

The geodesic completeness of compact Lorentzian manifolds admitting a timelike Killing vector field revisited: two new proofs

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ABSTRACT. Compact Lorentzian manifolds admitting a timelike Killing vector field are shown to be complete by means of two different proofs to the original one.

1. Introduction

Contrary to the Riemannian case, a compact Lorentzian manifold may be geodesically incomplete (see for instance [8, p. 193], [9], [10]). This striking fact motivated the search of sufficient assumptions under which compactness implies (geodesically) completeness of such a manifold or more generally of a compact indefinite Riemannian manifold [10]. On the other hand, an homogeneous indefinite Riemannian manifold needs not to be complete [8, p. 257]. However, in 1973, J.E. Marsden proved that any compact indefinite homogeneous Riemannian manifold must be complete [7], [1, Th. 4.2.22]. In Marsden's result the key idea is to show that the tangent bundle can be decomposed into a disjoint union of compact subsets which are invariant under the geodesic flow. Thus, any integral curve of the geodesic vector field remains in a compact subset of the tangent bundle, which implies that it (and hence its projection on the manifold) can be defined on all the real line.

It should be remarked that any tangent vector v at a point p of an homogeneous semi-Riemannian manifold (M, g) can be extended to a globally defined Killing vector field V . Note that if $g_p(v, v) < 0$ then $g(V, V) < 0$ near the point p . However, V is not timelike everywhere, in general. On the other hand, recall that a necessary and sufficient condition for a compact manifold to admit a Lorentzian metric is the existence of a nowhere zero vector field [8]. Thus it is natural to link

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the completeness of a compact Lorentzian manifold with the existence of a suitable timelike vector field.

In 1993, Y. Kamishima proved in that a compact Lorentzian manifold which admits a timelike Killing vector field and has constant sectional curvature must be complete [5]. His technique depends of the rich group machinery of spaces of constant sectional curvature and it is completely different from Marsden's one. With respect to the curvature assumption in Kamishima's result, B. Klinger proved in 1996 that every compact Lorentzian manifold of constant sectional curvature c must be complete [6], extending a previous result by Y. Carrière [3] when $c = 0$. Moreover, note that there is no compact Lorentzian manifold (M, g) of constant sectional curvature $c > 0$, which follows from Klinger's theorem and a classical result of E. Calabi and L. Markus [2].

In 1995, A. Romero and M. Sánchez proved that a compact Lorentzian manifold which admits a timelike conformal vector field must be complete [11]. Recall that a vector field is called conformal if any of its (local) flows consists of (local) conformal transformations. In particular, a Killing vector field is clearly conformal. The existence of a timelike conformal vector field imposes a serious restriction on the topology of an n -dimensional compact Lorentzian manifold, indeed, this implies that a compact Lorentzian manifold is topologically a Seifert fiber space. However, for a fixed topology the family of such Lorentzian metrics is very wide. In dimension two, a complete description of all Lorentzian metrics on a 2-tori which admit a nontrivial Killing vector field was given by M. Sánchez in [12].

The main aim of this note is to give two new proofs of the result in [11] in the case Killing,

A compact Lorentzian manifold which admits a timelike Killing vector field must be geodesically complete.

The two new proofs which follow could bring the researchers in analytical mechanics or second order ODEs on manifolds closer to this topic of Lorentzian geometry.

In Section 2 we will follow a different strategy from the one in [11]. In fact, we will pay attention to two numbers associated to each geodesic γ , the first one showing its causal character and the second one given by the conservation law (2.2). Using each couple of these numbers, we will construct a compact subbundle of the tangent bundle TM invariant under the geodesic flow and such that the velocity vector field of γ lies within it at any value of its parameter. Finally, according also to the causal character of the geodesics, we construct in Section 3 three subbundles of TM , each of them invariant under the geodesic flow and under suitable assumptions in order to apply Gordon's approach, concluding the completeness of the restriction of the geodesic vector field in each case.

2. Hamiltonian approach à la Marsden

Let (M, g) be a compact Lorentzian manifold which admits a timelike Killing vector field K . The geodesics of (M, g) can be characterized as solutions of the Hamiltonian system given by

$$H : TM \rightarrow \mathbb{R}, \quad H(p, v) = \frac{1}{2} g_p(v, v).$$

for all $(p, v) \in TM$. Consider

$$(2.1) \quad J : TM \rightarrow \mathbb{R}, \quad J(p, v) = g_p(K_p, v),$$

for all $(p, v) \in TM$. Taking into account that K is Killing, if γ is a geodesic in M , then we get

$$(2.2) \quad J(\gamma(t), \gamma'(t)) = \alpha.$$

for all t . Notice that the function J is the associated momentum function relative to the (complete) infinitesimal generator K of the action [1, Sect. 4.2], and $\alpha \in \mathbb{R}$ depends on the geodesic γ in M . Moreover, the conservation law (2.2) is just the Noether Theorem for the mechanical system.

Now, in (M, g) consider the time orientation defined by K . Thus, let us consider a future pointing unit timelike geodesic in (M, g) , i.e., a geodesic $\gamma : I \rightarrow M$, $0 \in I$, with $\gamma(0) = p \in M$ and $\gamma'(0) = v \in T_p M$, $g_p(v, v) = -1$ and $g_p(K_p, v) < 0$. We may assume $g(K, K) < -1$ (otherwise, we may change K to the Killing vector λK with a suitable positive number λ).

