Universidad de Oviedo
Universidá d'Uviéu
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Modern approaches to Quantum Field Theory

Enfoques modernos en Teoría Cuántica de Campos

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Tesis Doctoral

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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

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RESUMEN (en español)

La Teoría Cuántica de Campos es el formalismo más exitoso a la hora de modelizar y entender una amplia gama de fenómenos físicos, desde altas energías a materia condensada. Sin embargo, una vez alejados del régimen de validez de teoría de perturbaciones para constantes de acoplo débiles, no hay mucho que se pueda decir sobre su comportamiento. En esta tesis, abordamos el problema de TCC en régimen de acoplos fuertes de dos maneras: en la primera parte de la tesis estudiamos teorías supersimétricas, lo que permite encontrar resultados exactos. En particular, estudiamos los espacios de moduli y simetrías generalizadas de teorías gauge con grupos gauge disconexos. En la segunda parte, consideramos teorías sencillas de campos escalares sin supersimetría, y estudiamos el origen microscópico de la expansión en cargas grandes en varios ejemplos, así como posibles aplicaciones.

RESUMEN (en Inglés)

Quantum Field Theory is the most successful formalism to model and understand a wide range of phenomena from high energy to condensed matter physics. Nevertheless, when outside of the regime of validity of perturbation theory at weak coupling, little can be said about its behaviour. In this thesis, we approach the problem of QFT at strong coupling in two ways: in the first half of the thesis we study supersymmetric theories, which allows to find exact results. In particular, we study the moduli spaces and generalised symmetries of gauge theories with disconnected gauge groups. In the second half, we consider simple scalar field theories without supersymmetry, and study the microscopic origin of the large charge expansion in some examples, as well as several possible applications.

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Abstract

Quantum Field Theory is the most successful formalism to model and understand a wide range of phenomena from high energy to condensed matter physics. Nevertheless, when outside of the regime of validity of perturbation theory at weak coupling, little can be said about its behaviour. In this thesis, we approach the problem of QFT at strong coupling in two ways: in the first half of the thesis we study supersymmetric theories, which allows to find exact results. In particular, we study the moduli spaces and generalised symmetries of gauge theories with disconnected gauge groups. In the second half, we consider simple scalar field theories without supersymmetry, and study the microscopic origin of the large charge expansion in some examples, as well as several possible applications.

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La Teoría Cuántica de Campos es el formalismo más exitoso a la hora de modelizar y entender una amplia gama de fenómenos físicos, desde altas energías a materia condensada. Sin embargo, una vez alejados del régimen de validez de teoría de perturbaciones para constantes de acoplo débiles, no hay mucho que se pueda decir sobre su comportamiento. En esta tesis, abordamos el problema de TCC en régimen de acoplos fuertes de dos maneras: en la primera parte de la tesis estudiamos teorías supersimétricas, lo que permite encontrar resultados exactos. En particular, estudiamos los espacios de moduli y simetrías generalizadas de teorías gauge con grupos gauge disconexos. En la segunda parte, consideramos teorías sencillas de campos escalares sin supersimetría, y estudiamos el origen microscópico de la expansión en cargas grandes en varios ejemplos, así como posibles aplicaciones.

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Chapter 1

Introduction

Quantum Field Theory (QFT) is one of the main pillars of modern theoretical physics. Since its inception in the mid 20th century, it has been successful in providing a framework for understanding a wide range of phenomena, ranging from the description of fundamental particles and their interaction, to the behaviour of the universe during its very early stages (especially during inflation), and through the study of various condensed matter systems.

During its early stages, the theoretical development of QFT ran in parallel to the experimental discovery of new subatomic particles during the 40's and 50's. In consonance with this, the formalism was designed to systematically deal with the situations arising in particle accelerators, allowing one to compute probabilities of different outcomes. Two of the key advancements during this time were the precise formulation of perturbation theory, both in terms of Feynman diagrams as well as the path integral; as well as the handling of infinities by means of renormalization.

However, it soon became apparent that these systematic methods, rooted on the existence of a small parameter encoding the strength of the interactions, are not effective to capture the behaviour of systems at strong coupling. This means that one needs new tools and methods to analyze the physics when the coupling constants are big.

Strong coupling phenomena arises in many physical situations. The most prominent example in particle physics is QCD at low energies. While the field theoretic model for strong interactions benefits from asymptotic freedom (i.e. the coupling is small at very high energies), meaning it makes sense as a fundamental theory at the smallest scales; when the energy decreases the coupling grows, and one cannot make use of perturbative QCD to answer seemingly elementary questions such as the inner structure of the proton. Other examples include condensed matter systems such as high temperature superconductors or topological phases of matter.

To date, the problem of QFT at strong coupling is not one that has a single, systematic and unequivocal answer. The best that one can attempt is to find methods that work in some situations, relying on assumptions that are heavily example-dependant. Some approaches include numerical simulations on the lattice, the use of holographic duality, and/or integrability.

In this work, we explore two tools that can serve to make an inroad in this question. The first is supersymmetry, which allows in some cases the computation of observables in an exact way, namely without making any assumption on the coupling, and therefore can be used to learn about strong coupling phenomena. The second is the large charge expansion, which sometimes can be used to employ perturbation theory when the coupling constant is big, since the inverse of the large charge becomes the small parameter instead.

Each of these two methods has their own advantages and disadvantages. Assuming the existence of supersymmetry is a big restriction on the quantum field theory, and it seems it is not realised in models relevant for the description of nature, at least in a straightforward way (whether it might be broken and realised at higher energies in the context of fundamental interactions, or emergent in the context of condensed matter systems remains an open question). On the other hand, for the toy models that do have supersymmetry, we can usually rephrase the problem in geometric language, which allows to study more subtle aspects of QFT by e.g. making contact with String Theory. Conversely, the large charge expansion is a very general tool that can be utilized in many theories with or without supersymmetry, in such a way that one has good control over the performed approximation. However, one lacks the exactness of SUSY and what that brings to the table with regards to the finer print of QFT.

This thesis is divided in mainly two parts, corresponding to these two approaches. In the first half we will discuss supersymmetric gauge theories, in the case where the gauge group is disconnected. This is based on references [1–3], preceded by a brief review of the basics of 4d $\mathcal{N} = 2$ SUSY. In the second half we will study a double scaling limit involving large charge in the simple ϕ^4 theory. This is based on [4–6], and is also preceded by a brief review of generalities of the large charge expansion. Each of these chapters includes a section collecting some conclusions and various remarks, but we also will end with a joint conclusion chapter putting together the lessons from the two approaches and considering some possible outlook.

Chapter 2

Supersymmetric theories with disconnected gauge groups

One of the main tools that allow for a degree of control on the strongly coupled phenomena of QFTs is supersymmetry. While the original motivation for its introduction, as a possible solution to the naturalness problem in particle physics, has fallen from grace due to the lack of experimental evidence for it at sufficiently low energies; it is still the case that SUSY can be invoked as a simplifying assumption, producing toy models for QFTs that are much more constrained than their non-supersymmetric cousins, while still maintaining rich non-perturbative dynamics. The hope is that, through the study of these simpler models, on can extract lessons that apply to less symmetric QFTs relevant for the description of nature.

An important quantity in this discussion is the amount of SUSY, given by the number of fermionic symmetry generators which in 4d can range from $\mathcal{N}=1$ to $\mathcal{N}=4$. Naturally, the larger the quantity of supersymmetries, the more constraints we have in our theory: as an example, for the case of $\mathcal{N}=4$, it is believed that the theory is completely specified by a choice of gauge group, it has conformal symmetry, and there are no UV divergences. This also implies that a number of strongly coupled phenomena, which we would like to explore, are abstent in this case. In this thesis we will focus on 4d $\mathcal{N}=2$ theories, which represent a sweet spot between the quantitative control of non-perturbative effects and the presence of rich dynamical features.

The key insight is that for $\mathcal{N}=2$ theories a lot of physical information is encoded in the geometry of their moduli space of vacua, which in turn can be studied by means of complex geometry. In particular, the goal in this chapter will be to study, using these tools, gauge theories based on disconnected gauge groups. This is a topological property that doesn't affect the local dynamics of the theory and only plays a role when taking into account the more subtle aspects of QFT; as such, SUSY provides us with a foothold for the understanding of said aspects.

The organization of this chapter is as follows: in section 2.1, we will provide a brief introduction to 4d $\mathcal{N}=2$ SUSY, leaving out many important aspects and instead making a beeline toward a handful of relevant concepts for the study of moduli spaces which we shall use frequently in later sections. The rest of the chapter is divided in three parts. In section 2.2 we will discuss the contents of [1], where the possible choices of disconnected gauge group (with $\mathrm{SU}(N)$ as the identity component) were classified, and some aspects of their moduli space of vacua were investigated using Hilbert Series techniques. Section 2.3 is based on [2], where we study the Higgs Branch of those theories in depth, including Magnetic Quivers and its stratification structure as a symplectic singularity. Finally, in section 2.4 we study the generalised symmetries of these theories [3], and find that the disconnected gauge group leads to a non-invertible 1-form symmetry even in the non-supersymmetric case. We finish in section 2.5 with some concluding remarks.

2.1 Lightning review of 4d $\mathcal{N}=2$ supersymmetry

In this section, we review the basic concepts of 4d $\mathcal{N}=2$ supersymmetry, somewhat following [7] and [8]. In 2.1.1 we introduce the $\mathcal{N}=2$ multiplets and write the most generic lagrangian for a gauge theory with matter. In 2.1.2 we discuss the moduli spaces of vacua of these gauge theories and define their Higgs and Coulomb branches. In 2.1.3 we present the geometric tools that we will use in later sections for the study of moduli spaces.

2.1.1 Multiplets and lagrangians

In 4d, the Poincaré algebra is generated by the Lorentz transformations $M_{\mu\nu}$ as well as the translations P_μ. This spacetime symmetry can be enlarged by adding fermionic generators $\mathcal{Q}^m_\alpha$ and $\widetilde{\mathcal{Q}}^n_{\dot\alpha}$, giving rise to the super-Poincaré algebra. Here, $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are Weyl spinors, α and $\dot\alpha$ are (anti)chiral spinor indices; and $m,n=1,\dots,\mathcal{N}$ are R-symmetry indices, which in the $\mathcal{N}=2$ case will be $\mathrm{U}(2)_R=\mathrm{U}(1)_r\times\mathrm{SU}(2)_R$. The non-trivial commutation relations of the supercharges are

$$\left\{\mathcal{Q}^m_\alpha,\widetilde{\mathcal{Q}}^n_{\dot\beta}\right\}=2\left(\sigma^\mu\right)_{\alpha\dot\beta}\,\epsilon^{mn}P_\mu\,, \tag{2.1.1}$$

$$\left\{\mathcal{Q}^m_\alpha,\mathcal{Q}^n_\beta\right\}=\delta_{\alpha\beta}\epsilon^{mn}\mathbf{Z}\,, \tag{2.1.2}$$

where σ^μ are the Pauli matrices, ϵ^{mn} is the Levi-Civita tensor, and $\mathbf{Z}$ is called the central charge.

In order to construct a 4d $\mathcal{N}=2$ lagrangian, we need to discuss the irreducible representations of the SUSY algebra, of which there are two types: the vector multiplet and the hypermultiplet.

- In order to build the vector multiplet, we start with one complex scalar ϕ which is annihilated by all the supercharges with a given chirality,

$$\mathcal{Q}^m|\phi\rangle = 0\,. \tag{2.1.3}$$

  Acting once with the remaining supercharges $\widetilde{\mathcal{Q}}^n$ will give rise to two spinors λ^n; and acting twice to a vector field A_μ. This vector field can be used to describe a gauge theory; thus its superpartners λ^n are known as the gauginos, who together with ϕ will transform in the adjoint of the gauge group. Note that the two gauginos transform as a doublet of the R-symmetry, while the scalar ϕ is charged under $\mathrm{U}(1)_r$ but not $\mathrm{SU}(2)_R$.

- In order to build the hypermultiplet, we start with two complex scalars q and $\widetilde{q}$, which are annihilated by the supercharges in the following way:

$$\mathcal{Q}^1|q\rangle = \widetilde{\mathcal{Q}}^1|q\rangle = \mathcal{Q}^1|\widetilde{q}\rangle = \widetilde{\mathcal{Q}}^1|\widetilde{q}\rangle = 0\,. \tag{2.1.4}$$

  Acting with $\mathcal{Q}^2$, $\widetilde{\mathcal{Q}}^2$ now we can obtain two Weyl fermions ψ and $\widetilde{\psi}$, but any further action of the supercharges will annihilate the state, i.e. we cannot reach a spin-1 field. Therefore, the hypermultiplets will be used to describe matter fields. Note that in this case the scalars q, $\widetilde{q}$ transform under the $\mathrm{SU}(2)_R$ symmetry, but are uncharged under $\mathrm{U}(1)_r$.

An efficient way to write down $\mathcal{N}=2$ lagrangians is to use the $\mathcal{N}=1$ superspace formalism and impose that it has a $\mathrm{SU}(2)_R$ R-symmetry; this can be acomplished by restricting the form of the superpotential and the Kahler potential, and it will automatically result in supersymmetry enhancement to $\mathcal{N}=2$. Recall that in terms of $\mathcal{N}=1$ multiplets, a $\mathcal{N}=2$ vector multiplet is made from a $\mathcal{N}=1$ vector (vector field and gaugino) plus a $\mathcal{N}=1$ chiral (scalar and fermion) in the adjoint of the gauge group; likewise a $\mathcal{N}=2$ hypermultiplet is made of two $\mathcal{N}=1$ chiral multiplets of opposite chirality.

In order to use the $\mathcal{N}=1$ superspace formalism, we introduce grassmanian coordinates

$(\theta_\alpha, \bar{\theta}_{\dot\alpha})$ and auxiliary fields D and F^j. Then, we can define

$$Q^j = q^j + i\theta\psi^j + \theta\theta F^j\,, \tag{2.1.5}$$

$$V = -i\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D\,, \tag{2.1.6}$$

$$W_\alpha = -i\lambda_\alpha + iF_{\mu\nu}\left(\sigma^{\mu\nu}\theta\right)_\alpha + D\theta_\alpha + \theta\theta\left(\sigma^\mu\partial_\mu\bar{\lambda}_\alpha\right)\,, \tag{2.1.7}$$

which are called chiral, vector and gaugino superfields respectively. In these terms, any gauge and Lorentz invariant combination of the superfields can be integrated over superspace to produce a $\mathcal{N}=1$ supersymmetric lagrangian. Requiring $\mathrm{SU}(2)_R$ R-symmetry places further restrictions. For example, in order to write down the lagrangian for a $\mathcal{N}=2$ vector multiplet, we need to combine the $\mathcal{N}=1$ lagrangians for a vector and a chiral multiplets with a specific choice of Kahler potential and overall coefficients (due to the exchange of the two fermions). This gives rise to the lagrangian for $\mathcal{N}=2$ super Yang-Mills (SYM) theory:

$$\mathcal{L}^{\mathrm{SYM}}_{\mathcal{N}=2} = \frac{1}{8\pi i}\int d^2\theta \mathrm{Tr}\,\left(\tau W_\alpha W^\alpha + \text{ c.c.}\right) + \frac{\mathrm{Im}(\tau)}{4\pi}\int d^4\theta \mathrm{Tr}\,\left(\Phi^\dagger e^{\mathrm{adj}(V)}\Phi\right)\,, \tag{2.1.8}$$

where τ is the complexified gauge coupling including the theta angle,

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}\,, \tag{2.1.9}$$

and Φ is a $\mathcal{N}=1$ chiral superfield involving the scalar ϕ and one of the two fermions λ analogously to (2.1.5).

In a similar fashion, we can construct a lagrangian for a $\mathcal{N}=2$ hypermultiplet starting with the lagrangians for two chiral multiplets, and restricting the superpotential and the overall coefficients in such a way that we respect the $\mathrm{SU}(2)_R$ R-symmetry that exchanges the two scalars. This results in

$$\mathcal{L}^{\mathrm{matter}}_{\mathcal{N}=2} = \int d^4\theta\left[Q^\dagger e^{\rho(V)}Q + \widetilde{Q}^\dagger e^{\bar{\rho}(V)}\widetilde{Q}\right] + \int d^2\theta\,\left[\widetilde{Q}\rho(\Phi)Q + m\widetilde{Q}Q + \text{ c.c.}\right]\,, \tag{2.1.10}$$

where ρ is the representation of the gauge group where the matter fields live. A generic $\mathcal{N}=2$ lagrangian will be given by a choice of fields and a combination of (2.1.8) and (2.1.10).

One of the most important consequences of $\mathcal{N}=2$ SUSY are the so called *non – renormalization theorems*, which come about as follows: First, note that the only classically marginal parameter in the $\mathcal{N}=2$ lagrangian is the holomorphic coupling τ. Assuming the existence of a renormalization scheme where holomorphicity is preserved leads to the conclusion that the renormalization of τ is 1-loop exact to all orders in perturbation theory, as the l-loop contribution would be proportional to $\mathrm{Im}(\tau)^{1-l}$. Further assuming that this renormalization scheme also preserves the R-symmetry further constraints the a priori possible wave-function renormalization factors for the different kinetic terms, and indeed it turns out they cannot be non-trivial.

These non-renormalization theorems, and the extended supersymmetry, have important consequences for the study of the low energy dynamics of $\mathcal{N}=2$ theories. They imply that there must exist a holomorphic function $\mathcal{F}$ of the chiral superfield Φ in the vector multiplet, called the prepotential, such that the various quantities in the effective action depend on its derivatives. For example, the coupling τ will be

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial \phi^2} \,. \tag{2.1.11}$$

In turn, as discovered in the seminal works [9, 10], the prepotential can be completely determined by the geometry of the moduli space of the theory, which we set to discuss in what follows.

2.1.2 Supersymmetric moduli space of vacua

We have seen that in a $\mathcal{N}=2$ lagrangian we must have a collection of scalar fields, coming from both vector multiplets as well as hypermultiplets. It is then natural to ask whether they can take VEVs and what are the possible vacua. In fact, due to the non-renormalization theorems, the classical moduli space of vacua is not lifted by quantum corrections, as opposed to non-supersymmetric theories and even $\mathcal{N}=1$ theories. Moreover, as shown by Seiberg and Witten, SUSY restricts the mathematical propierties of the geometry of these spaces, which can be exploited to obtain a great deal of information about the full theory [9, 10].

In order to find the moduli space, we need to find the minima of the scalar potential $V(\phi, q, \widetilde{q})$. A direct computation shows that these are given by the following set of equations,

$$\left[\phi, \phi^\dagger\right] = 0 \,, \tag{2.1.12}$$

$$\begin{cases} \left. \left(q_j q^{\dagger j} - \widetilde{q}^\dagger_j \widetilde{q}^j \right) \right|_{\text{traceless}} & = 0 \,, \\ q_j \widetilde{q}^j |_{\text{traceless}} & = 0 \,, \end{cases} \tag{2.1.13}$$

$$
\begin{cases}
\phi q_i + M_i^j Q_j = 0\,, \\
\widetilde{q}^i \phi + M_j^i \widetilde{q}^j = 0\,, \\
\phi^\dagger q_j + M_i^{\dagger j} q_j = 0\,, \\
\widetilde{q}^i \phi^\dagger + M_j^{\dagger i} \widetilde{q}^j = 0\,.
\end{cases}
\tag{2.1.14}
$$

Let us now focus on the case where the masses are all equal to zero. In this case we immediately see that there are three qualitatively different types of solutions to the vacuum equations, which receive the names of *Coulomb branch*, *Higgs branch* and *mixed branch*.

