2x\circ' y)\circ' x-\Big(x^{2'}\circ' 2y+2x\circ'(y\circ' x)\Big)\Big)\end{aligned}$$

$$- \pi_2'\big(\big(x^{2'} \circ' y\big) \circ' e + \big((x \circ' e) \circ' y\big) \circ' x - \big(x^{2'} \circ' (y \circ' e) + (x \circ' e) \circ' (y \circ' x)\big)\big)$$
$$= \pi_2'(0) - \pi_2'\big(\big(x^{2'} \circ' y\big) \circ' e + \big((x \circ' e) \circ' y\big) \circ' x - \big(x^{2'} \circ' (y \circ' e) + (x \circ' e) \circ' (y \circ' x)\big)\big) = 0$$

since $\tilde{J}'$ satisfies the same conditions as $\tilde{J}$.

By injectivity of φ (2.7), $\pi_0(a-b) = 0$, and we have shown $\pi_\alpha(a-b) = 0$ for any $\alpha = 0, 1, 2$, i.e., $a - b = 0$, that is to say, (v) holds. □

3.5. Theorem. *If $J = J_2 \oplus J_1 \oplus J_0$ is a Peirce graded Jordan algebra, then $\tilde{J}$ satisfies* (3.3)(i)*. If the Peirce grading of J satisfies* (2.2)(1) *strictly, then $\tilde{J}$ satisfies* (3.3)(ii)*. If, in addition, J does not have 2-torsion, then $\tilde{J}$ is a Jordan algebra, and $J_\alpha = \tilde{J}_\alpha(e) \cap J$, for any $\alpha \in \{0, 1, 2\}$.*

Proof. By the definition (2.5) of $\tilde{J}$, for any $\alpha, \beta \in \Phi$, $x, y \in J$,

$$\begin{aligned}
&\big((x+\alpha e) \circ (y+\beta e)\big) \circ (x+\alpha e) - (x+\alpha e)^2 \circ (y+\beta e) \\
&\quad = \big(x \circ y + \beta(2x_2 + x_1) + \alpha(2y_2 + y_1) + 2\alpha\beta e\big) \circ (x+\alpha e) - \big(x^2 + \alpha(2x_2 + x_1) + \alpha^2 e\big) \circ (y+\beta e) \\
&\quad = (x \circ y) \circ x - x^2 \circ y + \alpha\big((2y_2 + y_1) \circ x + e \circ (x \circ y) - (2x_2 + x_1) \circ y\big) \\
&\qquad + \alpha^2\big(e \circ (2y_2 + y_1) - e \circ y\big) + \alpha\beta\big(e \circ (2x_2 + x_1) + 2e \circ x - e \circ (2x_2 + x_1)\big) \\
&\qquad + \alpha^2\beta(2e \circ e - e \circ e) + \beta\big((2x_2 + x_1) \circ x - e \circ x^2\big) \\
&\quad = (x \circ y) \circ x - x^2 \circ y + 2\alpha\big(y_2 \circ x_1 + y_1 \circ x_0 + x_2 \circ y_2 + \pi_2(x_1 \circ y_1)\big) \\
&\qquad + 2\alpha^2 y_2 + 2\alpha\beta(2x_2 + x_1) + 2\alpha^2\beta e + 2\beta\big(x_2^2 + x_2 \circ x_1 + \pi_0\big(x_1^2\big)\big),
\end{aligned} \tag{1}$$

using (3.4)(1)(2), (0.6), and (2.5). Now, (2.5)(b) and (1) with (0.5)(ix) for J imply (3.3)(i) for $\tilde{J}$.

Let us now assume that the Peirce grading of J satisfies (2.2)(1). Using that e is an idempotent of $\tilde{J}$ (2.5)(ii), we have