The curve in the tangent bundle TM given by $t \mapsto (\gamma(t), \gamma'(t))$ lies in the following subbundle of TM

$$\mathcal{T}M^\alpha = \bigcup_{p \in M} \mathcal{T}M_p^\alpha, \quad \mathcal{T}M_p^\alpha := \{v \in T_p M : g_p(v, v) = -1, J(p, v) = \alpha\},$$

where $\alpha = g_p(K_p, v) < 0$. Notice that the submanifold $\mathcal{T}M^\alpha$ of TM is invariant under the geodesic flow. If a unit timelike geodesic is past pointing, changing its parameter by its opposite we have a geodesic under the previous assumptions.

For each $p \in M$, $\mathcal{T}M_p^\alpha$ is the intersection of the future component of the hyperbolic space in $T_p M$ and the spacelike affine hyperplane $\{v \in T_p M : J(p, v) = \alpha\}$. Thus, $\mathcal{T}M_p^\alpha$ is homeomorphic to an $(n-2)$ -dimensional Euclidean sphere and, in particular, a compact subset of $T_p M$. Consequently, $\mathcal{T}M^\alpha$ is also compact and then the restriction of the geodesic vector field on $\mathcal{T}M^\alpha$ is complete. Therefore, γ may be extended as a geodesic on all \mathbb{R} .

A similar argument works when the geodesic γ is assumed to be spacelike or (future pointing) lightlike, yielding respectively the subbundles of TM

$$\mathcal{S}M^\beta := \{(p, v) \in TM : g_p(v, v) = 1, J(p, v) = \beta \in \mathbb{R}\},$$

$$\mathcal{L}M^\varepsilon := \{(p, v) \in TM : g_p(v, v) = 0, J(p, v) = \varepsilon < 0\}.$$

Both of them are invariant under the geodesic flow. The subspace $\{v \in T_p M : g_p(v, v) = 1\}$ of $T_p M$ is diffeomorphic to an $(n - 1)$ -dimensional De Sitter space-time whereas $\{v \in T_p M : g_p(v, v) = 0, g_p(K_p, v) < 0\}$ is diffeomorphic to an $(n - 1)$ -dimensional future light cone. Then, the intersection with the corresponding spacelike affine hyperplane is also homeomorphic to an $(n - 2)$ -dimensional Euclidean sphere and, in particular, a compact subset of $T_p M$. Consequently, the corresponding fiber bundles are compact, which ends the proof.

3. Approach à la Gordon

First recall the following technical result [4] (see also [1, Lemma 2.1.20]) to be used later.

LEMMA 3.1. *Let $X \in \mathfrak{X}(N)$ be a vector field on a manifold N . Suppose that*

- (i) *There exists a function $f \in C^\infty(N)$ such that $|X_p(f)| \leq \delta_1 |f(p)|$, for any $p \in N$, where δ_1 is a non-negative constant.*
- (ii) *There exists a proper function $h \in C^\infty(N)$ such that $|h(p)| \leq \delta_2 |f(p)|$, for any $p \in N$, where δ_2 is a positive constant.*

Then, X is complete.

Now, as in Section 2, let (M, g) be a compact Lorentzian manifold with a timelike Killing vector field K . Formula (2.2) suggests to take $N = TM$ and $f = J$, the momentum function, in order to use Lemma 3.1. In fact, assumption (i) is automatically satisfied by X the geodesic vector field, with $\delta_1 = 0$. However, no smooth function $h : TM \rightarrow \mathbb{R}$ satisfying (ii) is proper. Note that if such a function does exist then it takes the zero value. Let p_0 be a fixed arbitrary point of M and consider $K_{p_0}^\perp = \{v \in T_{p_0} M : g_{p_0}(K_{p_0}, v) = 0\}$. Clearly, $K_{p_0}^\perp \subset h^{-1}(0)$ and $K_{p_0}^\perp$ is closed and noncompact. Therefore, h is not proper. The following argument is a way to avoid this inconvenient.

Consider $N = \mathcal{P}M^\varepsilon := \bigcup_{p \in M} \mathcal{P}M_p^\varepsilon$, $\varepsilon = -1, 0, +1$, where

$$\mathcal{P}M_p^\varepsilon := \{v \in T_p M : g(v, v) = \varepsilon\}, \quad \text{for } \varepsilon = -1, +1,$$

$$\mathcal{P}M_p^0 := \{v \in T_p M : g(v, v) = 0, g_p(K_p, v) < 0\}.$$

Note that the subbundle $\mathcal{P}M^\varepsilon$ of TM is invariant under the geodesic flow. Denote by G^ε the restriction on $\mathcal{P}M^\varepsilon$ of the geodesic vector field G .

In order to apply Lemma 3.1, we take $f, h : \mathcal{P}M^\varepsilon \rightarrow \mathbb{R}$ defined as

$$f(p, v) = h(p, v) := J(p, v),$$

for all $(p, v) \in \mathcal{P}M^\varepsilon$. It is not difficult to see that f is proper.

We have

$$G^\varepsilon(f)_{(p,v)} = g\left(\frac{D}{dt}\Big|_{t=0}(K \circ \gamma)(t), \gamma'(t)\right) = 0,$$

where γ is the geodesic satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. Now, Lemma 3.1 with $\delta_1 = 0$ and $\delta_2 = 1$ allow us to end the proof.

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