- The Coulomb branch (CB) is obtained by setting all the q's to zero, in which case the only non-trivial equation that remains is (2.1.12). This is solved when ϕ takes values in the Cartan algebra of the gauge group. Such non-zero VEVs will result in adjoint Higgsing of the gauge group, generically to $\mathrm{U}(1)^{\mathrm{rank}(G)}$, which justifies the name "Coulomb branch". Note that, since the scalar in the vector multiplet is charged under the $\mathrm{U}(1)_r$ R-symmetry, only $\mathrm{SU}(2)_R$ is preserved along these vacua.

- The Higgs branch (HB) is obtained, on the other hand, by setting $\phi = 0$, which leaves us with equations (2.1.13). A point on this branch of the moduli space will be specified by the VEVs of the q's satisfying the equations, modulo gauge transformations, a construction known as a hyper-Kahler quotient. Therefore, the complex dimension of the space is $\dim_{\mathbb{C}}(\mathrm{HB}) = 2(n_H - n_V)$, with n_H and n_V the number of hypermultiplets and vector multiplets, as long as there are enough of the former to completely Higgs the gauge group. Since the q's are charged under $\mathrm{SU}(2)_R$, only $\mathrm{U}(1)_r$ is preserved along these vacua.

- If both hypermultiplet and vector multiplet scalars have non-zero VEVs, we are in a mixed branch. This can always be understood as a fibration of a Higgs branch type fiber over a Coulomb branch base. Along these vacua, the R-symmetry is completely broken.

There are two main steps in the study of the moduli space of a $\mathcal{N} = 2$ theory. The first is to understand the basic underlying algebraic variety, which contains information of, for example, the list of gauge invariant operators one can build in the theory in question. The second is to compute the metric on top of this variety, which is in turn the metric of the sigma model (which has the moduli space as a target) relevant for the low energy EFT description.

If we focus on the Coulomb branch, it is very common that the underlying algebraic variety is very simple. For example, if the gauge group is SU(N), it is parametrized by gauge invariant operators of the form Tr (ϕ^k), $k = 2, \ldots, N$; where we have taken care of the redundancy due to the Weyl group of SU(N), and with no relations between the different generators. This is to say, the CB is simply given by $\mathbb{C}^{N-1}/S_N$. More generally, for a theory with gauge group G it is very often the case that the CB is $\mathbb{C}^{\text{rank}(G)}/\mathcal{W}_G$, with $\mathcal{W}_G$ the Weyl group of G; although we should remark that this is not so for the theories that we will discuss in the following sections. Then, one asks what is the metric on top of this algebraic variety. It turns out that preserving $\mathcal{N} = 2$ SUSY restricts this metric such that the Coulomb branch is a special Kahler manifold, a fact which can be exploited in order to completely determine said metric and in some cases solve the full low energy EFT including strong coupling phenomena [9, 10].

Regarding the Higgs branch, due to the non-renormalization theorems and the fact that the hypermultiplet part of the lagrangian is blind to the gauge coupling τ, the metric on the moduli space is just the classical one. Concretely, if we denote the first term of (2.1.10) as

$$\int d^4\theta\, K(Q_i, \widetilde{Q}_j)\,, \tag{2.1.15}$$

then, after writing down the superfields in components,

$$g_{ij} = \frac{\partial^2 K}{\partial q_i \partial \widetilde{q}_j}\,, \tag{2.1.16}$$

which is the metric appearing on the low energy sigma model in the kinetic term of the scalars as

$$\mathcal{L} \supset g_{ij}\, \partial_\mu q^i \partial^\mu \widetilde{q}^j\,. \tag{2.1.17}$$

On the other hand, the underlying algebraic variety can be quite complicated. As we have mentioned before, the HB is obtained as

$$\left\{ \begin{array}{ll} (q_j q^{\dagger j} - \widetilde{q}_j^\dagger \widetilde{q}^j)\Big|_{\text{traceless}} & = 0 \\ q_j \widetilde{q}^j\big|_{\text{traceless}} & = 0 \end{array} \right\} \Big/ \, (G - \text{gauge transformation})\,. \tag{2.1.18}$$

This construction, known as a hyperKahler quotient, can produce a variety with any number of generators and various relations among them; in fact, a good part of the work of the following chapters consists of studying the algebraic structure of the HB in a collection of examples. The constraint imposed by $\mathcal{N}=2$ SUSY is that the variety must be hyperKahler, which means that it has a quaternionic structure. This is because of the following reason:

First, starting with a complex scalar $q = (\operatorname{Re} q, \operatorname{Im} q)$ and acting on it with the supersymmetry generators $\widetilde{\mathcal{Q}}_{\dot\alpha}\mathcal{Q}_\alpha$ results in something proportional to $(-\operatorname{Im} q, \operatorname{Re} q)$, i.e. it involves a multiplication by the matrix

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1.19}$$

which is a complex structure on the Higgs branch. Since we have $\mathcal{N}=2$, we will have two complex structures I and J. Moreover, since $\mathcal{Q}^{(c)} = c_1\mathcal{Q}^1 + c_2\mathcal{Q}^2$ is also a supersymmetry as long as $|c_1|^2 + |c_2|^2 = 1$, we can also produce a third complex structure K. This will be not independent from the previous two, rather I, J, K will satisfy precisely the quaternionic conditions,

$$I^2 = J^2 = K^2 = -1\,, \tag{2.1.20}$$

$$IJ = K = -JI\,, \tag{2.1.21}$$

$$JK = I = -KJ\,, \tag{2.1.22}$$

$$KI = J = -IK\,. \tag{2.1.23}$$

The fact that the Higgs branch is always restricted to be a hyperKahler cone, also known as a symplectic singularity, can also be used to obtain information about the theory in question. We will use this fact for example in section 2.3.

2.1.3 Tools for moduli spaces

If the theory is simple enough, studying the algebraic structure of its moduli space is a task that can be easily completed, i.e. one can find for example the set of equations that fully define the algebraic variety. This ceases to be the case when the theory grows in complexity, by which we mean increasing number of gauge groups, bigger representations of the matter fields, or even non-lagrangian theories, etc. In particular, in the following sections we will focus on the case where the gauge group has non-trivial topological properties.

In these instances, we need to figure out what characteristics of the moduli space we can determine, even if the full computation of the moduli space is not possible. This section is devoted to reviewing three tools with this purpose, that can be used to obtain partial information of the moduli space, and in particular of Higgs branches. They are the *Hilbert series*, *magnetic quivers* and *Hasse diagrams*.

Hilbert Series

It is a basic fact of algebraic geometry that an algebraic variety is completely specified by its coordinate ring, i.e. the ring of all the polynomials one can build from the coordinates modulo the relations imposed by the defining equations of the variety. When the full ring is not accessible, an intermediate observable is the *Hilbert series*, which is a generating function that counts the number of different monomials at each degree. That is to say, a variety whose coordiante ring has 1 monomial of degree 0 (the identity), n_1 of degree 1, n_2 of degree 2, and so on, will have a Hilbert Series

$$\mathrm{HS}(t) = 1 + n_1 t + n_2 t^2 + \cdots \,. \tag{2.1.24}$$

For example, the variety $\mathbb{C}$, which has coordinate ring $\mathbb{C}[X]$, will have Hilbert Series

$$\mathrm{HS}(t) = \frac{1}{1-t} = 1 + t + t^2 + \cdots \,. \tag{2.1.25}$$

If the variety is a moduli space of a $\mathcal{N}=2$ theory, say its Higgs branch, the coordinates on it are the VEVs of the scalars in the hypermultiplets. Then, finding the Hilbert Series is a two step process implementing the hyperKahler quotient (2.1.18) [11–13]. First, one needs to count all possible monomials built from the scalars, modulo the relations imposed by the F-terms. In the case where the number of hypermultiplets and vector multiplets is such that there is complete higgsing, this step is straightforward. Second, one needs to mod out by the action of the gauge group. This can be achieved by introducing the character of the representation of the gauge group that each field transforms in, and then integrating over said group with the corresponding Haar measure.

For example, for $\mathcal{N}=2$ SQCD with gauge group G, and with N_f hypermultiplets in the fundamental representation, the formula for the Higgs branch Hilbert series will look like

$$\mathrm{HS}(t) = \int_G d\mu_G \frac{\det(1-\chi_{\mathrm{adj}} t^2)}{\det(1-\chi_f t)^{N_f} \det(1-\chi_{\bar{f}} t)^{N_f}} \,, \tag{2.1.26}$$

where χ_ρ is the diagonal matrix in the Cartan of the gauge group whose trace is the usual character of the representation ρ and $d\mu_G$ is the Haar measure of the group G. Here, the denominator takes care of counting all the possible symmetrized products between the scalars q_i and $\widetilde{q}_i$, and the numerator imposes the F-term relations, which themselves are polynomials of degree 2 and transform in the adjoint of G. Note that, even if an analogous formula to (2.1.26) can be written for most lagrangian theories, the evaluation of the integral grows quickly in computational complexity when the rank of the gauge group increases.

Since the coordinates used in our moduli space come directly from the fields in the lagrangian, the Hilbert series has the direct physical interpretation of counting the number of gauge invariant operators, and it can be used even outside the realm of supersymmetry in the context of EFTs (see e.g. [14–18]). Moreover, (2.1.26) can be refined to include fugacities for the flavour symmetry, which in turn will allow us to know in which representations of the global symmetry group the gauge invariants will transform. This has nice applications, such as exploring possible symmetry enhancements, or computing the global form of the global symmetry group.

Magnetic Quivers

One of the many advantages that supersymmetry brings to the table are dualities, which is the statement that two apparently different theories are really different descriptions of the same theory. Since many SUSY theories can be constructed as compactifications or brane systems in String Theory, it is often the case that dualities between SUSY theories are inherited from string dualities. One such example is 3d mirror symmetry, an infrared duality relating two 3d $\mathcal{N}=4$ theories stating that their moduli spaces are equal, with the Higgs branch of the first being equal to the Coulomb branch of the second, and the Higgs branch of the second equal to the Coulomb branch of the first.

Magnetic Quivers represent a generalization of this statement, exploiting the fact that the Higgs branch is invariant under changing dimension across $d=3,\ldots,6$. If we denote our original 4d $\mathcal{N}=2$ as $\mathcal{T}^{4d}$, in whose Higgs branch we are interested; then a magnetic quiver for said Higgs branch is a quiver theory[1] denoted $\mathcal{MQ}$, which understood as a 3d $\mathcal{N}=4$ theory has a Coulomb branch equal to the Higgs branch of $\mathcal{T}$. In other words, we have the following equality of moduli spaces,

$$\mathrm{HB}\left(\mathcal{T}^{4d}\right)=\mathrm{CB}\left(\mathcal{MQ}^{3d}\right)\,. \tag{2.1.27}$$

[1]A quiver theory is a lagrangian theory specified by a total gauge group which is a direct product of gauge groups, and matter fields which transform in the bifundamental representations of those; and therefore can be completely encoded in an ordered graph called a *quiver*.

This generalizes 3d mirror symmetry in the sense that there is no requirement for the Coulomb branch of $\mathcal{T}$ and the Higgs branch of $\mathcal{MQ}$ to be related, and therefore it shouldn't be regarded as a duality between two theories. Still, the equality of moduli spaces presents several interesting possibilities on itself. One of the main ones concerns the ability to compute Hilbert series for theories where evaluating the integral (analogous to) (2.1.26) is not feasible. This is thanks to the *monopole formula* [19]. It states that, since the 3d $\mathcal{N} = 4$ Coulomb branch is the space of dressed monopole operators in the theory, whose conformal dimension is known, then the Hilbert series can be computed as

$$\mathrm{HS}(t) = \frac{1}{|\mathcal{W}|} \sum_{m \in \mathbb{Z}^r} \sum_{\gamma \in \mathcal{W}(m)} \frac{t^{2\Delta(m)}}{\det(1 - t^2 \gamma)} , \tag{2.1.28}$$

where r is the total rank of the gauge group, $|\mathcal{W}|$ is the order of the Weyl group and $\mathcal{W}(m)$ is the subgroup of the Weyl group that leaves the weight m invariant. Lastly, $\Delta(m)$ is the conformal dimension of the monopole corresponding to the weight m, whose expression in terms of m is known but depends on the theory. If one can derive the magnetic quiver for a Higgs branch from string theory arguments (such as brane constructions, etc.), this will allow the computation of the Hilbert series beyond the reach of the Molien integral.

Hasse Diagrams

We have seen in section 2.1.2 that Higgs branches are always hyperKahler cones, also known as symplectic singularities. As a consequence, they have the propierty that they always admit a foliation structure, a fact studied by mathematicians in [20,21]. This means that the Higgs branch can be divided into a collection of *symplectic leaves* (i.e. subspaces in the foliation where the symplectic form is preserved) which have a partial order given by inclusion of their closures. In physical terms, this foliation contains the information of the structure of possible partial higgsings [22].

Hasse diagrams are a very generic tool that allows to intuitively depict a partial order relation in any set. In the context of Higgs branches of supersymmetric theories, the bottom point of the Hasse diagram will correspond to the origin of the Higgs branch, where all the VEVs are zero and the gauge group remains unhiggesd. The top of the Hasse diagram will correspond to a generic point on the Higgs branch, where the different hypermultiplet scalars take various VEVs, the gauge group is completely higgsed (or, if we don't have enough matter, as higgsed as possible) and the only remaining massless fields are a collection of free hypermultiplets. Intermediate points in the Hasse diagram will correspond to the possible partial higgsings, or in mathematical terms to the symplectic leaves of the foliation. They will be characterized by the remaining gauge group and massless matter fields. In other words, the Hasse diagram encodes the possible phases of the theory and Higgs branch flows.

Even outside the lagrangian context, when one cannot directly explore the mechanism of partial higgsing, Hasse diagrams can be derived using string theory constructions or magnetic quivers. If one has a construction of the theory as a brane system, investigating the possible movements of the branes in the different transverse directions allows to explore the Hasse diagram. Likewise, it can also be obtained in the context of geometric engineering by studying the possible resolutions and deformations (see e.g. [23]). From a magnetic quiver, the Hasse diagram can also be computed via the algorithm of *quiver subtraction* [22].

2.2 Discrete gauge theories of charge conjugation

Gauge symmetry governs the dynamics of a huge variety of systems, ranging form Condensed Matter to Particle Physics. Very often, when discussing gauge symmetry, one implicitly refers to symmetries associated to continuous groups. However, gauge theories based on discrete groups (*a.k.a.* discrete gauge theories), while perhaps more exotic, are also very interesting. Indeed, in Condensed Matter (or lattice models), discrete gauge theories play a relevant role. For instance, in 2d, a web relating various well-known dualities was recently described in [24] by including appropriate $\mathbb{Z}_2$ gaugings using previous results in [25, 26]. Also in 3d discrete gauge theories play a relevant role. For example, the $\mathbb{Z}_2$ Ising model for one-half spins in a squared lattice is dual to a lattice $\mathbb{Z}_2$ gauge theory [27]. Moreover, it admits a phase whose continuum limit is realized by the same doubled Chern-Simons theory appearing in the description of certain topological phases of electrons in [28].

In turn, in High Energy Physics, discrete gauge theories also play a relevant role. In many cases, discrete global symmetries – a prominent example being R-parity in supersymmetric (SUSY) models – are needed to achieve phenomenologically viable scenarios. Yet, if only global symmetries, their constraints would be washed out by Quantum Gravity effects. This suggests, as first discussed in [29], that discrete symmetries must be gauged at a fundamental level. The subject was recently revived in [30], where it was argued that in a consistent theory of Quantum Gravity such as String Theory all global symmetries, including discrete ones, are expected to be gauged. Indeed, String Theory quite often produces gauged discrete symmetries. For instance, in the presence of NS5 branes, there can be discrete $\mathbb{Z}_k$ gauge symmetries as discussed in [31]. These gauged discrete symmetries have also been quite extensively discussed in the context of String Phenomenology (see *e.g.* [32]). More recently, [33] conjectured discrete S_N gauge symmetries in 6d Conformal Field Theories (CFT's) needed in order to correctly reproduce their operator spectrum.

On a seemingly separate line, the traditional approach to Quantum Field Theory (QFT) is based on perturbation theory, specifically through the computation of correlation functions using Feynman diagrams. While this approach is very successful to compute observables such as scattering amplitudes, it misses a great deal of the beautiful subtleties of Quantum Field Theory. Indeed, by definition perturbation theory considers small fluctuations around the (trivial) vacuum, and hence it is essentially blind to the global structure of the group, which at most enters as a superselection rule. Nevertheless, interesting Physics may be hiding in the global structure of the group, despite being local Physics blind to it. A particular example is the case of $\mathrm{O}(N)$ theories, which can be regarded as the composition of a continuous gauge $\mathrm{SO}(N)$ symmetry and a discrete gauge $\mathbb{Z}_2$ symmetry. Thus, this is yet another context in which discrete gauge theories may appear, in this case as part of a larger and disconnected gauge group.

Also in the realm of High Energy Physics, the gauging of discrete symmetries has been argued to play a very relevant role in the construction of $\mathcal{N} = 3$ SCFTs in 4d. The first examples of these were constructed in [34] starting with $\mathcal{N} = 4$ SYM where the complexified Yang-Mills coupling is tuned to a self-S-dual point. At those points a subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ mapping the theory to itself appears as an extra discrete global symmetry. As shown in [34–37], quotienting by a well-chosen combination of Γ and a subgroup of the R-symmetry – which amounts to gauging a discrete group – precisely breaks the supersymmetry down to exactly $\mathcal{N} = 3$. Hence, it is the discrete gauging which is breaking the supersymmetry down to $\mathcal{N} = 3$.

In [35] this strategy was generalized to a systematic study of gaugings of discrete symmetries preserving at least $\mathcal{N} = 2$ SUSY (mostly restricting to rank one theories). In this way a beautiful landscape of theories interrelated among them emerged. In the case of theories based on an $\mathrm{SU}(N)$ gauge group, a natural such discrete symmetry to quotient by is charge conjugation. As noted in [35], gauging charge conjugation can be subtle and all supersymmetries can be broken. The ultimate reason for this lies in the fact that charge conjugation is essentially akin to complex conjugation ($\sim \mathbb{Z}_2$), the outer automorphism of $\mathrm{SU}(N)$. It is then intuitive that the combined group incorporating $\mathrm{SU}(N)$ and charge conjugation transformations cannot simply be the direct product $\mathrm{SU}(N) \times \mathbb{Z}_2$ but rather the semidirect product

$\mathrm{SU}(N) \rtimes \mathbb{Z}_2$.[2] This conflicts with the two-step procedure of first considering a SU(N) gauge theory and then gauging its $\mathbb{Z}_2$ charge conjugation symmetry, which implicitly assumes a direct product structure.