$$\begin{aligned}
\big((x+\alpha e)^2 \circ (y+\beta e)\big) \circ (x+\alpha e) &= \big(\big(x^2 + \alpha x \circ e + \alpha^2 e\big) \circ (y+\beta e)\big) \circ (x+\alpha e) \\
&= \big(x^2 \circ y\big) \circ x + \alpha\big(\big(x^2 \circ y\big) \circ e + \big((x \circ e) \circ y\big) \circ x\big) \\
&\quad + \alpha^2\big((e \circ y) \circ x + \big((x \circ e) \circ y\big) \circ e\big) \\
&\quad + \alpha^3 (e \circ y) \circ e + \beta\big(x^2 \circ e\big) \circ x + \alpha\beta\big(\big(x^2 \circ e\big) \circ e + \big((x \circ e) \circ e\big) \circ x\big) \\
&\quad + \alpha^2\beta\big((e \circ e) \circ x + \big((x \circ e) \circ e\big) \circ e\big) + \alpha^3\beta(e \circ e) \circ e,
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
(x+\alpha e)^2 \circ \big((y+\beta e) \circ (x+\alpha e)\big) &= \big(x^2 + \alpha x \circ e + \alpha^2 e\big) \circ \big((y+\beta e) \circ (x+\alpha e)\big) \\
&= x^2 \circ (y \circ x) + \alpha\big(x^2 \circ (y \circ e) + (x \circ e) \circ (y \circ x)\big) \\
&\quad + \alpha^2\big(e \circ (y \circ x) + (x \circ e) \circ (y \circ e)\big) \\
&\quad + \alpha^3 e \circ (y \circ e) + \beta x^2 \circ (e \circ x) + \alpha\beta\big(x^2 \circ (e \circ e) + (x \circ e) \circ (e \circ x)\big) \\
&\quad + \alpha^2\beta\big(e \circ (e \circ x) + (x \circ e) \circ (e \circ e)\big) + \alpha^3\beta e \circ (e \circ e).
\end{aligned} \tag{3}$$

The equality between (2) and (3) follows from (0.5)(iv) applied to J, and (3.4). Moreover, any scalar extension of $\tilde{J}$ comes from the corresponding scalar extension of J (naturally Peirce graded, and satisfying (2.2)(1) since J satisfies it strictly) through the construction (2.5), hence it also satisfies the equality (2) = (3), i.e., $\tilde{J}$ satisfies (3.3)(ii).

By (3.3), the polynomials $4j_1$, $8j_2$, $8j_3$, $32j_4$, $16j_5$, and $64j_6$ vanish strictly on $\tilde{J}$, hence, for any linearization p of j_i, $i = 1, \dots, 6$, $2^k p$ vanishes on $\tilde{J}$, for some k. Now let us assume that J does not have 2-torsion, and recall the natural projections μ, τ defined in (2.5). We have that $0 = \mu(2^k p(\tilde{J})) = 2^k \mu(p(\tilde{J}))$ implies $\mu(p(\tilde{J})) = 0$ since $\mu(p(\tilde{J})) \subseteq J$. On the other hand, since τ is a quadratic algebra homomorphism (2.5), $\tau(p(\tilde{J})) \subseteq p(\tau(\tilde{J})) \subseteq p(\Phi e) = 0$ because Φe is a Jordan algebra. Thus $p(\tilde{J}) = 0$, and we have shown that the polynomials j_i, $i = 1, \dots, 6$, vanish strictly on $\tilde{J}$, i.e., $\tilde{J}$ is a Jordan algebra. Finally, the equalities $J_\alpha = \tilde{J}_\alpha(e) \cap J$, for any $\alpha \in \{0, 1, 2\}$, are immediate consequences of (0.8) and (2.5). □

3.6. The unitization $\hat{J}$ of J (0.4) is a particular case of $\tilde{J}$ when we consider in J the trivial Peirce grading given by $J_2 = J$, $J_1 = J_0 = 0$, which obviously satisfies (2.2)(1) strictly, and write $e = 1$. In this sense, (3.5) can be viewed as a generalization of [5, Theorem 5] in absence of 2-torsion.

3.7. Notice that if J does not have 2-torsion and satisfies (2.2)(1), then it automatically satisfies it strictly, which happens with any identity of degree at most three in each of its variables.

3.8. Corollary. *A Peirce grading $J = J_2 \oplus J_1 \oplus J_0$ of a Jordan algebra without 2-torsion J satisfies* (2.2)(1) *if and only if there exists a Jordan algebra $\tilde{J}$ with an idempotent e such that J is a subalgebra of $\tilde{J}$, and $J_\alpha = \tilde{J}_\alpha(e) \cap J$, for any $\alpha \in \{0, 1, 2\}$.*

Proof. The "only if" follows from (3.5) and (3.7), while the "if" is proved in (2.2). □

In the following results, we will show that basic conditions of regularity on a Peirce graded algebra automatically imply (2.2)(1).