In [38] a fresh approach to the problem was taken, namely, first constructing a Lie group which incorporates both SU(N) transformations as well as charge conjugation, and then using it to build gauge theories. Such groups had been introduced in the mathematical literature under the name of *principal extensions* (in this case of SU(N)) in the past [39,40], and made a brief appearance in the Physics literature in [41–43] in the context of branes wrapping group manifolds. A more related set-up is that discussed in [44], where after symmetry breaking one ends up with a remainder discrete charge conjugation symmetry which can produce Alice strings. While this was mostly with an eye on the $U(1)$ case, the non-abelian version plays a relevant role as well in the process of constructing orientifold theories, as discussed in [45]. These groups have two disconnected components and are very similar to the orthogonal gauge theories briefly alluded above. Indeed, O($2N$) is a principal extension of SO($2N$) [46] (since $\mathrm{O}(2N+1) = \mathrm{SO}(2N+1) \times \mathbb{Z}_2$, this case is much more tractable and less interesting). Surprisingly, as shown in [38] and soon after in [36,37] for other discrete gaugings of the like, it turns out that the gauge theories based on principal extension of SU(N) provide the first examples of four-dimensional $\mathcal{N} = 2$ theories with non-freely generated Coulomb branches. While *a priori* no argument forbids theories with non-freely generated Coulomb branches (and indeed their putative properties had been studied [47]), in view of the lack of explicit examples it was widely believed that such theories would not exist. The theories proposed in [36–38] then provide the first counterexamples to that conjecture.

In [38] $\mathcal{N} = 2$ SUSY theories where considered as proof-of-concept for gauge theories based on principal extensions, with the bonus that the first theories with non-freely generated Coulomb branches were discovered. Yet in principle one may construct gauge theories in arbitrary dimensions with any SUSY (including no SUSY) based on principal extension groups. In this section we re-consider with more detail the construction of such groups. It turns out that the correct way to think about these groups is as extensions of SU(N) by its (outer) automorphism group (recall, $\sim \mathbb{Z}_2$). A detailed analysis shows that actually there are exactly two possible extensions, corresponding to two possible disconnected Lie groups which we dub $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$. In a precise way that we describe below, these two types correspond

[2] Schematically, this can be easily seen by writing any element in the Cartesian product $\mathrm{SU}(N) \times \mathbb{Z}_2$ as $g(U, \gamma)$, where γ is either the identity or complex conjugation ($\sim$ charge conjugation) and U a SU(N) matrix. A short computation shows that the multiplication rule is that of a semidirect product (more details in section (2.2.1)).

to the classification of symmetric spaces of type A. To our knowledge, the existence of these two possible extensions and their construction has not appeared before. We then go on and explicitly construct gauge theories based on them, concentrating, like in [38] as a proof-of-concept, on $\mathcal{N}=2$ SQCD-like theories. In particular, we analyze certain protected sectors of the operator spectrum using different limits of the Superconformal Index (SCI). This allows us to show that the corresponding Coulomb branches are isomorphic as complex algebraic varieties and both are not freely generated. As a by-product of the explicit construction of the Lagrangian of the theories, we can understand from first principles the global symmetry pattern emerging from the Higgs branch Hilbert series computation in [38].

The organization of this section is as follows: we start in section (2.2.1) describing the principal extensions of $\mathrm{SU}(N)$ as extensions of the $\mathbb{Z}_2$ outer automorphism group of $\mathrm{SU}(N)$ by $\mathrm{SU}(N)$. As anticipated, we find exactly two such possibilities which are in one-to-one correspondence with the Cartan classification of symmetric spaces of type A. In section (2.2.2) we study aspects of the representation theory of these groups, paying special attention to the fundamental and the adjoint representations as well as to some of the invariants which can be formed out of them. We also construct the Weyl integration formula over the $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ groups. In section (2.2.3) we turn to Physics and construct $\mathcal{N}=2$ SQCD gauge theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$. As a by-product, we will provide an *a priori* understanding of the global symmetry groups of the resulting theories. In section (2.2.4) we turn to the quantitative analysis of the theories using various limits of the superconformal index as diagnostics tool. To that matter, the integration formula previously developed in section (2.2.2) plays a very relevant role. In particular, we will find that the $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ theories have non-freely generated Coulomb branches. Finally, we conclude in (3.4.4) with some final remarks and open lines. For the interest of the reader, we postpone to the appendices several technical details.

2.2.1 Construction of two disconnected groups

The groups we are interested in are extensions of $\mathbb{Z}_2$ by a Lie group, which in this section we will take to be $\mathrm{SU}(N)$. In [38], the so-called *principal extension* were considered, but it turns out that although the group of outer automorphisms of $\mathrm{SU}(N)$ is $\mathrm{Out}(\mathrm{SU}(N)) \cong \mathbb{Z}_2$, there are in some cases more than one (in fact, exactly two) inequivalent ways of constructing a semi-direct product of $\mathrm{SU}(N)$ by $\mathrm{Out}(\mathrm{SU}(N))$. This section aims at studying this issue in depth.

For concreteness, we will start with a pedestrian approach to the problem, and we will see that the two disconnected groups arise in a natural way. Then we will provide a more abstract, but also more rigorous construction, of the semi-direct products. As we will explain, they are built from *involutive* outer automorphisms (IOA) of $\mathrm{SU}(N)$, i.e. automorphisms which are their own inverse. In turn, we will see that these are classified by real forms of the complex Lie algebra $\mathfrak{sl}(N,\mathbb{C})$, or equivalently by symmetric spaces. This last feature will also help us understand the global symmetry of the gauge theories constructed in later sections.

Explicit matrix realization

Let's construct the (disconnected) gauge group of an $\mathrm{SU}(N)$ theory in which charge conjugation is gauged as well. In such a theory, the lowest-dimensional non-trivial representation has dimension $2N$, so we will construct our group as a $2N\times 2N$ matrix group. It has a subgroup, denoted G, which is isomorphic to $\mathrm{SU}(N)$ in the fundamental plus antifundamental representation:

$$G=\left\{\mathbf{U}=\left(\begin{array}{cc} M & 0 \\ 0 & M^{\star} \end{array}\right)\middle| M\in\mathrm{SU}(N)\right\}\cong\mathrm{SU}(N)\,, \tag{2.2.1}$$

where the star denotes complex conjugation. The charge conjugation is a $\mathbb{Z}_2$ group which exchanges the fundamental and antifundamental of $\mathrm{SU}(N)$, so it has to be of the form

$$\Gamma_A=\left\{\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),\left(\begin{array}{cc} 0 & A \\ A^{-1} & 0 \end{array}\right)\right\}\cong\mathbb{Z}_2\,, \tag{2.2.2}$$

where $A\in\mathrm{U}(N)$ is a matrix on which we will come back later. The total gauge group, which we call $\widetilde{G}_A$, is the image of the Cartesian product $G\times\Gamma_A$ under the multiplication map,

$$\widetilde{G}_A=\{g\gamma\mid g\in G,\gamma\in\Gamma_A\}\ . \tag{2.2.3}$$

We have added the subscript A to insist on the fact that this depends on the matrix A chosen above. The product in this group is simply matrix multiplication. Thus for two elements $g\gamma$ and $g'\gamma'$ of $\widetilde{G}_A$, we have

$$g\gamma\cdot g'\gamma'=g\gamma g'\gamma'=(g\varphi_\gamma(g'))(\gamma\gamma')\,, \tag{2.2.4}$$

where the last rewriting is necessary for the product to be manifestly in the form $g''\gamma''$. This is a semi-direct product structure, with

$$\begin{array}{rccc} \varphi_\gamma: & G & \to & G \\ & \mathbf{U} & \mapsto & \gamma\,\mathbf{U}\,\gamma^{-1} \end{array} \tag{2.2.5}$$

Note that for the non-trivial element $\gamma \in \Gamma_A$ one has

$$\varphi_\gamma(\mathbf{U}) = \begin{pmatrix} A\,M^\star A^{-1} & 0 \\ 0 & A^{-1}\,M\,A \end{pmatrix} . \tag{2.2.6}$$

Writing down the requirement that this matrix belongs to G defined in (2.2.1) leads to the condition that $A\,A^\dagger = 1$ together with $A^{-1}\,M\,A = (A\,M^\star A^{-1})^\star$ for all $M \in \mathrm{SU}(N)$. From the last condition it follows that $M\,(A\,A^\star) = (A\,A^\star)\,M$, which is solved for $A\,A^\star = \lambda\,1$. Since $A \in \mathrm{U}(N)$, by multiplying on the left by $A^\dagger$, we could as well write $A^\star = \lambda\,A^\dagger$. Likewise, we could take the complex conjugate of the equation to write $A^\star\,A = \lambda^\star\,1$. Multiplying now on the right by $A^\dagger$ leads to $A^\star = \lambda^\star\,A^\dagger$, which requires $\lambda \in \mathbb{R}$. Then, since $A \in \mathrm{U}(N)$, by taking the determinant, $|\det(A)|^2 = \lambda^N = 1$. Hence, we obtain

$$A\,A^\star = \begin{cases} \pm 1 & \text{for } N \text{ even} \\ +1 & \text{for } N \text{ odd}\,. \end{cases} \tag{2.2.7}$$

Thus, all in all, we have found a family of matrix groups given by (2.2.3) where

- for odd N one needs $A^T = A$. This defines a group that we call $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$.
- for even N we have two cases:
  - $A = A^T$: this defines a group that we call $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ (the even N version).
  - $A = -A^T$: this defines a group that we call $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$.

While this gives an intuitive construction of two different groups $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ and $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$, several questions are left unanswered: why did we choose to represent $\widetilde{G}_A$ in the specific form (2.2.3)? Is the symmetry property of the matrix A enough to characterize entirely the groups $\widetilde{G}_A$? Do this construction really yield two non-isomorphic groups?

As for the last point, a preliminary observation is that had the two groups $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ and $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$, for N even, been conjugated one to the other, there should be an invertible $2N\times 2N$ matrix $\mathbf{X}$ such that $\gamma_{\mathrm{II}} = \mathbf{X}\,\gamma_{\mathrm{I}}\,\mathbf{X}^{-1}$, where we denoted by $\gamma_{\mathrm{I,II}}$ the non-trivial element of Γ_A in the two cases. A natural ansatz for the matrix $\mathbf{X}$ is

$$\mathbf{X} = \begin{pmatrix} X & 0 \\ 0 & X^\star \end{pmatrix} , \tag{2.2.8}$$

for some $X \in \mathrm{SU}(N)$. A short computation shows that the condition for both choices to

$$\begin{array}{c} \text{Split extensions of } \mathbb{Z}_2 \text{ by } \mathrm{SU}(N) \\ \updownarrow \\ \text{Involutive automorphisms of } \mathrm{SU}(N) \\ \updownarrow \\ \text{Involutive automorphisms of } \mathfrak{sl}(N,\mathbb{C}) \\ \updownarrow \\ \text{Real forms of } \mathfrak{sl}(N,\mathbb{C}) \\ \updownarrow \\ \text{Symmetric spaces associated with} \\ \text{the compact group } G_c = \mathrm{SU}(N) \end{array}$$

Figure 2.1: Schematic representation of the method of classification of split extensions of $\mathbb{Z}_2$ by $\mathrm{SU}(N)$, covered in sections 2.2.1 and 2.2.1.

be conjugated translates into $A_{\mathrm{II}} = X\,A_{\mathrm{I}}\,X^T$, where $A_{\mathrm{I,II}}$ denotes in the obvious way the A matrix for the corresponding choice. Transposing this equation leads to $A_{\mathrm{II}} = -X\,A_{\mathrm{I}}\,X^T$, which shows that such X does not exist. While this hints that indeed both choices are two different groups, it strongly relies on a specific representation. In the rest of this section will offer more formal arguments that indeed there are exactly two extensions $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ and $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$ of $\mathrm{SU}(N)$ for N even, and only one for N odd, that can be obtained by gauging the outer automorphism, and the labels I and II are related to the Cartan classification of real forms of $\mathfrak{sl}(N,\mathbb{C})$. Figure 2.1 gives an overview of the logical steps that we will follow.

Extensions and semidirect products

The semidirect products that we consider are extensions of $\mathbb{Z}_2$ by $\mathrm{SU}(N)$. In order to point out the subtleties of this construction, we begin with a review of some general theory of extensions of discrete groups by Lie groups (more details can be fount in [48], Chapter 18). Let $\tilde{G}$ be any Lie group, connected or not. Then $\tilde{G}$ is the extension of the discrete groups of its connected components, called $\pi_0(\tilde{G})$, by its identity component, called G. This means that there is a short exact sequence

$$1 \longrightarrow G \xrightarrow{\ \iota\ } \tilde{G} \xrightarrow{\ q\ } \pi_0(\tilde{G}) \longrightarrow 1\,. \tag{2.2.9}$$

Note that the groups introduced in the previous section fit this structure: constructing the maps $\iota(g) = g\,\mathrm{id}_{\Gamma_A}$ (which maps an element $g \in G$ into $\widetilde{g} \in \widetilde{G}$) and $q(\widetilde{g}) = \gamma$ (which maps an element $\widetilde{g} = g\gamma \in \widetilde{G}$ into $\Gamma_A \cong \mathbb{Z}_2$), it is clear that $\mathrm{Ker}(q) = \mathrm{Im}(\iota)$.

In the following we will restrict ourselves to *split extensions*, or equivalently (due to the "splitting lemma") to the situation where $\tilde{G}$ is a semidirect product $G \rtimes \pi_0(\tilde{G})$. The assumption that the extension is split means that there exist a Lie group morphism $\sigma : \pi_0(\tilde{G}) \to \tilde{G}$ such that $q \circ \sigma = \mathrm{id}_{\pi_0(\tilde{G})}$. Note that for any extension (split or not) we can construct a map $C_G : \tilde{G} \to \mathrm{Aut}(G)$ defined by $C_G(\tilde{g})(g) = \iota^{-1}(\tilde{g}\iota(g)\tilde{g}^{-1})$ for all $g \in G$ and $\tilde{g} \in \tilde{G}$. Using now the splitting morphism σ we can form a homomorphism $S = C_G \circ \sigma$, which is precisely what is needed to build a semi-direct product $G \rtimes_S \pi_0(\tilde{G})$.

Note as well that indeed the groups introduced in the previous section do fit in this structure, since we can construct a map from Γ_A into $\widetilde{G}$ as $\sigma(\gamma) = \mathrm{id}_G\, \gamma$ which clearly satisfies that $q \circ \sigma = \mathrm{id}_{\Gamma_A}$. Hence, when regarded as a sequence, indeed $\widetilde{G}$ is split and, consequently, there is a semidirect product structure (which on the other hand we explicitly constructed). In that language, the homomorphism S corresponds to the φ_γ in eq. (3.3.7).

Finally, we add a last ingredient to the construction, namely the group of outer automorphisms of G, $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, with the obvious map $[\cdot] : \mathrm{Aut}(G) \to \mathrm{Out}(G)$. For $\varpi \in \pi_0(\tilde{G})$, one can show that the formula $s(\varpi) = [C_G(\tilde{g})]$ where $\tilde{g} \in \tilde{G}$ is chosen such that $q(\tilde{g}) = \varpi$ defined a group homomorphism $s : \pi_0(\tilde{G}) \to \mathrm{Out}(G)$, called the *characteristic homomorphism* of the extension. This is summarized by the diagram

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r, "\iota"] & \tilde{G} \arrow[r, bend left, "q"] \arrow[d, "C_G"] & \pi_0(\tilde{G}) \arrow[l, "\sigma"] \arrow[r] \arrow[dl, "S"] \arrow[d, "s"] & 1 \\
& & \mathrm{Aut}(G) \arrow[r] & \mathrm{Out}(G) &
\end{tikzcd} \tag{2.2.10}$$

Two equivalent extensions of $\pi_0(\tilde{G})$ by G define the same s.[3] Obviously, there are exactly two possible homomorphisms s when $\pi_0(\tilde{G}) \cong \mathrm{Out}(G) \cong \mathbb{Z}_2$: the trivial morphism, giving the direct product $\tilde{G} = G \times \mathbb{Z}_2$, and the identity morphism, giving a semi-direct product. However the converse is not true, and we will see that two inequivalent extensions correspond to the identity morphism $s : \mathbb{Z}_2 \to \mathbb{Z}_2$. By definition, the classification of semidirect products $G \rtimes_S \pi_0(\tilde{G})$ reduce to the classification of the possible maps S. In the case at hand, where $\pi_0(\tilde{G}) \cong \mathbb{Z}_2$, S has to be its own inverse, and thus this means that we need to classify the *involutive* automorphisms of G. This is the main lesson that we learn from this abstract development: *classifying the different extensions is equivalent to classifying the involutive automorphisms of* G, which will then be our next task. In particular, the semidirect products will be associated to outer involutive automorphisms.

[3] See Lemma 18.1.6 in [48].

Real forms and Antiinvolutions

There is a very elegant theory of involutive automorphisms in complex Lie algebras, connecting them to real forms – this is what will allow us to classify them. We begin with a quick reminder of some aspects of the theory of real and complex Lie algebras, referring to [49] for more details. Given a real Lie algebra $\mathfrak{g}_0$ we denote by $\mathfrak{g}_0(\mathbb{C})$ the corresponding complexification, which is uniquely defined by the bracket

$$[x_1 + iy_1, x_2 + iy_2] = [x_1, x_2] - [y_1, y_2] + i([x_1, y_2] + [y_1, x_2]),\ \forall\ x_1, x_2, y_1, y_2 \in \mathfrak{g}_0\,. \quad (2.2.11)$$

Conversely, a *real structure* σ of $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$ is defined to be an involutive antilinear automorphism, or *antiinvolution* for short. This means that a real structure is an automorphism which satisfies

$$\begin{cases} \sigma(\alpha x + \beta y) = \alpha^* \sigma(x) + \beta^* \sigma(y) & \forall\ x, y \in \mathfrak{g}_0 \quad \forall \alpha, \beta \in \mathbb{C} \\ \sigma^2(z) = z & \forall\ z \in \mathfrak{g}\,, \end{cases} \quad (2.2.12)$$

where α^* denotes the complex conjugate of α. Finally, a real subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ is called a real form of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$.

It's important to note that although the complexification of a real Lie algebra is unique, there might be several real forms for a given complex Lie algebra, and these can be classified using the real structures. Indeed, on the complex Lie algebra $\mathfrak{g}$ there is a bijection between real structures and real forms:

- Given a real structure $\sigma : \mathfrak{g} \to \mathfrak{g}$ we can construct the corresponding real form

$$\mathfrak{g}^\sigma := \{X \in \mathfrak{g} \mid \sigma(X) = X\}\,. \quad (2.2.13)$$

- Conversely, given a real form $\mathfrak{g}_0$ of a complex Lie algebra $\mathfrak{g}$, we can construct the corresponding real structure σ as the complex conjugation, $\sigma(x + iy) = x - iy \quad \forall\ x, y \in \mathfrak{g}_0$.