3.9. Lemma. *Let $J = J_2 \oplus J_1 \oplus J_0$ be a Peirce graded Jordan algebra. Then $z \circ J = z^2 = 0$, for any $z = x_1^3 - \pi_2(x_1^2) \circ x_1$, with $x_1 \in J_1$.*

Proof. Using (0.6)(iv), $x_1^2 = a_2 + a_0$, where $a_2 = \pi_2(x_1^2) \in J_2$, $a_0 = \pi_0(x_1^2) \in J_0$. If $y_0 \in J_0$, then $\{x_1, a_2, y_0\} = \{a_0, x_1, y_0\} = 0$ by (0.6)(ii)(iii), hence

$$\begin{aligned}\big(\pi_2\big(x_1^2\big) \circ x_1\big) \circ y_0 &= (a_2 \circ x_1) \circ y_0 = \{a_2, x_1, y_0\} + \{x_1, a_2, y_0\} \quad \big(\text{by } (0.5)(\text{ii})\big)\\ &= \{a_2, x_1, y_0\} = \{a_2, x_1, y_0\} + \{a_0, x_1, y_0\} = \big\{x_1^2, x_1, y_0\big\} = x_1^3 \circ y_0\end{aligned}$$

by (0.5)(i), which implies $z \circ J_0 = 0$.

By (0.7), also $z \circ J_2 = 0$. Let $y_1 \in J_1$

$$\begin{aligned}\big(\pi_2\big(x_1^2\big) \circ x_1\big) \circ y_1 &= (a_2 \circ x_1) \circ y_1 = \{a_2, x_1, y_1\} + \{x_1, a_2, y_1\} \quad \big(\text{by } (0.5)(\text{ii})\big)\\ &= \pi_2\big(\big\{x_1^2, x_1, y_1\big\}\big) + \pi_0\big(\big\{x_1, x_1^2, y_1\big\}\big) \quad \big(\text{by } (2.1)(\text{iv})\big)\\ &= \pi_2\big(x_1^3 \circ y_1\big) + \pi_0\big(x_1^3 \circ y_1\big) \quad \big(\text{by } (0.5)(\text{i})\big)\\ &= x_1^3 \circ y_1,\end{aligned}$$

which implies $z \circ J_1 = 0$, and we have shown $z \circ J = 0$.

Now,

$$\begin{aligned}\big(\pi_2\big(x_1^2\big) \circ x_1\big)^2 &= (a_2 \circ x_1)^2 = a_2 \circ U_{x_1} a_2 + U_{a_2} x_1^2 + U_{x_1} a_2^2 \quad \big(\text{by } (0.5)(\text{v})\big)\\ &= U_{a_2} x_1^2 + U_{x_1} a_2^2 \quad \big(\text{by } (0.6)(\text{i})(\text{v})\big)\end{aligned}$$

$$
\begin{aligned}
&= U_{a_2}(a_2 + a_0) + U_{x_1} a_2^2 = U_{a_2} a_2 + U_{x_1} a_2^2 \quad \big(\text{by (0.6)(i)}\big) \\
&= U_{a_2 + a_0} a_2 + U_{x_1} a_2^2 \quad \big(\text{by (0.6)(i)(iii)}\big) \\
&= U_{x_1^2} a_2 + U_{x_1} a_2^2 = U_{x_1} U_{x_1} a_2 + U_{x_1} a_2^2 \quad (\text{by QJ5}) \\
&= U_{x_1}\big(U_{x_1} a_2 + a_2^2\big) = U_{x_1}\big(\pi_0\big(U_{x_1} x_1^2\big) + \pi_2\big(x_1^4\big)\big) \quad \big(\text{by (2.1)(v)(vi)}\big) \\
&= U_{x_1}\big(\pi_0\big(x_1^4\big) + \pi_2\big(x_1^4\big)\big) \quad \big(\text{by (0.5)(vi)}\big) \\
&= U_{x_1} x_1^4 \quad \big(\text{by (0.6)(iv)(v)}\big) \\
&= x_1^6, \qquad (1)
\end{aligned}
$$

hence

$$
\begin{aligned}
z^2 &= \big(x_1^3 - \pi_2\big(x_1^2\big) \circ x_1\big)^2 = \big(x_1^3\big)^2 + \big(\pi_2\big(x_1^2\big) \circ x_1\big)^2 - x_1^3 \circ \big(\pi_2\big(x_1^2\big) \circ x_1\big) \\
&= x_1^6 + x_1^6 - \big(z + \big(\pi_2\big(x_1^2\big) \circ x_1\big)\big) \circ \big(\pi_2\big(x_1^2\big) \circ x_1\big) \quad \big(\text{by (0.5)(vi) and (1)}\big) \\
&= x_1^6 + x_1^6 - \big(\pi_2\big(x_1^2\big) \circ x_1\big) \circ \big(\pi_2\big(x_1^2\big) \circ x_1\big) \quad (\text{since } z \circ J = 0) \\
&= x_1^6 + x_1^6 - 2\big(\pi_2\big(x_1^2\big) \circ x_1\big)^2 = 0
\end{aligned}
$$

by (1). □

3.10. Lemma. *Let J be a Jordan algebra, $z \in J$.*

(i) $z \circ J = z^2 = 0 \Rightarrow z^3 \circ J = (z^3)^2 = (z^3)^3 = 0$.
(ii) $z \circ J = z^3 = 0 \Rightarrow U_z U_J z = 0$.
(iii) $z \circ J = z^2 = z^3 = 0 \Rightarrow U_{U_z x} J = 0$, *for any $x \in J$.*