Moreover two real forms $\mathfrak{g}_0, \mathfrak{g}_1$ of $\mathfrak{g}$ are isomorphic if and only if the corresponding real structures σ_0, σ_1 are conjugate by an automorphism of $\mathfrak{g}$, i.e. there exists $\alpha \in \mathrm{Aut}\ \mathfrak{g}$ such that $\sigma_1 = \alpha\sigma_0\alpha^{-1}$.

We now focus on a rank r semisimple complex Lie algebra $\mathfrak{g}$. It admits a canonical system of generators (h_i, e_i, f_i) for $i = 1, \ldots, r$. One can show that [49] there exists a unique real structure, which we call τ from now on, such that

$$\tau(h_i) = -h_i\,, \qquad \tau(e_i) = -f_i\,, \qquad \tau(f_i) = -e_i\,. \tag{2.2.14}$$

The associated real form is a *compact* real form.[4]

Cartan Class	Real form $\mathfrak{sl}(N,\mathbb{C})^\sigma$	Real structure σ	Involution θ
AI	$\mathfrak{sl}(N,\mathbb{R})$	$X \mapsto X^*$	$\theta_{\mathrm{I}} : X \mapsto -X^T$
AII (N even)	$\mathfrak{sl}(N/2,\mathbb{H})$	$X \mapsto -J_N X^* J_N$	$\theta_{\mathrm{II}} : X \mapsto J_N X^T J_N$
$AIII, AIV$	$\mathfrak{su}(p, N-p)$	$X \mapsto -I_{p,N-p}(X^*)^T I_{p,N-p}$	$X \mapsto I_{p,N-p} X I_{p,N-p}$

Table 2.1: The three (types of) real forms of the complex Lie algebra $\mathfrak{sl}(N,\mathbb{C})$. The second line exists only when N is even. In the third line, $p = 0, 1, \ldots, [N/2]$. For each real form, we indicate the corresponding real structure σ and the corresponding involution $\theta = \sigma\tau$ with $\tau : X \mapsto -(X^*)^T$.

We now use the compact real structure τ to associate to any real structure σ the automorphism

$$\theta = \sigma\tau\,. \tag{2.2.16}$$

It is clear that θ is a linear (as opposed to antilinear) automorphism, but in general it is not an involution. However, we have seen above that up to replacing the real form $\mathfrak{g}_0$ by an other isomorphic real form, we can conjugate the corresponding real structure σ by any automorphism of $\mathfrak{g}$, and Cartan proved that at least one of these conjugates gives rise to an involutive θ. Therefore, in the case of semisimple complex Lie algebras, to each real form one can associate an involutive automorphism. This non-trivial statement is the key step to obtain Cartan's theorem: two real forms $\mathfrak{g}_0, \mathfrak{g}_1$ of $\mathfrak{g}$ are isomorphic if and only if the corresponding involutions θ_0, θ_1 are conjugate to each other, i.e. $\exists\ \alpha \in \mathrm{Aut}(\mathfrak{g}) \mid \theta_1 = \alpha\theta_0\alpha^{-1}$.

With this theorem at hand, we can now, given a list of inequivalent real forms of a semisimple complex Lie algebra, obtain the corresponding classification of inequivalent involutive automorphisms. Let us work out the case of $\mathfrak{g} = \mathfrak{sl}(N,\mathbb{C})$. In that case, it is easy to check that[5] $\tau(X) = -(X^\star)^T$ for $X \in \mathfrak{g}$. The real forms of $\mathfrak{sl}(N,\mathbb{C})$ can be read on Cartan's classification, and come in three different types; it is then a simple task to compute the associated involution θ in each case. This is summarized in Table 2.1. We have used the

[4]There exists also a unique real structure ς which fixes the system of generators,

$$\varsigma(h_i) = h_i\,, \qquad \varsigma(e_i) = e_i\,, \qquad \varsigma(f_i) = f_i\,. \tag{2.2.15}$$

The associated real form is the *split* real form.

[5]The split real form ς is the usual complex conjugation.

notations

$$I_{p,q} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}, \qquad J_N = \begin{pmatrix} 0 & -1_{N/2} \\ 1_{N/2} & 0 \end{pmatrix}, \quad (N \text{ even}). \tag{2.2.17}$$

One representant of each conjugacy class of involution of $\mathfrak{sl}(N, \mathbb{C})$ is presented in the last column of Table 2.1. Note that the involutions corresponding to the real forms $\mathfrak{su}(p, N-p)$ are inner, and we are left with precisely two conjugacy classes of outer involutive automorphisms when N is even, and only one class when N is odd.

There is a bijective correspondence between the simple real Lie algebras and the irreducible noncompact symmetric spaces of noncompact type which we briefly review in Appendix 5.A.1 (see [50]); this relates the involution of the algebra θ to an involution Θ on the group. This correspondence is illustrated in the case of the compact group $\mathrm{SU}(N)$ in Table 2.2. Although we will not exploit the symmetric spaces duality, we want to point out the involutions Θ and the subgroups K left invariant by Θ. One can check that in all cases the involutions Θ induce on the corresponding Lie algebras the involutions θ of Table 2.1. As for the compact subgroups K, they will turn out to determine the global symmetry of gauge theories based on the semidirect products of $\mathrm{SU}(N)$ by Θ.

From now on, we focus on the first two lines of Tables 2.1 and 2.2, and borrowing names from the Cartan classification, we define the two following groups:

$$\begin{aligned} \widetilde{\mathrm{SU}}(N)_{\mathrm{I}} &= \mathrm{SU}(N) \rtimes_{\Theta_{\mathrm{I}}} \mathbb{Z}_2 \,, \\ \widetilde{\mathrm{SU}}(N)_{\mathrm{II}} &= \mathrm{SU}(N) \rtimes_{\Theta_{\mathrm{II}}} \mathbb{Z}_2 \,, \qquad (N \text{ even}) \,. \end{aligned} \tag{2.2.18}$$

Note that these are indeed the groups constructed in the previous subsection, thus confirming the claim that indeed there are the two possible extensions of $\mathbb{Z}_2$ by $\mathrm{SU}(N)$.

Cartan Class	G	K	$\dim K$	Involution Θ
AI	$\mathrm{SL}(N, \mathbb{R})$	$\mathrm{SO}(N)$	$\frac{1}{2}N(N-1)$	$g \mapsto (g^{-1})^T$
AII (N even)	$\mathrm{SL}(N/2, \mathbb{H})$	$\mathrm{Sp}(N/2)$	$\frac{1}{2}N(N+1)$	$g \mapsto -J_N (g^{-1})^T J_N$
$AIII, AIV$	$\mathrm{SU}(p, N-p)$	$S(\mathrm{U}(p) \times \mathrm{U}(N-p))$	$p^2 + (N-p)^2 - 1$	$g \mapsto I_{p,N-p}\, g\, I_{p,N-p}$

Table 2.2: The three (types of) symmetric spaces for which $G_c = \mathrm{SU}(N)$. In each case we indicate the dual group G, the compact subgroup K and the lift to the group $\mathrm{SU}(N)$ of the involutions θ in Table 2.1. One can check that K is the subgroup of G fixed by Θ.

A construction of automorphisms

Now we explain how to construct explicitly automorphisms in the various classes corresponding to the lines of Table 2.1. We will use a method based on the Weyl group.

General theory Consider a simple complex Lie algebra $\mathfrak{g}$. Let $\phi : \Phi \to \Phi$ be an isomorphism of the root system Φ, and let Δ be a set of simple roots in Φ. The isomorphism ϕ extends in a trivial way on the Cartan subalgebra $\mathfrak{h}$, giving an isomorphism $\theta : \mathfrak{h} \to \mathfrak{h}$, and we want to extend it to the whole Lie algebra $\mathfrak{g}$. To do this, let us first choose a non-zero element X_α in each root space $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$. We also choose a family of non-zero complex numbers c_α for α simple. Then (see [51], Theorem 14.2) there exist a unique isomorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ that extends $\theta : \mathfrak{h} \to \mathfrak{h}$ and such that

$$\theta(X_\alpha) = c_\alpha X_{\phi(\alpha)} \tag{2.2.19}$$

for every *simple* root $\alpha \in \Delta$.

The Weyl group W, generated by reflections with respect to the hyperplanes orthogonal to the simple roots in $\mathfrak{h}^*$, corresponds to a set of automorphisms of the root system, and by the construction of the previous paragraph, gives rise to inner automorphisms of $\mathfrak{g}$. Outer automorphisms will arise from root system isomorphisms that are not in the Weyl group.

The $\mathfrak{sl}(N, \mathbb{C})$ case In the case of $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{C})$ with $N \geq 3$ the root system isomorphisms that are not in the Weyl group are of the form $-w$ for $w \in W$. If we choose $c_\alpha = 1$ for all the simple roots, then on can generate automorphisms in all possible classes from $W \cup (-W)$. It should be noted that the class of an involutive Lie algebra automorphism associated to a given root system automorphism depends on the choice of the c_α, as illustrated by the example below.

The $\mathfrak{sl}(4, \mathbb{C})$ example Let us illustrate this with the concrete example of $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$. We express root system automorphisms as matrices in the basis of the simple roots. Thus the corresponding Weyl group is the order 4! group generated by the three simple reflections

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}. \tag{2.2.20}$$

Out of these automorphisms, we now generate Lie algebra involutions. We first choose $c_\alpha = 1$. Then we find that exactly ten give rise to (inner) involutions of $\mathfrak{su}(4)$, and their type from Table 2.1 can be read from the multiplicity of the eigenvalue 1, which can be 15 (for $p = 0$), 9 (for $p = 1$) or 7 (for $p = 2$). The identity corresponds to $p = 0$, six automorphisms correspond to $p = 1$, namely

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \tag{2.2.21}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{2.2.22}$$

and three automorphisms correspond to $p = 2$, namely

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{2.2.23}$$

Now we turn to the outer automorphisms, generated by $-W$. Here there are exactly six involutive outer automorphisms, and their type can be read from the multiplicity of the eigenvalue 1, which is 6 for type I and 10 for type II. We find four involutions of type I, namely

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \tag{2.2.24}$$

and two of type II, namely

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}. \tag{2.2.25}$$

Now we can do the same exercise with $c_\alpha = -1$. In that case, there are only 6 inner involutions generated by the Weyl group, two of them of type $p = 1$ and four of them of

Value of c_α	N odd	N even
$c_\alpha = +1$	I	II
$c_\alpha = -1$	–	I

Table 2.3: Type of outer involutive automorphisms generated by the flip of the Dynkin diagram of A_{N-1} as a function of the parity of N and of the choice of the constant c_α, taken to be the same for all the simple roots.

type $p = 2$. There are 10 outer involutions generated by $-W$, all of them of type I.

The flip involution Let us focus on a particular element of $-W$, namely the *flip* defined by

$$\alpha_i \to \alpha_{N-i} \,. \tag{2.2.26}$$

When N is odd, the flip is of course always of type I. On the other hand, it turns out that when N is even, the flip generates an outer involutive automorphism of type I when we choose $c_\alpha = -1$, while it generated an outer involutive automorphism of type II when we choose $c_\alpha = +1$. This observation gives us a definition of the two groups $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ that just differs by a sign, namely, we use (2.2.18) where $\Theta_{\mathrm{I,II}}$ is the flip defined using for all simple root α

$$c_\alpha \equiv c = \begin{cases} -1 & \text{for type I} \\ +1 & \text{for type II} \end{cases} \tag{2.2.27}$$

This is summarized in Table 2.3. It is easy to prove by recursion on the height[6] of the root α that the extensions of the flip to the Lie algebra are defined by

$$\theta_{\mathrm{I,II}}\left(X_\alpha\right) = -(-c)^{\mathrm{ht}(\alpha)} X_{\phi(\alpha)} \,, \tag{2.2.28}$$

for any root α (in the case of simple roots, this reduces to (2.2.19)). The corresponding Lie group morphisms are called $\Theta_{\mathrm{I,II}}$.

[6]We recall that the height of a root α, denoted by $\mathrm{ht}(\alpha)$, is the sum of its coefficients when expressed in the basis of simple roots.

The fundamental representation is given by (2.2.3), where the matrix A is

$$A = \begin{pmatrix} & & & & 1 \\ & & & -c & \\ & & (-c)^2 & & \\ & \cdots & & & \\ \cdots & & & & \\ (-c)^{N-1} & & & & \end{pmatrix} . \tag{2.2.29}$$

One checks that these matrices satisfy the symmetry properties encountered in section 2.2.1.

2.2.2 Representations, invariants and integration measure

Having established the existence of the two groups $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$, our next task is to study them. An aspect of primary interest are their representations, in particular the fundamental and the adjoint, as they will provide the basic building blocks to construct gauge theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$. In turn, the possible invariants which can be constructed out of them will also play a role, as they will either enter the construction of the Lagrangian of the theories or because they will be identified with gauge-invariant operators. The latter can be systematically constructed by computing index-like (we will be more precise below) generating functions such as the Higgs branch Hilbert series or the Coulomb branch limit of the index, for which a necessary tool is the integration measure on these groups.

Representations

Let us begin with some general remarks, aiming at understanding the representations of $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ from those of $\mathrm{SU}(N)$. To that matter, we adapt the discussion of [52] (section VI.7), regarding *induced representations*. Let us define $\widetilde{G} = \widetilde{\mathrm{SU}}(N)$ and $G = \mathrm{SU}(N)$. We have $\widetilde{G}/G \simeq \mathbb{Z}_2$. For $k \in \mathbb{Z}_2$, we define $\Omega(k)$ the representation of $\widetilde{G}/G$,

$$\begin{aligned} \Omega(k) \quad : \quad & \widetilde{G}/G \times \mathbb{C} \to \mathbb{C} \\ & (x, z) \mapsto (-1)^k z \, . \end{aligned} \tag{2.2.30}$$

Using the canonical projection $\widetilde{G} \to \widetilde{G}/G$, $\Omega(k)$ can also be seen as a representation of $\widetilde{G}$.

Now we consider two constructions:

- From a representation V of $\widetilde{G}$, one can construct other representations $V \otimes \Omega(k)$ of $\widetilde{G}$

	Type A	Type B
G-representations U	all U_x isomorphic	all U_x distinct
$\widetilde{G}$-representations V	all $V \otimes \Omega(k)$ distinct	all $V \otimes \Omega(k)$ isomorphic

Table 2.4: Types of representations related by induction and restriction. See theorem VI.7.3 of [52]: If U is a representation of G of type A, the induced representation of $\widetilde{G}$ is $\mathrm{ind}^{\widetilde{G}} U = \bigoplus_k V \otimes \Omega(k)$. If U_x, $x \in \widetilde{G}/G$ are of type B, they all induce the same representation on $\widetilde{G}$, $\mathrm{ind}^{\widetilde{G}} U_x = V$.

for $k \in \widetilde{G}/G$. Note that $V \otimes \Omega(0) \cong V$.

- From a representation U of G, one can construct other representations U_x of G for $x \in \widetilde{G}/G$. These are defined by the action of G on U_x given by $g \cdot u = \widetilde{g}\, g\, \widetilde{g}^{-1} u$ where $\widetilde{g} \in \widetilde{G}$ is such that $x \in g\,G$.

These representations can be partitioned into two types, according to Table 2.4. The reason for this classification is that induction and restriction relate representations of the same type. Now consider a representation of $\mathrm{SU}(N)$, which we call U_0, with Dynkin labels $[\lambda_1, \cdots, \lambda_{N-1}]$, (this means that these are the coefficients of the highest weight in the basis of fundamental weights). Since we are working with the flip involution (2.2.26), the twisted representation by the non-trivial element of $\widetilde{G}/G$, U_1, has Dynkin labels $[\lambda_{N-1}, \cdots, \lambda_1]$. Therefore we are in type A if and only if $\lambda_i = \lambda_{N-i}$ for all i. As a consequence:

- If an $\mathrm{SU}(N)$ representation U has $\lambda_i = \lambda_{N-i}$ for all i, then the induced representation on $\widetilde{\mathrm{SU}}(N)$ is reducible and can be written $(V \otimes \Omega(0)) \oplus (V \otimes \Omega(1))$.

- If an $\mathrm{SU}(N)$ representation U has $\lambda_i \neq \lambda_{N-i}$ for some i, then the induced representation on $\widetilde{\mathrm{SU}}(N)$ is irreducible (and is the same as the induced representation from $[\lambda_{N-1}, \cdots, \lambda_1]$).

For instance, we have

- The fundamental $[1, 0, \cdots, 0]$ of $\mathrm{SU}(N)$ induces a unique irreducible representation of $\widetilde{\mathrm{SU}}(N)$. It has dimension $2N$.

- The adjoint $[1, 0, \cdots, 0, 1]$ of $\mathrm{SU}(N)$ induces a reducible representation of $\widetilde{\mathrm{SU}}(N)$, which decomposes into two irreducibles.

Let us now explicitly construct the fundamental and the adjoint representations, which will be relevant for our later purposes.

The fundamental representation A particularly important representation will be the fundamental representation. It corresponds to the the matrix representation introduced in section 2.2.1, which acts on a $2N$ dimensional complex space $\mathbb{C}^N \times \mathbb{C}^N$. Note that we may alternatively think of this space as $\mathbb{C}^N \times (\mathbb{C}^\star)^N$, thus making explicit that $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ representations comprise a fundamental and antifundamental of the connected component $\mathrm{SU}(N)$. The elements of this space are of the form

$$\mathbf{Q} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \qquad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}. \tag{2.2.31}$$

It is useful to introduce a "conjugate"

$$\overline{\mathbf{Q}} = \mathbf{Q}^T \, \mathbf{\Gamma_0} \,, \qquad \text{with} \qquad \mathbf{\Gamma_0} = \begin{pmatrix} 0 & 1 \\ -c\,1 & 0 \end{pmatrix}. \tag{2.2.32}$$

Then, for a generic $\widetilde{\mathbf{U}} \in \widetilde{G}$, $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ transform as

$$\mathbf{Q} \to \widetilde{\mathbf{U}}\,\mathbf{Q}\,, \qquad \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}\,\widetilde{\mathbf{U}}^\dagger\,. \tag{2.2.33}$$

The adjoint representation Another very important representation for our purposes is the adjoint representation. Given the matrix representation in section 2.2.1, an element $\mathbf{\Phi}$ in the adjoint representation is the $2N \times 2N$ block-diagonal matrix (recall that $\Phi \in \mathfrak{su}(N)$, so $\Phi^\dagger = \Phi$ and the Lie algebra automorphism –complex conjugation for hermitean generators– is $\Phi \mapsto -\Phi^\star$)

$$\mathbf{\Phi} = \begin{pmatrix} \Phi & 0 \\ 0 & -\Phi^\star \end{pmatrix}. \tag{2.2.34}$$

Under $\widetilde{G}$ it transforms as

$$\mathbf{\Phi} \to \widetilde{\mathbf{U}}\,\mathbf{\Phi}\,\widetilde{\mathbf{U}}^\dagger\,. \tag{2.2.35}$$

For future purposes, it is interesting to note that

$$\mathbf{\Gamma_0}\,\mathbf{\Phi}^T\,\mathbf{\Gamma_0} = c\,\mathbf{\Phi}\,. \tag{2.2.36}$$

Note that since one block is complex-conjugated of the other, the number of degrees of freedom is really N^2-1 as it should be for the adjoint. On the other hand, expressing the adjoint in this way turns out to be most convenient for our latter purposes of constructing gauge theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ due to the transformation properties expressed as (2.2.35).

Invariants

Having explicitly constructed the fundamental and the adjoint representations, we now study the invariants which can be constructed out of them. To that matter, let us consider F copies of the fundamental representation in addition to an adjoint representation. To set notation, we will denote $\widetilde{G}$ indices by $\widetilde{\alpha}$ with $\widetilde{\alpha}=1,\cdots,2N$; and "global symmetry indices" by I with $I=1,\cdots,F$. To be explicit with the notation, the fundamentals will be $(\mathbf{Q}_I)^{\widetilde{\alpha}}$. Note that it follows that the indices of the conjugate are $(\overline{\mathbf{Q}}_I)_{\widetilde{\alpha}}$.

With the transformation rules described above for these representations, we may construct all possible group invariants made out of them. Let us stress that the list of such group invariants is infinite and we will not attempt for an exhaustive classification. Instead, we will focus on the ones which will be of uttermost relevance for our purposes. Indeed, we use a gauge-theoretic inspired naming with an eye on applications to gauge theories. Such most relevant invariants are

1. Meson-like invariants: consider

$$\mathbf{M}_{IJ}=(\overline{\mathbf{Q}}_I)_{\widetilde{\alpha}}\,(\mathbf{Q}_J)^{\widetilde{\alpha}}\equiv\overline{\mathbf{Q}}_I\,\mathbf{Q}_J\,. \tag{2.2.37}$$

   It is clear that such quantity is an invariant of the group action, using (2.2.33). Moreover, a short computation[7] shows that, as a $F\times F$ matrix

$$\mathbf{M}_{IJ}=-c\,\mathbf{M}_{JI}\,. \tag{2.2.38}$$

2. Baryon-like invariants: introduce the ϵ-like tensor $\Upsilon_{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_N}$ such that

$$\Upsilon_{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_N}=\begin{cases}\epsilon_{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_N} & \text{if}\qquad \widetilde{\alpha}_i\in 1,\cdots N\;\;\forall i\\ \epsilon_{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_N} & \text{if}\qquad \widetilde{\alpha}_i\in N+1,\cdots 2N\;\;\forall i\\ 0 \qquad \text{otherwise}\end{cases}\qquad; \tag{2.2.39}$$

   where $\epsilon_{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_N}$ is the standard ϵ-tensor in $\mathrm{SU}(N)$. Then we have the baryon-like invari-

[7] $\mathbf{M}_{IJ}=\overline{\mathbf{Q}}_I\,\mathbf{Q}_J=(\overline{\mathbf{Q}}_I\,\mathbf{Q}_J)^T=\mathbf{Q}_J^T{\mathbf{\Gamma_0}}^T\mathbf{Q}_I=-c\mathbf{Q}_J^T\mathbf{\Gamma_0}\mathbf{Q}_I=-c\,\mathbf{M}_{JI}\,.$

ants $\mathbf{B}_{I_1\cdots I_F}$ given by

$$\mathbf{B}_{I_1\cdots I_F} = (\mathbf{Q}_{I_1})^{\widetilde{\alpha}_1}\cdots(\mathbf{Q}_{I_F})^{\widetilde{\alpha}_F}\,\Upsilon_{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_N}\,. \tag{2.2.40}$$

Note that $\mathbf{B}_{I_1\cdots I_F}$ is completely antisymmetric on its F indices.

3. Superpotential-like invariants: consider

$$\overline{\mathbf{Q}}_I\,\mathbf{\Phi}\,\mathbf{Q}_J\,. \tag{2.2.41}$$

It is clear that such quantity is an invariant. Note that we may replace $\mathbf{\Phi}$ by $\mathbf{\Phi}^n$, since the n-power of an adjoint still transforms in the same way. Moreover, as a $F\times F$ matrix, we have

$$\overline{\mathbf{Q}}_I\,\mathbf{\Phi}\,\mathbf{Q}_J = c\,\overline{\mathbf{Q}}_J\,\mathbf{\Phi}\,\mathbf{Q}_I\,. \tag{2.2.42}$$

4. Coulomb branch-like invariants: consider

$$\mathrm{Tr}\,\mathbf{\Phi}^{2n}\,. \tag{2.2.43}$$

It is clear that these quantities are invariant under the group transformations above.

Note that these are “holomorphic" invariants in that they do not make use of complex conjugation. On top of them, and explicitly using complex conjugation, we can construct the “non-holomorphic" quantity (which we will dub Kähler-like)

$$\mathbf{Q}_I^\dagger\,\mathbf{Q}_J\,, \tag{2.2.44}$$

which is also invariant under the above transformations.

The integration measures

In this section, we consider only the case N even (for N odd, we refer to [38]). In order to be able to compute index-like quantities for gauge theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$, we need the integration measures on said groups. Recall that the standard way of defining the Haar measure of a *connected* Lie group, grounded on the fact that conjugation of elements of the maximal torus of the group is surjective onto the full group, doesn’t apply to our situation. Instead, to be able to integrate over the disconnected component of $\widetilde{\mathrm{SU}}(N)$ we use Lemma

2.1 of [40], namely the fact that the map

$$\begin{aligned} \varphi : \mathrm{SU}(N)/S_0(\Theta) \times S_0(\Theta) &\to \mathrm{SU}(N)\Theta \\ (yS_0(\Theta), z) \quad &\mapsto \quad yz\Theta y^{-1} \end{aligned} \tag{2.2.45}$$

where $S_0(\Theta)$ is the subgroup of the maximal torus of $\mathrm{SU}(N)$ left invariant by the involution Θ, is surjective onto the component of $\widetilde{\mathrm{SU}}(N)$ disconnected from the identity. Therefore, we can use φ as a change of variables, and turn the integration over $\mathrm{SU}(N)\Theta$ into one over $S_0(\Theta)$. The measure arises from the Jacobian of the change of variables,

$$\det\left(\mathrm{d}\varphi\right)(y,z) = \det\left(\mathrm{Ad}(z\Theta)^{-1} - \mathrm{Id}\right)\Big|_{\mathfrak{su}(N)/\mathfrak{s}_0(\theta)} , \tag{2.2.46}$$

where $\mathfrak{s}_0(\theta)$ is the Lie algebra of $S_0(\Theta)$. The Jacobian (2.2.46) can be easily calculated from the data in the root system, since the involution Θ is completely defined by the flip ϕ (2.2.26) of the roots and the sign c introduced in (2.2.27). As in [38], we use an adapted parametrization for the fugacities,

$$z^\lambda = \left(\prod_{i=1}^{\frac{N}{2}-1} z_i^{\lambda_i+\lambda_{N-i}}\right)\left(\prod_{i=1}^{\frac{N}{2}-1} z_{\frac{N}{2}+i}^{\lambda_i-\lambda_{N-i}}\right) z_{\frac{N}{2}}^{\lambda_{\frac{N}{2}}} \tag{2.2.47}$$

If a root α is fixed by ϕ, the corresponding element of the Lie algebra X_α is transformed to $-(-c)^{\mathrm{ht}(\alpha)}X_\alpha = cX_\alpha$ since the height is necessarily odd, and it will contribute $(1 - cz^{-\alpha})$ to the determinant (2.2.46). On the other hand, if α is exchanged with $\phi(\alpha)$, their contribution will come from the determinant of the block matrix

$$\det\begin{pmatrix} -1 & -(-c)^{\mathrm{ht}(\alpha)}z^{-\alpha} \\ -(-c)^{\mathrm{ht}(\alpha)}z^{-\phi(\alpha)} & -1 \end{pmatrix} = 1 - z^{-\alpha-\phi(\alpha)} , \tag{2.2.48}$$

where we have used (2.2.28). In total, the integration measure is

$$\mathrm{d}\mu^{-}_{\mathrm{I,II}}(z) = \prod_{\alpha=\phi(\alpha)}\left(1 - cz^{-\alpha}\right)\prod_{\alpha\neq\phi(\alpha)}\left(1 - z^{-(\alpha+\phi(\alpha))}\right)^{1/2}\prod_{j=1}^{N/2}\frac{\mathrm{d}z_j}{2\pi i z_j} . \tag{2.2.49}$$

In (2.2.49), the products run over the positive roots. In the second product, the power $\frac{1}{2}$ takes care of the fact that each pair of roots is counted twice. The integration over $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$

for N even is then obtained by taking an average,

$$\int_{\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}} d\mu(X) f(X) = \frac{1}{2}\left(\int_{\mathrm{SU}(N)} d\mu^+(z) f(z) + \int_{\mathrm{SU}(N)\Theta_{\mathrm{I,II}}} d\mu^-_{\mathrm{I,II}}(z) f(\Theta_{\mathrm{I,II}}(z))\right) , \quad (2.2.50)$$

where f is a function defined on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ which is invariant under conjugation and $d\mu^+$ is the standard Haar measure of $\mathrm{SU}(N)$.

Real and pseudo-real representations

Having constructd a group measure allows us to construct an indicator –the so-called Frobenius-Schur indicator– sensible to the reality properties of the representations. This is very useful since, in Physics language, allows us to discern whether we have an orthogonal, unitary or symplectic global symmetry. We first quote the Frobenius-Schur theorem (see [50], Theorem 43.1). Consider an irreducible representation ρ of a compact group G, and compute the quantity

$$\mathrm{FS}(\rho) = \int_G \chi_\rho(g^2)\mathrm{d}g\,, \quad (2.2.51)$$

where χ_ρ is the character of the representation. Then

$$\mathrm{FS}(\rho) = 1 \iff \rho \text{ is real} \quad (2.2.52)$$

$$\mathrm{FS}(\rho) = 0 \iff \rho \text{ is complex} \quad (2.2.53)$$

$$\mathrm{FS}(\rho) = -1 \iff \rho \text{ is pseudo-real} \quad (2.2.54)$$

Using this, combined with the integration formula, we can investigate the properties of our representations. Let us focus on the fundamental representation of $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$, for N even. Using the measure (2.2.49), we evaluate

$$\mathrm{FS}(\mathrm{Fund}) = -c\,. \quad (2.2.55)$$

This means that the fundamental representation is real in type I and pseudo-real in type II. This will have consequences in the next section, when we will study 4d $\mathcal{N}=2$ gauge theories with fundamental matter in hypermultiplets: the unitary global symmetry that exchanges copies of the fundamental is enhanced to

- Symplectic global symmetry when the representations are real, i.e. in type I;
- Orthogonal global symmetry when the representations are pseudo-real, i.e. in type II.

2.2.3 Construction of $\mathcal{N} = 2$ gauge theories

We now explicitly construct gauge theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ groups. For definiteness, we will construct 4d $\mathcal{N} = 2$ SQCD-like theories.

A point which is worth emphasizing is that the $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ groups are not just the direct product of $SU(N)$ and the charge conjugation $\mathbb{Z}_2$. This bars the simple construction of general gauge theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ as the extra gauging of $\mathbb{Z}_2$ in a standard $SU(N)$ $\mathcal{N} = 2$ theory, a procedure which would be tantamount to considering the direct product $SU(N) \times \mathbb{Z}_2$ which in general would not be consistent. An intuitive reason is that complex conjugation cannot be disjoint from gauge transformations since these are in general complex. The advantage of constructing the $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ groups is that this problem is *ab initio* circumvented and hence the standard technology to construct gauge theories can be directly imported.

Matter content

The relevant multiplets to construct our theories are

- Vector multiplet:
  The vector multiplet contains, in 4d $\mathcal{N} = 1$ language, a vector multiplet and a chiral multiplet in the adjoint. The latter will be described by an adjoint superfield which we will denote by $\mathbf{\Phi}$ with the transformation properties described in section 2.2.2.

- (Fundamental) Hypermultiplet:
  In order to construct hypermultiplets, let us take two chiral superfields, say $\mathbf{A}$ and $\mathbf{B}$, transforming in the fundamental representation as described in section 2.2.2. Out of, say, $\mathbf{B}$, we can construct the corresponding $\overline{\mathbf{B}}$. Let us now consider constructing a chiral superfield out of $\mathbf{A}$, and another chiral superfield out of $\mathbf{B}$, but instead taking its barred cousin, *i.e.* $\overline{\mathbf{B}}$. Let us insist once again that *both* $\mathbf{A}$ and $\overline{\mathbf{B}}$ are chiral superfields of the *same* chirality. Thus, we may construct a hypermultiplet out of them, *i.e.*

$$\mathcal{H} = (\mathbf{A},\, \overline{\mathbf{B}})\,. \tag{2.2.56}$$

  Note that both $\mathbf{A}$ and $\overline{\mathbf{B}}$ provide $2N$ degrees of freedom, so that $\mathcal{H}$ contains 4N degrees of freedom.

Having described the basic ingredients, the construction of the Lagrangian follows the standard techniques in supersymmetric gauge theories. The kinetic terms will come from a

Kähler potential. A natural candidate is the "non-holomorphic" invariant (2.2.44) above, *i.e.* (we quote the free case; we will comment on the gauged version below)

$$K = \mathbf{A}^{\dagger}\,\mathbf{A} + \overline{\mathbf{B}}\,\overline{\mathbf{B}}^{\dagger}\,. \tag{2.2.57}$$

Let us now turn to the superpotential W. Since it is an integration over half of superspace, it can only involve the chiral fields in $\mathcal{H}$. Assuming a number F of hypermultiplets, the natural W can be constructed out of (2.2.42),

$$W = \overline{\mathbf{B}}_J\,\mathbf{\Phi}\,\mathbf{A}_I\,\mathbf{G}^{IJ}\,. \tag{2.2.58}$$

with $\mathbf{G}^{IJ}$ a suitable matrix of couplings which would be fixed by the requirement of $\mathcal{N}=2$ SUSY.

So far we have evaded the gauge sector. By construction, only the part of the gauge group connected to the identity will contribute with a field in the Lagrangian, while the disconnected part of the gauge group will appear as a superselection rule (see [29, 53] for early discussions, and [35] for a more recent account). Thus, the vector multiplet will be the standard one associated to the gauge transformations in the $\mathrm{SU}(N)$ part of $\widetilde{\mathrm{SU}}(N)$.

Smaller representations

So far we have assumed $\mathbf{A}\neq\mathbf{B}$. But nothing prevents us from taking $\mathbf{A}=\mathbf{B}=\mathbf{Q}$. In this case, the hypermultiplet becomes $\mathcal{H}=(\mathbf{Q},\,\overline{\mathbf{Q}})$. Note that this cannot be done with a standard $\mathrm{SU}(N)$ hypermultiplet: indeed, if we want to construct invariants of $\mathrm{SU}(N)$ we need to consider a hypermultiplet $(Q,\,\tilde{Q})$ with Q a fundamental of $\mathrm{SU}(N)$ and $\tilde{Q}$ an antifundamental. If we wanted "$Q=\tilde{Q}$", we would have to set $\tilde{Q}\sim Q^{\star}$, and hence it would be a chiral superfield of the other chirality. The crucial difference is now that in the fundamental of $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ there is both the $\mathbf{N}$ and the $\overline{\mathbf{N}}$ of the connected $\mathrm{SU}(N)$ part, and the construction of the second element of the hyper –the equivalent to $\tilde{Q}$– does not involve complex conjugation but rather a simple transposition, and hence does not change the chirality of the superfield. Since the degrees of freedom are half of the standard hyper, it would be perhaps more appropriate to call this $2N$ dimensional representation a half-hypermultiplet (note that in fact this is the same number of dof. as a full hypermultiplet of $SU(N)$).

All in all, we can write the theory for F half-hypermultiplets. The W is just the obvious particularization of (2.2.58), *i.e.*

$$W = \overline{\mathbf{Q}}_J \, \mathbf{\Phi} \, \mathbf{Q}_I \, \mathbf{G}^{IJ} \tag{2.2.59}$$

and using the symmetry property (2.2.42) fixes the matrix $\mathbf{G}$ to be either symmetric or antisymmetric. This allows us to immediately read the global symmetry of the theory:

- $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$: $\mathbf{G}$ is antisymmetric. The global symmetry is $\mathrm{Sp}(\frac{F}{2})$.
- $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$: $\mathbf{G}$ is symmetric. The global symmetry is $\mathrm{SO}(F)$.

This is in perfect agreement with the result derived using the Frobenius-Schur indicator in section 2.2.2, and it will be confirmed by the explicit computation of the Higgs branch Hilbert series. Moreover, it is also suggested by table 2.2 – the $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ behaves in this respect as its subgroup K would. If K is of orthogonal type, then the global symmetry will be symplectic, and vice versa.

Note that for the type I extensions the case of odd F is not well-defined. The issue is manifest in the simplest case of $F = 1$, where it is simply impossible to write a non-vanishing W. Since for any odd F one can write $F = 2f + 1$, this very same argument suggests that type I theories with odd number of flavors do not exist as a $\mathcal{N} = 2$ theories. In the following we will restrict our attention to even F for type I theories.

Dynamics

In the following we will be interested in SCDQ theories with $\widetilde{SU}(N)_{\mathrm{I,II}}$ gauge group and F fundamental half-hypers. As discussed above, the vector multiplet only contains a gauge field for the connected part of the gauge symmetry, while the disconnected part only enters as a superselection rule. As a consequence, the Lagrangian of the theory is just identical to that of its $SU(N)$ SQCD cousin. Hence, the Feynman rules will just be the same, and consequently, all local Physics will be identical to that of SQCD with the only extra addition that one has to impose the constraints arising from gauge invariance under the disconnected part of the gauge group (see *e.g.* [29, 35, 53]).

An important consequence of these observations is that all (local) anomalies are just identical to those in SQCD, which, in particular, implies that pure gauge anomalies automatically vanish (a consequence of being a non-chiral theory). Note however that in general there may be 't Hooft anomalies associated to global symmetries (including mixed gauge-$U(1)_R$ anomalies, which vanish in the conformal case). In addition, there may be anomalies associated to the disconnected part of the gauge group. It would be very interesting to undertake a detailed analysis of this point.

Another very important consequence is that the β function will just be the same as in SQCD. Thus, in particular we can tune N and F and restrict to well-behaved 4d QFT's. In particular, we can choose N and F so that our theories become conformal. This will be the most interesting case, since the gauge dynamics will greatly simplify due to the absence of a strong coupling scale and the full power of conformal invariance will provide very useful tools to analyze the theories. In particular, by means of the SCI we can study their spectrum in both the Coulomb and Higgs branches as we will do below. Note however that the Higgs branch is non-renormalized [54], and thus when, studying the Higgs branch, the requirements on N and F may be dropped (more on this below).