Proof. (i) For any $x \in J$,

$$
\begin{aligned}
z^3 \circ x &= \big\{z^2, z, x\big\} \quad \big(\text{by (0.5)(i)}\big) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\big(z^3\big)^2 &= z^6 = U_{z^2} z^2 \quad \big(\text{by (0.5)(vi)}\big) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\big(z^3\big)^3 &= z^9 = U_{z^2} z^5 \quad \big(\text{by (0.5)(vi)}\big) \\
&= 0.
\end{aligned}
$$

(ii) For any $x \in J$,

$$
\begin{aligned}
U_z U_x z &= U_{z \circ x} z - z \circ U_x z^2 - U_x z^3 \quad \big(\text{by (0.5)(vii)}\big) \\
&\subseteq U_{z \circ J} z - z \circ J - U_J z^3 = 0.
\end{aligned}
$$

(iii) For any $x, y \in J$,

$$\begin{aligned} U_{U_z x} y &= U_z U_x U_z y \quad \text{(by QJ6)} \\ &= U_z U_{x \circ z} y - \{z^2 \circ x, y, U_z x\} + \{x, z^2, y\} \circ U_z x \\ &\quad - \{y, z, U_z U_x z\} - U_{z^2} U_x y \quad \big(\text{by (0.5)(viii)}\big) \\ &= 0 \end{aligned}$$

by the hypotheses and (ii). □

3.11. Theorem. *If $J = J_2 \oplus J_1 \oplus J_0$ is a Peirce graded Jordan algebra, then* (2.2)(1) *holds strictly when J is under any of the following circumstances*:

(i) *J does not have 2-torsion and does not contain nonzero invisible elements,*
(ii) *J is special and semiprime,*
(iii) *J is nondegenerate.*

Proof. We remark that any of the properties (i)–(iii) on J is inherited by the scalar extension $J \otimes_\Phi \tilde{\Phi}$ of J when $\tilde{\Phi}$ is the unital, associative, commutative ring of Φ-polynomials in an infinite set of variables. Then, if we show that (2.2)(1) holds on J, it will also hold on $J \otimes_\Phi \tilde{\Phi}$, which implies (2.2)(1) holds strictly on J.

Let $x_1 \in J_1$, and $z = x_1^3 - \pi_2(x_1^2) \circ x_1$. We have to show that, assuming (i), (ii) or (iii), $z = 0$.

(i) By (3.9), $z \circ J = 0$, which implies that z is invisible (cf. [8]) since J does not have 2-torsion. Hence $z = 0$.

(ii) Since J is special, we can find an associative algebra A such that it is an envelope of J, i.e., $J \leqslant A^{(+)}$ and A is generated as an associative algebra by J. Let I be the ideal of A generated by z. Since $z^2 = z \circ J = 0$, it can be readily seen that $II = 0$. Then $L = I \cap J$ is an ideal of J such that $U_L L \subseteq III = 0$, which, by semiprimeness of J, implies $L = 0$. Hence $z \in I \cap J = L = 0$.

(iii) Notice that $t := z^3$ satisfies $t \circ J = t^2 = t^3 = 0$ by (3.10)(i). Hence, for any $x \in J$, $U_t x$ is an absolute zero divisor of J by (3.10)(iii). By nondegeneracy of J, $U_t x = 0$ for any $x \in J$, i.e., t itself is an absolute zero divisor of J, hence $t = 0$ again by nondegeneracy. Thus, $z^3 = 0$, and (3.10)(iii) implies that $U_z x$ is an absolute zero divisor of J, for any $x \in J$, but this implies, as above, that $z = 0$. □

3.12. Corollary. *If $J = J_2 \oplus J_1 \oplus J_0$ is a Peirce graded Jordan algebra without 2-torsion in any of the situations* (3.11)(i)–(iii)*, then $\tilde{J}$ is a Jordan algebra with an idempotent e such that J is a subalgebra of $\tilde{J}$, and $J_\alpha = \tilde{J}_\alpha(e) \cap J$, for any $\alpha \in \{0, 1, 2\}$.*

3.13. Further comments. The condition of absence of 2-torsion can be removed in (3.5) (hence in (3.8) and (3.12)). In that general setting, the proof is much more involved and lengthy and will be the subject of a forthcoming paper.

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