2.2.4 The spectrum of the theory

One aspect of basic interest is the operator content of the theories based on $\widetilde{G}$ and their relations. As discussed above, we can restrict to well-defined QFT's by choosing N and F so that the theory is at least asymptotically free. Nevertheless, in order to avoid the complicated gauge dynamics associated to the strong coupling scale of the gauge group, we can further focus on SCFT's. In that case, due to superconformal invariance, we have the powerful tool of the SCI to analyze the operator spectrum of the theory. While the full index is a complicated function, in particular limits it simplifies and allows to study in detail both the Coulomb branch and the Higgs branch of the theory.

Regarding the Coulomb branch, we can study the operator content through the so called Coulomb branch limit of the superconformal index [55].

In turn, for the Higgs branch, we can consider the Hall-Littlewood limit of the index. On general grounds, for a theory corresponding to a quiver with no loops, it is clear that the computation of such Hall-Littlewood limit of the index coincides with the computation of the Higgs branch Hilbert series, which is a counting of gauge-invariant operators made out of hypermultiplets [56].[8] Note however that, due to supersymmetry, the Higgs branch remains classical [54]. Hence the computation of the Higgs branch Hilbert series using the classical Lagrangian even beyond the conformal window provides us a sensible description of the Higgs branch in the full quantum theory. Thus, when computing the Higgs branch Hilbert

[8] The way this comes about is as follows: for a theory such as SQCD, the vector multiplet contribution to the index is through the gaugino, and it precisely coincides with the would-be contribution of the F-term constraint to the Hilbert series. On the other hand, the hypermultiplet contribution is just identical in both the Hall-Littlewood limit of the index and the Hilbert series.

series, we will not restrict ourselves to theories in the conformal window. To be specific, we will consider below theories with gauge group $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ with F half-hypermultiplets (in the sense discussed above) only when $F \geq 2(N-1)$. The $F < 2N$ region is more difficult to study in part because at a generic point on the Higgs branch, the gauge group may not be completely Higgsed, which means that we can not use the letter-counting formula (2.2.71). A complete treatment of that question has appeared in [57] for gauge groups $\mathrm{SU}(N)$, but the extension of the techniques used in [57] to disconnected gauge groups remains an open problem.

Warm-up: the free theory

Let us first consider the free theory by sending the Yang-Mills coupling to zero. The spectrum of the theory will consist of gauge-invariant operators (as Gauss' law is kept as a constraint) with no other relation. Focusing on the Higgs branch –*i.e.* on operators made out of hypermultiplet fields–, to lowest order the gauge invariants are the mesons $\mathbf{M}_{IJ}$ (we assume N big enough so that baryon-like operators appear at high dimensions), which are either a symmetric (for $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$) or an antisymmetric (for $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$) $F \times F$ matrix. Thus, introducing a fugacity t to count dimensions, we should expect the first non-trivial contribution a (unrefined) partition function counting operators to be $t^2\, \frac{F\,(F\pm 1)}{2}$.[9]

Let us consider the next order t^4. For definitness, say we have odd N –so we have $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$. In that case, $\mathbf{M}$ is a symmetric matrix and hence has $d_S = \frac{F\,(F+1)}{2}$ entries. To order t^4 we will have the symmetrized product of those, *i.e.* $\frac{d_S\,(d_S+1)}{2}$. In turn, while at order t^2, the $d_A = \frac{F\,(F-1)}{2}$ antisymmetric pieces of $\mathbf{M}$ are projected out, their symmetrized squares, *i.e.* $\frac{d_A\,(d_A+1)}{2}$, survive at order t^4. Hence the t^4 coefficient is expected to be

$$\frac{1}{2}\Big[\frac{F\,(F+1)}{2}\,\Big(\frac{F\,(F+1)}{2}+1\Big)\Big] + \frac{1}{2}\Big[\frac{F\,(F-1)}{2}\,\Big(\frac{F\,(F-1)}{2}+1\Big)\Big] = \frac{F^2\,(F^2+3)}{4}\,. \quad (2.2.60)$$

Note that, for $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$ the roles of symmetric and antisymmetric are exchanged. Nevertheless this has no effect on the t^4 coefficient. Hence, all in all, we expect

- $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$:

$$\mathrm{HS}_{\mathrm{I}}^{\mathrm{free}}(t) = 1 + \frac{F\,(F+1)}{2}\,t^2 + \frac{F^2\,(F^2+3)}{4}\,t^4 + o(t^4)\,; \quad (2.2.61)$$

[9] As argued above, the Hall-Littlewood limit index of the index coincides with the Higgs branch Hilbert series for the theories at hand. Nevertheless, strictly speaking, if though as the Hall-Littlewood limit of the index, the contribution of vector multiplet and hypermultiplet comes weigthed by a fugacity τ^R. However, since the operators to count satisfy the BPS bound $\Delta = 2R$, the difference between Higgs branch Hilbert series and Hall-Littlewood limit of the index is just a simple redefinition of the fugacity $t^2 \leftrightarrow \tau$.

- $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$:

$$\mathrm{HS}^{\text{free}}_{\mathrm{II}}(t) = 1 + \frac{F\,(F-1)}{2}\,t^2 + \frac{F^2\,(F^2+3)}{4}\,t^4 + o(t^4)\,. \tag{2.2.62}$$

Note that this implicitly assumes N large enough. Indeed, if $N \leq 4$, the baryon would contribute to the order t^4. As we will see below, indeed the integration formula allows us to recover this expectation from the computation of the Higgs branch Hilbert series for the free theory.

Note that the t^2 term is somewhat special, in that its contributions come from scalars in conserved current multiplets (moment maps of the global symmetry). Since these are, by construction, in the adjoint representation of the global symmetry, the coefficient of t^2 provides a cross-check of the global symmetry of the theory. Indeed, for type I that coefficient coincides with the dimension of the adjoint of $\mathrm{Sp}(\frac{F}{2})$, while for type II it coincides with the dimension of the adjoint of $\mathrm{SO}(F)$.

Using the measure on the groups developed above we can cross-check (and extend to arbitrary order) the expectations above. The (free theory) Higgs branch Hilbert series reads

$$\mathrm{HS}^{\text{free}}_{(N,F)}(t) = \int_G d\eta_G(X)\frac{1}{\det(1-t\Phi_{\text{Fund}}(X))^F}\,. \tag{2.2.63}$$

As discussed above, the integral splits into the sum of the connected and disconnected part, and the measures are the ones found in section 2.2.2. It is then easy to show that indeed the expectation above for the first few terms is recovered. In order not to clutter the presentation, as an example, we quote the results for $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$, and $F = 2,4,6$ [10]

$$\begin{aligned}
\mathrm{HS}^{\text{free}}_{(4,2)}(t) =& \mathrm{PE}[3t^2+t^4] = 1+3t^2+7t^4+o\left(t^4\right)\ , \\
\mathrm{HS}^{\text{free}}_{(4,4)}(t) =& \frac{1-t^2+16t^4-10t^6+25t^8-5t^{10}+6t^{12}}{\left(1-t^2\right)^{17}\left(1+t^2\right)^6} = 1+10t^2+77t^4+o\left(t^4\right)\ , \\
\mathrm{HS}^{\text{free}}_{(4,6)}(t) =& \frac{1}{\left(1-t^2\right)^{33}\left(1+t^2\right)^{14}}\Big(1+2t^2+124t^4+435t^6+3393t^8+11034t^{10}+38282t^{12}+ \\
& 91513t^{14}+195923t^{16}+326359t^{18}+476999t^{20}+554635t^{22}+569026t^{24}+ \\
& 465194t^{26}+334666t^{28}+190410t^{30}+95283t^{32}+35694t^{34}+12626t^{36}+2599t^{38}+
\end{aligned}$$

[10] The plethystic exponential (PE) of a function $f(x)$ such that $f(0)=0$ is defined as

$$\mathrm{PE}[f(x)] = \exp\left(\sum_{n=1}^{\infty}\frac{f(x^n)}{n}\right)\,.$$

$$734t^{40} + 45t^{42} + 15t^{44}\Big) = 1 + 21t^2 + 366t^4 + o\left(t^4\right) \quad ,$$

and the results for $N = 6$ and $F = 2, 4, 6, 8$:

$$\begin{aligned}
\mathrm{HS}^{\mathrm{free}}_{(6,2)}(t) =& \mathrm{PE}[3t^2 + t^4] = 1 + 3t^2 + 7t^4 + o\left(t^4\right) \quad , \\
\mathrm{HS}^{\mathrm{free}}_{(6,4)}(t) =& 1 + 10t^2 + 76t^4 + o\left(t^4\right) \quad , \\
\mathrm{HS}^{\mathrm{free}}_{(6,6)}(t) =& 1 + 21t^2 + 351t^4 + o\left(t^4\right) \quad , \\
\mathrm{HS}^{\mathrm{free}}_{(6,8)}(t) =& 1 + 36t^2 + 1072t^4 + o\left(t^4\right) \quad .
\end{aligned}$$

We observe that the first three terms in each of these examples match the expected result given by (2.2.61), once the additional baryons that appear in the t^4 term for $N = 4$, $F \geq 4$ are taken into account. In particular, this provides a confirmation of our expectations on the global symmetry to add to the computation of the Frobenius-Schur indicator as described above.

Furthermore, while both the numerator of the Hilbert series for the component connected with the identity and the numerator of the Hilbert series for the component not connected with the identity are palindromic in the above examples, in general the full Hilbert series $\mathrm{HS}^{\mathrm{free}}_{(N,F)}(t)$ has not a palindromic numerator. Note however that in the free limit we are considering there is no *a priori* reason for the Hilbert series to be palindromic (for instance, upon removing the W the theory is effectively not even $\mathcal{N} = 2$). Moreover, in general, the ring of invariants as a quite involved structure and the corresponding Highest Weights Generating function (HWG) [58] does not seem to be given by a complete intersection.

The full theory: Coulomb branch operators

The Coulomb branch index is a counting of operators on the Coulomb branch of a CFT, and thus can be thought as a Hilbert series for the Coulomb branch. Note that the hypermultiplets only enter this computation through ensuring that we have a CFT, but otherwise they are blind to the computation of the Coulomb branch index. Thus, we will assume the matter content to be such that the theory has vanishing beta functions. From the transformation properties of the adjoint representations described above, it is clear that for either $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ the Coulomb branch will only count operators of the form $\mathrm{Tr}\,\Phi^{2n}$. Thus it is clear that [38]

$$\mathrm{HS}^{C}_{N}(t) = \frac{1}{2}\Big[\prod_{n=2}^{N}\frac{1}{1-t^2} - \prod_{n=2}^{N}\frac{1}{1-(-t)^i}\Big]\,. \tag{2.2.64}$$

This can be explicitly verified using the integration formula. On general grounds, for a theory with gauge group G, the Coulomb branch index (or Coulomb branch Hilbert series) reads

$$\mathrm{HS}_N^C(t) = \int_G d\eta_G(X) \frac{1}{\det(1 - t\Phi_{\mathrm{Adj}}(X))} \,, \tag{2.2.65}$$

Using the formula (2.2.65) we get

$$\mathrm{HS}_4^C(t) = \mathrm{PE}[t^2 + t^4 + t^6] \,, \tag{2.2.66}$$

$$\mathrm{HS}_6^C(t) = \mathrm{PE}[t^2 + t^4 + 2t^6 + t^8 + t^{10} - t^{16}] \,, \tag{2.2.67}$$

which indeed agrees with (2.2.64).

Note that eq.(2.2.64) shows that the Coulomb branch is identical as a complex variety for both $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$. Moreover, it immediately follows that both families of theories provide explicit examples of consistent $\mathcal{N} = 2$ QFT's with non-freely generated Coulomb branches, thus extending [38].

The full theory: Higgs branch operators

Let us now look to the operators in the Higgs branch for the $\mathcal{N} = 2$ theories. Before delving in a full computation of the generating function of such operators, let us first obtain by hand the lowest lying such operators. To that matter we now need to additionally mod out by the F-terms. Note first that the F-terms can be computed by forgetting the vanishing trace requirement on the adjoint and adding a Lagrange multiplier $\lambda\,\mathrm{Tr}\,\mathbf{\Phi}$ to the W. Then, the F-terms are essentially

$$\mathbf{Q}_I\,\overline{\mathbf{Q}}_J\,\mathbf{G}^{IJ} = \mathbf{1}\,, \tag{2.2.68}$$

where the free indices are in color space.

It is clear that the F terms will enter first at order t^4. Thus, the coefficient of t^2 is just like in the free theory, and hence the same comment on the fact that it dictates the global symmetry of the theory applies. In turn, at order t^4 we need to take F terms into account. Eq.(2.2.68) essentially means that, when squaring $\mathbf{M}$ to construct the terms contributing to t^4, one combination of them, times a antisymmetric $F \times F$ matrix ($\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$) or symmetric matrix ($\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$) can be dropped. Hence, we should expect the t^4 term in the $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$ case to be that of the free theory minus $\frac{F\,(F+1)}{2}$; while for $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ it should be that of the free theory minus $\frac{F\,(F-1)}{2}$. That is, we expect

- $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$:

$$\mathrm{HS}^{\mathrm{I}}(t) = 1 + \frac{F\,(F+1)}{2}\,t^2 + \frac{F\,(F+1)}{2}\Big(\frac{F\,(F-1)}{2}+1\Big)\,t^4 + o(t^4)\ \ ; \tag{2.2.69}$$

- $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$:

$$\mathrm{HS}^{\mathrm{II}}(t) = 1 + \frac{F\,(F-1)}{2}\,t^2 + \frac{F\,(F-1)}{2}\Big(\frac{F\,(F+1)}{2}+1\Big)\,t^4 + o(t^4)\ \ . \tag{2.2.70}$$

Just like in the free case, we can explicitly test this expectation and extend it to arbitrary orders in t by explicitly computing the Higgs branch Hilbert series (recall, identical to the Hall-Littlewood limit of the index) using the Haar measure and the technology developed above. It generically reads

$$\mathrm{HS}_{(N,F)}(t;q_i) = \int_G d\eta_G(X)\frac{\det(1-t^2\Phi_{\mathrm{Adj}}(X))}{\det(1-t[1,0,...,0]\times\Phi_{\mathrm{Fund}}(X))}\ \ , \tag{2.2.71}$$

where the $\{q_i\}$ are set of global symmetry fugacity and $[1,0,...,0]$ is the Dynkin label for the fundamental representation of the global symmetry group.

In order to give a flavor of the computation, let us make explicit the ingredients in (2.2.71) in the simplest example where the two outer involutions Θ_{I} and Θ_{II} are different, which is SU(4). Let's begin by choosing the following basis for the $\mathfrak{su}(4)$ Lie-algebra

$$\begin{aligned}\{&h_1,h_2,h_3,X_{\alpha_1},X_{\alpha_1+\alpha_2},X_{\alpha_2},X_{\alpha_1+\alpha_2+\alpha_3},X_{\alpha_2+\alpha_3},X_{\alpha_3},\\ &X_{-\alpha_1},X_{-\alpha_1-\alpha_2},X_{-\alpha_2},X_{-\alpha_1-\alpha_2-\alpha_3},X_{-\alpha_2-\alpha_3},X_{-\alpha_3}\}\ \ ,\end{aligned} \tag{2.2.72}$$

where the h_i, for $i=1,2,3$, denote generators of the Cartan subalgebra, while the α_i are the associated simple roots. For the type II extension, according to the discussion in section

2.2.1, the flip involution will act in the different representations of the Lie algebra as

$$\Phi_{\mathrm{Adj}}(\Theta_{\mathrm{II}}) = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \end{pmatrix}, \tag{2.2.73}$$

$$\Phi_{\mathrm{Fund}}(\Theta_{\mathrm{II}}) = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}. \tag{2.2.74}$$

The measures for the connected and non-connected component with the identity have already been discussed in [38], and read

$$\mathrm{d}\mu_4^+(z) = \frac{\mathrm{d}z_1}{2\pi i z_1}\frac{\mathrm{d}z_2}{2\pi i z_2}\frac{dz_3}{2\pi i z_3}\left(1-\frac{z_1^2 z_3^2}{z_2}\right)\left(1-z_2 z_3^2\right)\left(1-\frac{z_2^2}{z_1^2}\right)\left(1-z_1^2\right)\left(1-\frac{z_2}{z_3^2}\right)\left(1-\frac{z_1^2}{z_2 z_3^2}\right), \tag{2.2.75}$$

$$\mathrm{d}\mu_{4,\mathrm{II}}^-(z) = \frac{\mathrm{d}z_1}{2\pi i z_1}\frac{\mathrm{d}z_2}{2\pi i z_2}\left(1-\frac{z_1^4}{z_2^2}\right)\left(1-z_2^2\right)\left(1-\frac{z_2^2}{z_1^2}\right)\left(1-z_1^2\right). \tag{2.2.76}$$

The second involutive outer automorphism is completely analogous except for the sign c

introduced in (2.2.27). The matrix $\Phi_{\text{Adj}}(\Theta_{\text{I}})$ reads

$$\Phi_{\text{Adj}}(\Theta_I) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.2.77}$$

On the other hand the matrix $\Phi_{\text{Fund}}(\Theta_{\text{I}})$ acting on the fundamental representation reads

$$\Phi_{\text{Fund}}(\Theta_{\text{I}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.2.78}$$

In this case, the measure of the disconnected part, with the same parametrization of the fugacities (2.2.47), is

$$\mathrm{d}\mu^{-}_{4,\text{I}}(z) = \frac{\mathrm{d}z_1}{2\pi i z_1}\frac{\mathrm{d}z_2}{2\pi i z_2}\left(1+z_1^2\right)\left(1-z_2^2\right)\left(1-\frac{z_1^4}{z_2^2}\right)\left(1+\frac{z_2^2}{z_1^2}\right) . \tag{2.2.79}$$

Note the somewhat unusual + signs that appear in the measure, as a consequence of (2.2.49).

The Higgs branch Hilbert series of SQCD

It is now straightforward to put all ingredients in place and explicitly evaluate (2.2.71) to obtain the full, refined, Higgs branch Hilbert series (or Hall-Littlewood index). Due to the lengthy –and rather non-illuminating– expressions, here we will quote some such examples of the type I projection for even N cases, referring to [38] for type I for odd N and type II examples.

$N = 4$ and $F = 8, 10$

$$\begin{aligned}
\mathrm{HS}^{\mathrm{I}}_{(4,8)}(t;q_i) &= 1 + [2,0,0,0]_{C_4}t^2 + \Big([4,0,0,0]_{C_4} + 2[0,2,0,0]_{C_4} + 2[0,0,0,1]_{C_4} \\
&\quad + 3[0,1,0,0]_{C_4} + 3\Big)t^4 + o(t^4)\ , \\
\mathrm{HS}^{\mathrm{I}}_{(4,10)}(t;q_i) &= 1 + [2,0,0,0,0]_{C_5}t^2 + \Big([4,0,0,0,0]_{C_5} + 2[0,2,0,0,0]_{C_5} + 2[0,0,0,1,0]_{C_5} \\
&\quad + 3[0,1,0,0,0]_{C_5} + 3\Big)t^4 + o(t^4)\ ,
\end{aligned}$$

where $\{q_i\}$ denote a set of global symmetry fugacities.

$N = 6$ and $F = 12, 14$

$$\begin{aligned}
\mathrm{HS}^{\mathrm{I}}_{(6,12)}(t;q_i) &= 1 + [2,0,0,0,0,0]_{C_6}t^2 + \Big([4,0,0,0,0,0]_{C_6} + 2[0,2,0,0,0,0]_{C_6} \\
&\quad + [0,0,0,1,0,0]_{C_6} + 2[0,1,0,0,0,0]_{C_6} + 2\Big)t^4 + o(t^4)\ , \\
\mathrm{HS}^{\mathrm{I}}_{(6,14)}(t;q_i) &= 1 + [2,0,0,0,0,0,0]_{C_7}t^2 + \Big([4,0,0,0,0,0,0]_{C_7} + 2[0,2,0,0,0,0,0]_{C_7} \\
&\quad + [0,0,0,1,0,0,0]_{C_7} + 2[0,1,0,0,0,0,0]_{C_7}\Big)t^4 + o(t^4)\ .
\end{aligned}$$

As can be seen in these expressions –as well as in the analogous ones in [38]–, at order t^2 we find the character of the representation of the adjoint of the predicted global symmetry group. Since such contribution is precisely coming from the conserved global symmetry current multiplet (in fact from one of the scalars in the multiplet $\sim$ moment maps), and the latter must be in the adjoint by definition, this provides a further check on our expectations.

The full unrefined Higgs branch Hilbert series

Upon unrefining one can find a slightly more manageable form of the Higgs branch Hilbert series.

As for the component connected to the identity, which is identical to standard SQCD, the Highest Weight Generating function (HWG) is known exactly [38, 58]

$$\mathrm{HWG}^{+}_{(N,F)}(t;\mu_i) = \mathrm{PE}\left[t^2 + \sum_{i=1}^{N-1} t^{2i}\mu_i\mu_{F-i} + t^N(\mu_N + \mu_{F-N})\right] . \tag{2.2.80}$$

Here the $\{\mu_i\}$ denote a set of highest weight fugacities for the $\mathrm{SU}(F)$ global symmetry group. Then, using (2.2.80), we can obtain the expression of the corresponding Hilbert Series $\mathrm{HS}^{+}_{(N,F)}(t)$ for the component connected with the identity. On the other hand the Hilbert Series for the component non-connected with the identity $\mathrm{HS}^{(\mathrm{I},\mathrm{II}),-}_{(N,F)}(t)$ can be explicitly computed performing the integration with the corresponding measure.

As an explicit example, let us consider the case of $\widetilde{\mathrm{SU}}(3)_I$ with $F = 6$, and, in order not to clutter the presentation, postpone to appendix 5.A.2 a longer list of examples.

$N = 3$ and $F = 6$

$$\begin{aligned}\mathrm{HS}^{+}_{(3,6)}(t) = \frac{1}{(1-t)^{20}(1+t)^{16}(1+t+t^2)^{10}}\Big(&1 + 6t + 41t^2 + 206t^3 + 900t^4 + 3326t^5 + \\ &10846t^6 + 31100t^7 + 79677t^8 + 183232t^9 + 381347t^{10} + 720592t^{11} + 1242416t^{12} + \\ &1959850t^{13} + 2837034t^{14} + 3774494t^{15} + 4624009t^{16} + 5220406t^{17} + 5435982t^{18} + \\ &\ldots + \text{ palindrome } + \ldots + t^{36}\Big) ,\end{aligned}$$

$$\mathrm{HS}^{\mathrm{I},-}_{(3,6)}(t) = \frac{1 + 2t^2 + 16t^4 + 23t^6 + 59t^8 + 46t^{10} + 59t^{12} + 23t^{14} + 16t^{16} + 2t^{18} + t^{20}}{(1-t^2)^{12}(1+t^2)^8} ,$$

$$\begin{aligned}\mathrm{HS}^{\mathrm{I}}_{(3,6)}(t) = \frac{1}{(1-t)^{20}(1+t)^{16}(1+t^2)^8(1+t+t^2)^{10}}\Big(&1 + 6t + 34t^2 + 144t^3 + 647t^4 + 2588t^5 + \\ &9663t^6 + 31988t^7 + 97058t^8 + 268350t^9 + 687264t^{10} + 1628374t^{11} + 3598201t^{12} + \\ &7421198t^{13} + 14364220t^{14} + 26130494t^{15} + 44837750t^{16} + 72656468t^{17} + 111456702t^{18} + \\ &162010222t^{19} + 223544610t^{20} + 292994926t^{21} + 365233973t^{22} + 433158422t^{23} + \\ &489154949t^{24} + 526027956t^{25} + 538960928t^{26} + \ \ldots \ + \text{palindrome } + \ldots + t^{52}\Big) .\end{aligned}$$

Note that again the coefficient of the t^2 term is just the expected one to reproduce the predicted global symmetry. Moreover, we observe that the dimension of the pole at $t = 1$ is the same for both $\mathrm{HS}^{+}_{(N,F)}(t)$ and $\mathrm{HS}^{\mathrm{I}}_{(N,F)}(t)$. A similar feature, for a different type of disconnected group, was observed in [46]. Moreover both the numerator of $\mathrm{HS}^{+}_{(N,F)}(t)$ and the numerator of $\mathrm{HS}^{\mathrm{I},-}_{(N,F)}(t)$ are given by a palindromic polynomial.

2.2.5 Conclusions

Because of a number of reasons, ranging from Condensed Matter inspirations to SUSY QFT, there has recently been interest in gauging discrete symmetries in Quantum Field Theory. In this section we have discussed the case of the charge conjugation symmetry in gauge theories based on $\mathrm{SU}(N)$ gauge groups in a systematic manner (systematically extending [44] and its more recent stringy version [45]). A key observation is that charge conjugation symmetry involves the outer automorphism of the $\mathrm{SU}(N)$ group, which is essentially complex conjugation and it is isomorphic to $\mathbb{Z}_2$. Since complex conjugation is non-trivially intertwined with the standard gauge transformations, it turns out that the appropriate framework for that is to construct a larger group which, from the beginning, includes both standard gauge transformations as well as complex conjugation on equal footing. More precisely, these two actions form a semidirect product group which can be thought as an extension of the outer automorphism group by the connected component. In this case this amounts to the extension of $\mathbb{Z}_2$ by $\mathrm{SU}(N)$. Quite surprisingly, and to our knowledge unnoticed in the literature, it turns out that the possible such extensions are in one-to-one correspondence with the Cartan classification of symmetric spaces (in this case of type A). Thus, in the case at hand it turns out that there are exactly two such groups including a gauged version of charge conjugation. Mirroring the terminology for symmetric spaces, we have dubbed these $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ (recall that the $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$ only exists for even N). This extends [38], which, in the newest terminology, only considered $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ for odd N and $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$ for even N.

In this section we provided an explicit construction of the $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ groups. As a by-product, we can explicitly write down the transformation properties of the fundamental and adjoint representations. Since these are the building blocks for 4d $\mathcal{N} = 2$ SQCD-like theories, we can explicitly write down the Lagrangian and understand, from first principles, the global symmetry pattern. We find that the global symmetry for a $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$ theory with F (half)-hypermultiplets is $\mathrm{Sp}(\frac{F}{2})$ –which requires F to be even–, while for a $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$ theory with F (half)-hypermultiplets it is $\mathrm{SO}(F)$. Also, the precise description of the groups allows us to write down a Haar measure and ultimately to explicitly compute indices counting operators which characterize some branches of the moduli space of the theory. Indeed, this way we can not only check that the expected global symmetry pattern emerges; but also that both $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$ SQCD theories have non-freely generated Coulomb branches. This is very interesting as it provides examples of non-freely generated $\mathcal{N}=2$ Coulomb branches.

In this section we focused, as a proof-of-concept, on 4d $\mathcal{N}=2$ SQCD-like theories based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$. Nevertheless, it is clear that we are just scratching the tip of an iceberg. Staying in the, perhaps tamest, realm of $\mathcal{N}=2$ theories, it would be interesting to study the String/M-theory realization. The close relative of gauging the CP symmetry has been considered in string-theoretic constructions in the past ([59]. See *e.g* [60] for a more recent discussion). While most of these constructions were typically devised with an eye on phenomenologically viable string-inspired scenarios, it is tempting to guess that our construction could fit along those lines. Another natural embedding in String Theory is through an orientifold construction, where indeed our groups play a role at intermediary steps as discussed in [45]. Yet another promising avenue would be embedded our theories into the class $\mathcal{S}$ framework, perhaps yielding a connection to the constructions in [61, 62]. It would also be very interesting to explore landmark aspects of discrete gauge theories such as codimension 2 defects [53], as well as other dimensionalities and other SUSY's (including no SUSY). In particular, in other dimensions it may be that new interesting phenomena are possible. For instance, given that $\pi_0(\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}})=\mathbb{Z}_2$, one may imagine a discrete θ parameter in a SUSY QM based on $\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}$. Also, one may consider extending the construction to $\mathrm{U}(N)$ groups. Since the latter have a non-trivial fundamental group, the corresponding extended versions may lead to interesting phenomena. There may be also a parallel to the Pin groups, in particular upon considering quotients by subgroups of the center. Also more exotic constructions, similar to the 2d $\mathrm{O}(N)_\pm$ orbifolds as in [63], may be possible in 2d. It would also be very interesting to explore dynamical aspects of these theories, perhaps using localization to compute correlation functions along the lines of [64].

2.3 Discrete gauging and Hasse diagrams

In this section, we continue the study of gauge theories with disconnected gauge groups. The study of gauge theories based on connected and simply connected Lie groups, like $\mathrm{SU}(N)$, has been a very active field of research early on, as they play a central role in the description of high energy physics, with the focus being really on the Lie algebra. However it was soon acknowledged that the global structure of the gauge group, beyond its Lie algebra properties, also plays a crucial physical role. The importance of the fundamental group π_1 has recently been abundantly discussed in the context of supersymmetric gauge theories [65] and the standard model [66].

By contrast, the importance of the group π_0 of connected components has been less investigated, even though early studies pointed out its physical relevance [29, 67, 68], and more recent works connect *finite* gauge groups and dualities in quantum field theory [24, 69]. It should also be mentioned that discrete symmetries appear prominently in the context of higher form symmetries [70], see for instance [71] for a recent account of the class $\mathcal{S}$ case. Another example of the relevance of discrete gaugings is provided by the discovery of new types of 4d $\mathcal{N}=3$ SCFTs [36, 37]. These theories are constructed starting with the 4d $\mathcal{N}=4$ SYM theory with complexified coupling constant τ tuned to a self dual point of the $\mathrm{SL}(2,\mathbb{Z})$ S-duality group. It turns out that, for these specific values of τ, extra discrete subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SU}(4)_R$ are global symmetries of the theories and act in a no trivial way on the supercharges. The gauging of these subgroups breaks the initial amount of supersymmetry down to exactly 12 supercharges, leading this way to 4d $\mathcal{N}=3$ strongly coupled SCFTs. As discussed in [36, 37], these 4d $\mathcal{N}=3$ SCFTs are different from those obtained using the S-fold construction [34, 72].

Here we are interested in a particular form of discrete symmetry: any simple connected Lie group has a (sometimes trivial) group of outer automorphisms. The automorphisms can intervene in compactification by twisting along cycles, and this can be used to engineer theories with non simply laced gauge groups from string / M theory [73–76]. This is also a much perused tool in F-theory since its early days [77]. The outer automorphisms for a simple complex Lie algebra correspond to the symmetries of its Dynkin diagram, and the the resulting non simply laced algebra is obtained by folding it. In particular it was studied how the Superconformal Index (SCI) [78, 79] of a $4d$ $\mathcal{N}=2$ class $\mathcal{S}$ theory is affected by the twist of this symmetry. These theories are obtained starting with the $6d$ $\mathcal{N}=(2,0)$ theory on $S^3 \times S^1 \times \Sigma_2$ labelled by a simply laced Dynkin diagram Γ and performing a compactification over the punctured Riemann surface Σ_2. In [62] the authors considered the evaluation of the SCI twisted by the outer automorphism group along the S^1. Another possibility is to introduce twisted punctures in class $\mathcal{S}$ theories [80–82]; in [83] it was studied how the SCI of type D theories is affected by twist lines on Σ_2. Similar ideas are considered in [84], where 3d mirror theories of class $\mathcal{S}$ theories of type A_{2N} with twisted punctures compactified on S^1 are derived.

Another possibility offered by outer automorphisms is to promote them to gauge symmetries, in effect *extending* the gauge group and making it disconnected. This class of disconnected groups is called *principal extension*, that is to say the disconnected gauge group $\widetilde{G}$ is obtained taking the semidirect product between the connected gauge group G and the discrete outer automorphism group Γ of the Dynkin diagram

$$\widetilde{G} \simeq G \rtimes \Gamma \,. \tag{2.3.1}$$

As amply discussed below, the equation (2.3.1) is not sufficient to define the group $\widetilde{G}$: it is necessary to provide an explicit action of Γ on G. While this construction could seem a bit abstract a well known example is provided by the $\mathrm{O}(2N)$ group that is isomorphic to $\mathrm{SO}(2N) \rtimes \mathbb{Z}_2$, where the discrete group $\mathbb{Z}_2$ acts on the Dynkin diagram of type D_N algebra flipping its two final simple roots. It is then natural to extend the same construction also to the case of type A_{N-1} Lie algebra, that is still endowed with a non-trivial $\mathbb{Z}_2$ outer automorphism group. In this case the $\mathbb{Z}_2$ acts on the set of roots $\{\alpha_i\}$ by reflection , i.e. $\alpha_i \leftrightarrow \alpha_{N-i+1}$. The corresponding disconnected group is denoted by $\widetilde{\mathrm{SU}}(N) \simeq \mathrm{SU}(N) \rtimes \mathbb{Z}_2$. From a physical perspective the gauging of this $\mathbb{Z}_2$ corresponds to gauging the charge conjugation symmetry. The study of SCFTs with $\widetilde{\mathrm{SU}}(N)$ gauge groups was initiated in [85] and further extended in [1]. In both these works we focused on a 4d $\mathcal{N} = 2$ context and consider SQCD-like theories, with a $\mathcal{N} = 2$ vector multiplet transforming under the adjoint representation of $\widetilde{\mathrm{SU}}(N)$ and matter provided by $\mathcal{N} = 2$ hypermultiplets in the fundamental representation of the gauge group.

In the discretely gauged theory with $\widetilde{\mathrm{SU}}(N)$ gauge group the gauge and the matter fields transform under representations of the disconnected gauge group. This is the place in which the different global structures of the groups play a crucial role since, in general, the representations of $\widetilde{\mathrm{SU}}(N)$ differ from representations of $\mathrm{SU}(N)$. Moreover it was observed in [1] that when N is even there are two non equivalent ways of performing the gauging of the $\mathbb{Z}_2$ symmetry, that give rise to two distinct gauge groups, that have been denoted by $\widetilde{\mathrm{SU}}(N)_I$ and $\widetilde{\mathrm{SU}}(N)_{II}$ respectively. On the other hand, when N is odd, there is only one possibility corresponding to $\widetilde{\mathrm{SU}}(N)_I$. From a mathematical point of view these two possibilities, arising in the N even case, are related to the fact that the complexified Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ admits two distinct real forms that give rise to two non equivalent ways of gauging charge conjugation.[11]

All the theories that we study are endowed with a moduli space of vacua parametrized

[11] See Section 2.3 of [1] for details on that point, and Appendix 5.B for a compendium of the essential definitions.

by BPS chiral scalar gauge invariant operators. It is then natural to investigate how the discrete gauging action affects these spaces. From a physical point of view the fact that the gauge group has become larger introduces further restrictions on the types of gauge invariant operators that we can construct and therefore, we expect a modification of the geometric structure of the corresponding moduli space. A systematic way to characterize the geometry of these moduli spaces is provided by the *Plethystic program* [11, 12], with the central notion of Hilbert series, a generating function that counts the chiral operators present in the theory according to their conformal dimension and other quantum numbers [19, 86]. The extension of these tools, in the context of principal extensions, was performed in [46] and we employ them in our analysis.

Moreover, even if the complete characterization of the full moduli space of vacua is in general very difficult, for a 4d $\mathcal{N}=2$ gauge theory we can identify two particular subbranches, namely the *Coulomb branch* and *Higgs branch*. Specifically the Coulomb branch arises when we give a vacuum expectation value (VEV) to the complexified scalar inside the $\mathcal{N}=2$ vector multiplet. For the theories discussed in this work the computation of the Hilbert series of the Coulomb branch was performed in [1, 85]. Remarkably it was found that the Coulomb branch of these theories is not freely generated. On the other hand the Higgs branch is parameterized by the VEVs of the scalar fields inside the $\mathcal{N}=2$ hypermultiplets. In general if there is enough matter in the theory a generic VEV completely breaks the gauge group. Nevertheless we can also give a VEV only to a subset of the scalar fields, this way the gauge group could be broken to a non-trivial subgroup. This partial Higgs mechanism is naturally described by a partial order diagram, called the *Hasse diagram*, where each node of the diagram is related to the subgroup of the initial gauge group that is left unbroken by the Higgs mechanism. The systematic study of the Higgs branch of theories with 8 supercharges using Hasse diagrams was initiated in [22] and further analysed in [23, 87–96]. The Higgs branch Hasse diagram in turns reveals the geometric structure of the Higgs branch as a symplectic singularity, the nodes being in correspondence with symplectic leaves, and the links representing elementary transverse slices. In this section we aim to move a further step in this direction and we analyse how the structure of the Higgs branch of the SQCD-like theories with $\widetilde{\mathrm{SU}}(N)_I$ or $\widetilde{\mathrm{SU}}(N)_{II}$ gauge groups is revealed by the partial Higgsing procedure described above. In particular our first main result is the derivation of the Hasse diagrams for Type I and Type II gauging in Figure 2.4 and Figure 2.6. This is based on a careful analysis of representations of $\widetilde{\mathrm{SU}}(N)_{I/II}$ groups, their characters and branching rules.

The Higgs branch of certain 4d $\mathcal{N}=2$ theories can be equivalently described as the

Coulomb branch of 3d $\mathcal{N} = 4$ quiver gauge theories. When this is the case, the quiver is called a *magnetic quiver* for that Higgs branch [97–101]. Our second main result is a magnetic quiver for the Higgs branch of $\widetilde{\mathrm{SU}}(N)_I$ theories, in the form of a *wreathed quiver*, as introduced in [91]. As a check of our conjecture we compute the 3d $\mathcal{N} = 4$ Coulomb branch Hilbert series of that quiver and find perfect agreement with the Higgs branch Hilbert series of the corresponding $\widetilde{\mathrm{SU}}(N)_I$ theory that was computed in [1, 85]. The computation is performed using the *monopole formula* originally introduced in [19] and generalized to wreathed quivers in [91].

The present section is organized as follows. In Section 2.3.1 we introduce the notion of characters for representations of disconnected groups and we discuss the derivation of the branching rules relevant for the partial Higgsing mechanisms discussed in this work. In Section 2.3.2 we briefly review the notion of Hasse diagram and we discuss its construction for type I and type II discretely gauged theories. In Section 2.3.3 we review the generalization of the monopole formula in the context of 3d $\mathcal{N} = 4$ wreathed quiver gauge theories and we apply it to theories of type I providing a candidate magnetic quiver. The appendices gather basic definitions and technicalities regarding $\widetilde{\mathrm{SU}}(N)$ groups.

2.3.1 Characters and branching rules for disconnected groups

In this section we develop tools that allow to use the theory of characters of Lie groups in the context of disconnected groups, focusing on the examples of $\mathrm{O}(N)$ and $\widetilde{\mathrm{SU}}(N)$. This allows to compute tensor products, and more importantly branching rules, which are needed to compute Hasse diagrams in the next section.

Writing the characters for a group G (connected or not) requires firstly the identification of irreducible representations $\rho : G \to \mathrm{GL}(V)$, and secondly the choice of a subgroup $T \subset G$ parametrized by fugacities (which can assume continuous or discrete values). The character is then the function $\chi_\rho : T \to \mathbb{C}$ defined by $\chi_\rho(t) = \mathrm{Tr}(\rho(t))$ for $t \in T$. The new feature of this analysis for disconnected groups G is the appearance of discrete fugacities in T. This can be seen as a fusion between the usual theories of characters of connected Lie group on one side, and of representation theory of finite groups (here the component group of G) on the other side. Here we consider only the simplest non trivial case (2.3.1) where $\Gamma = \mathbb{Z}_2$, which has character table

$$\begin{array}{c|cc} \epsilon & 1 & -1 \\ \hline \chi_{\mathbf{1}} & 1 & 1 \\ \chi_\epsilon & 1 & -1 \end{array} \tag{2.3.2}$$

but the principles would remain valid for a larger component group. In the character table (2.3.2), the two $\mathbb{Z}_2$ elements are denoted by $\epsilon = \pm 1$. and rows of this table contain the characters of its two irreducible representations.[12]

Representations and characters for $O(N)$

Groups $O(2N)$

We start with the very simple example of O(2) to set up the concepts and notations in a framework where everything can be written explicitly. This group is a semidirect product $\mathrm{SO}(2) \rtimes \mathbb{Z}_2$, so an element of O(2) can be written as a pair $(g, \epsilon) \in \mathrm{SO}(2) \times \mathbb{Z}_2$. The semidirect product is specified by the Θ_ϵ automorphism of SO(2) defined by

$$\Theta_1 \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \Theta_{-1} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.3.3}$$

Note that Θ_{-1} is the conjugation by the reflection matrix $\mathrm{Diag}(-1, 1)$. The fundamental representation is

$$\mathrm{O}(2) = \left\{ \left. \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right| \theta \in T^1 \right\} \cup \left\{ \left. \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \right| \theta \in T^1 \right\}. \tag{2.3.4}$$

Setting $z = e^{i\theta}$, the trace of the matrices in the identity component is $z + z^{-1}$ while the trace vanishes in the disconnected component. Therefore the character can be written as a function of z and ϵ as

$$\chi^{\mathrm{O}(2)}_{\mathrm{Fundamental}}(z, \epsilon) = \left(\frac{1+\epsilon}{2} \right) (z + z^{-1}) = \begin{cases} z + z^{-1} & \text{if } \epsilon = 1 \\ 0 & \text{if } \epsilon = -1 \end{cases}. \tag{2.3.5}$$

The character has two fugacities, one continuous variable z and one discrete variable ϵ, and they span the *fugacity group*.

Consider now the adjoint representation, i.e. the action of $(g, \epsilon) \in \mathrm{O}(2)$ on $a \in \mathbb{R}$ given by

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \mapsto (g, \epsilon) \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} (g, \epsilon)^{-1}. \tag{2.3.6}$$

[12] We slightly abuse notation in denoting by the same symbol ϵ two related objects, namely the generic element of $\mathbb{Z}_2$ (which plays the role of a discrete fugacity, satisfying $\epsilon^2 = 1$) and the non-trivial irreducible representation of $\mathbb{Z}_2$. With this choice the character of the ϵ representation is ϵ.

This is $a \mapsto a$ for $\epsilon = 1$ and $a \mapsto -a$ for $\epsilon = -1$. Therefore the corresponding character reads

$$\chi^{\mathrm{O}(2)}_{\mathrm{Adjoint}}(z, \epsilon) = \left(\frac{1+\epsilon}{2}\right) \times (1) + \left(\frac{1-\epsilon}{2}\right) \times (-1) = \epsilon\,. \tag{2.3.7}$$

We note an interesting fact: the adjoint representation is not the same as the trivial representation. There are two inequivalent representations of dimension 1, while there is only one of dimension 2.

Consider now $\mathrm{O}(4) = \mathrm{SO}(4) \rtimes \mathbb{Z}_2$, the first example in which the choice of the subgroup of fugacities T is not straightforward. As maximal torus of $\mathrm{SO}(4)$ we choose matrices of the form

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix} \tag{2.3.8}$$

The trace of this matrix is $z_1 + z_1^{-1} + z_2 + z_2^{-1}$ with $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. However we now need to specify how the semidirect product is defined, as there is no way to make this choice symmetric in z_1 and z_2. We *choose* to define Θ_{-1} as the conjugation by the reflection $\mathrm{Diag}(-1, 1, 1, 1)$. As a consequence, the trace of an element with $\epsilon = -1$ in the fundamental representation is $z_2 + z_2^{-1}$. The symmetry between z_1 and z_2 is broken. The embedding $\mathrm{O}(2) \subset \mathrm{O}(4)$ is obtained by sending $z_2 \to 1$, while sending $z_1 \to 1$ gives the embedding $\mathrm{SO}(2) \subset \mathrm{O}(4)$. The reader is encouraged to check this on the characters of the fundamental and adjoint representations of $\mathrm{O}(4)$ which read

$$\chi^{\mathrm{O}(4)}_{\mathrm{Fundamental}}(z_1, z_2, \epsilon) = \left(\frac{1+\epsilon}{2}\right)(z_1 + z_1^{-1}) + (z_2 + z_2^{-1}) \ , \tag{2.3.9}$$

$$\chi^{\mathrm{O}(4)}_{\mathrm{Adjoint}}(z_1, z_2, \epsilon) = \left(\frac{1+\epsilon}{2}\right)\left(2 + (z_1 + z_1^{-1})(z_2 + z_2^{-1})\right) \ . \tag{2.3.10}$$

One can generalize these computations to $\mathrm{O}(2N) = \mathrm{SO}(2N) \rtimes \mathbb{Z}_2$, with Θ_{-1} given by conjugation by $\mathrm{Diag}(-1, 1, \cdots, 1)$. The characters of the fundamental and adjoint representations of $\mathrm{O}(2N)$ are

$$\chi^{\mathrm{O}(2N)}_{\mathrm{Fundamental}}(z_i, \epsilon) = \left(\frac{1+\epsilon}{2}\right)(z_1 + z_1^{-1}) + \sum_{i=2}^{N}(z_i + z_i^{-1}) \ , \tag{2.3.11}$$

$$\chi^{\mathrm{O}(2N)}_{\mathrm{Adjoint}}(z_i, \epsilon) = \left(\frac{1+\epsilon}{2}\right)\left(2 + (z_1 + z_1^{-1}) \sum_{2 \leq j \leq N} (z_j + z_j^{-1})\right)$$

$\mathrm{O}(2N)$	$\longrightarrow$	$\mathrm{O}(2N-1)\times\mathrm{O}(1) = (\mathrm{SO}(2N-1)\times\mathbb{Z}_2)\times\mathbb{Z}_2$
$F_{\mathrm{O}(2N)}$	$\longmapsto$	$\big(F_{\mathrm{SO}(2N-1)}\otimes\epsilon\otimes\mathbf{1}\big)\oplus\big(\mathbf{1}_{\mathrm{SO}(2N-1)}\otimes\mathbf{1}\otimes\epsilon\big)$
$\mathrm{Adj}_{\mathrm{O}(2N)}$	$\longmapsto$	$\big(\mathrm{Adj}_{\mathrm{SO}(2N-1)}\otimes\mathbf{1}\otimes\mathbf{1}\big)\oplus\big(F_{\mathrm{SO}(2N-1)}\otimes\epsilon\otimes\epsilon\big)$

$\mathrm{SO}(2N+1)$	$\longrightarrow$	$\mathrm{O}(2N)$
$F_{\mathrm{SO}(2N+1)}$	$\longmapsto$	$F_{\mathrm{O}(2N)}\oplus\mathbf{1}_{\mathrm{O}(2N)}$
$\mathrm{Adj}_{\mathrm{SO}(2N+1)}$	$\longmapsto$	$\mathrm{Adj}_{\mathrm{O}(2N)}\oplus F_{\mathrm{O}(2N)}$

Table 2.5: Summary of branching rules for $\mathrm{O}(N)$ groups.

$$+(N-2)+\sum_{2\leq i<j\leq N}(z_i+z_i^{-1})(z_j+z_j^{-1})\,. \tag{2.3.12}$$

Note that the group $\mathrm{O}(2N)$ is simple for $N\geq 3$, and in those cases the trivial, fundamental and adjoint representations are given by Dynkin labels $[0,\dots,0]$, $[1,0,\dots,0]$ and $[0,1,0,\dots,0]$ respectively. These are invariant under the exchange of the Dynkin labels for the two spinor nodes, so from each of these representation one can build another inequivalent one by tensoring with ϵ.[13] In the case $N=2$, the group $\mathrm{O}(2N)$ is not simple, and accordingly the adjoint representation corresponds to Dynkin labels $[2,0]\oplus[0,2]$; from the general arguments given in [1, Sec. 3.1], it follows that the character should vanish for $\epsilon=-1$, and this is indeed the case in (2.3.12).

Groups $O(2N+1)$ and Branching rules

The group $\mathrm{O}(2N+1)$ is a direct product $\mathrm{SO}(2N+1)\times\mathbb{Z}_2$ so the characters factorize. Using the characters one can check the branching rules for orthogonal groups. The branching rules $\mathrm{O}(2N+1)\to\mathrm{O}(2N)$ are obtained by restricting the $\mathrm{O}(2N+1)$ characters to $\mathrm{O}(2N)$ characters, with no change in the fugacities. The branching rules $\mathrm{O}(2N)\to\mathrm{O}(2N-1)$ are obtained by setting $z_N\to 1$. The results are presented in Table 2.5.

Representations and characters for $\widetilde{\mathrm{SU}}(N)$

We can apply similar techniques to express characters of representations of $\widetilde{\mathrm{SU}}(N)_{I/II}$. Definitions and notations for these groups are gathered in Appendix 5.B. As for $\mathrm{O}(N)$, we first

[13] These representations are sometimes called "pseudo". From the trivial or scalar representation one builds the pseudo-scalar, and from the fundamental or vector one builds the pseudo-vector.

have two one-dimensional representations:

1. The trivial representation, with character $\chi_1^{\widetilde{\mathrm{SU}}(N)} = 1$.

2. The ϵ representation, with character $\chi_\epsilon^{\widetilde{\mathrm{SU}}(N)} = \epsilon$.

Let us now move to higher dimensional representations. As explained in [1], representations of $\mathrm{SU}(N)$ induce representations of $\widetilde{\mathrm{SU}}(N)$ according to the following rule. Let R be a representation of $\mathrm{SU}(N)$ with highest weight λ. If $\lambda = [\lambda_1, \ldots, \lambda_{N-1}]$ is invariant under the permutation $\lambda_i \leftrightarrow \lambda_{N-i}$ then there are two corresponding representations of $\widetilde{\mathrm{SU}}(N)$, both of dimension $\dim(R)$, which differ by a tensor product with the ϵ representation; if the highest weight is not invariant under that permutation, then there is a single corresponding representation of $\widetilde{\mathrm{SU}}(N)$, of dimension $2\dim(R)$.

In this work we focus on the representations induced by the fundamental and the adjoint of $\mathrm{SU}(N)$. These are

3. The fundamental representation. This is a $2N$ dimensional representation which we denote by $(F \oplus \overline{F})$. Let us emphasize that despite this notation, this is an *irreducible* representation, as the $\mathbb{Z}_2$ element mixes the F and $\overline{F}$ of the starting $\mathrm{SU}(N)$ group.

4. The adjoint representation, of dimension $N^2 - 1$.

5. The tensor product of the adjoint representation with ϵ representation, of dimension $N^2 - 1$.

To write down characters for these representations, we need to pick a group of fugacities. For characters in $\mathrm{SU}(N)$, the natural choice is to consider the subgroup of diagonal matrices $\mathrm{U}(1)^{N-1}$. In the case of the disconnected group $\widetilde{\mathrm{SU}}(N)$, it turns out that the choice of an appropriate fugacity subgroup is a subtle problem that is discussed at length in Appendix 5.B. In a nutshell, the reason for which the choice is subtle is that certain subgroups, called *Cartan subgroups*, are well suited for representation theory (e.g. an extension of the Weyl character formula is valid) but do not commute with the embedding of smaller disconnected groups like $\widetilde{\mathrm{SU}}(N-1) \subset \widetilde{\mathrm{SU}}(N)$. In the present section, we are interested in branching rules for that embedding, so we pick instead $\mathcal{T} = \{(z_1, \ldots, z_{N-1}, \epsilon)\} = \mathrm{U}(1)^{N-1} \rtimes \mathbb{Z}_2$ as defined in (5.B.23), where the first factor corresponds to diagonal matrices in $\mathrm{SU}(N)$ and the semidirect product is the one that serves to define the extension $\widetilde{\mathrm{SU}}(N)$. With these choices, the representations and their characters are summarized in Table 2.6.

For instance, in the fundamental representation the trace of the matrix corresponding to an element $(g, -1)$ is clearly 0, so that the character is the product of the the corresponding

Representation	Value on $(g,1)$	Value on $(g,-1)$
Trivial	1	1
ϵ	1	-1
$F \oplus \overline{F}$	$\begin{pmatrix} g & 0 \\ 0 & \Theta_{-1}(g) \end{pmatrix}$	$\begin{pmatrix} 0 & g \\ \Theta_{-1}(g) & 0 \end{pmatrix}$
Adj	$X \mapsto gXg^{-1}$	$X \mapsto g\theta_{-1}(X)g^{-1}$
Adj $\otimes\, \epsilon$	$X \mapsto gXg^{-1}$	$X \mapsto -g\theta_{-1}(X)g^{-1}$

Representation	Character
Trivial	1
ϵ	ϵ
$F \oplus \overline{F}$	$\left(\frac{1+\epsilon}{2}\right) \sum_{i=0}^{N-1} \left(\frac{z_i}{z_{i+1}} + \frac{z_{i+1}}{z_i} \right)$
Adj	$\begin{cases} \left(\frac{1+\epsilon}{2}\right) \left(-1 + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \frac{z_i}{z_{i+1}} \frac{z_{j+1}}{z_j} \right) + (1-N)\left(\frac{1-\epsilon}{2}\right) & \text{Type } I \\ \left(\frac{1+\epsilon}{2}\right) \left(-1 + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \frac{z_i}{z_{i+1}} \frac{z_{j+1}}{z_j} \right) + (1+N)\left(\frac{1-\epsilon}{2}\right) & \text{Type } II \end{cases}$
Adj $\otimes\, \epsilon$	$\begin{cases} \left(\frac{1+\epsilon}{2}\right) \left(-1 + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \frac{z_i}{z_{i+1}} \frac{z_{j+1}}{z_j} \right) - (1-N)\left(\frac{1-\epsilon}{2}\right) & \text{Type } I \\ \left(\frac{1+\epsilon}{2}\right) \left(-1 + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \frac{z_i}{z_{i+1}} \frac{z_{j+1}}{z_j} \right) - (1+N)\left(\frac{1-\epsilon}{2}\right) & \text{Type } II \end{cases}$

Table 2.6: The first table shows the representations of $\widetilde{\mathrm{SU}}(N)$ used in this work, by giving the explicit action of elements of the form (g,ϵ) for $\epsilon = \pm 1$. For the 1-dimensional representations, this is a number; for the $F \oplus \overline{F}$ representation we give a $2N \times 2N$ matrix, and for the adjoint and ϵ-adjoint we give the action on an element X in the Lie algebra $\mathfrak{g}$. The second table gives the characters expressed in terms of fugacities $(z_1, \dots, z_{N-1}, \epsilon) \in \mathcal{T}$ with the convention $z_0 = z_N = 1$.

$\mathrm{SU}(N)$ character by the projector $\frac{1+\epsilon}{2}$. As another example, for the adjoint representation the action of $(g,-1)$ on the Lie algebra is $X \mapsto g\theta_{-1}(X)g^{-1}$, see equation (5.B.13). As shown in section 5.B.3, the computation of the character reduces to the computation of the trace of θ_{-1}, which is given in equations (5.B.14) and (5.B.15).

The characters of Table 2.6 allow to compute branching rules. For instance the branching rules for fundamentals are

$$\begin{aligned} \chi_{(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N)}}\big|_{z_{N-1}\to 1} &= \left(\frac{1+\epsilon}{2}\right) \sum_{i=0}^{N-2} \left(\frac{z_i}{z_{i+1}} + \frac{z_{i+1}}{z_i} \right) + 2\left(\frac{1+\epsilon}{2}\right) \\ &= \chi_{(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N-1)}} + 1 + \epsilon\,. \end{aligned} \tag{2.3.13}$$

$\widetilde{\mathrm{SU}}(N)_I$	$\longrightarrow$	$\widetilde{\mathrm{SU}}(N-1)_I$
$(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N)_I}$	$\longmapsto$	$(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N-1)_I}\oplus\epsilon\oplus\mathbf{1}$
$\mathrm{Adj}_{\widetilde{\mathrm{SU}}(N)_I}$	$\longmapsto$	$\mathrm{Adj}_{\widetilde{\mathrm{SU}}(N-1)_I}\oplus(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N-1)_I}\oplus\epsilon$
ϵ	$\longmapsto$	ϵ

$\widetilde{\mathrm{SU}}(2N)_{II}$	$\longrightarrow$	$\widetilde{\mathrm{SU}}(2N-2)_{II}$
$(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(2N)_{II}}$	$\longmapsto$	$(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(2N-2)_{II}}\oplus 2\times\epsilon\oplus 2\times\mathbf{1}$
$\mathrm{Adj}_{\widetilde{\mathrm{SU}}(2N)_{II}}$	$\longmapsto$	$\mathrm{Adj}_{\widetilde{\mathrm{SU}}(2N-2)_{II}}\oplus 2\times(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(2N-2)_{II}}\oplus\epsilon\oplus 3\times\mathbf{1}$
ϵ	$\longmapsto$	ϵ

$\widetilde{\mathrm{SU}}(N)_{I,II}$	$\longrightarrow$	$\mathrm{SU}(N)$
$(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N)}$	$\longmapsto$	$F_{\mathrm{SU}(N)}\oplus\overline{F}_{\mathrm{SU}(N)}$
$\mathrm{Adj}_{\widetilde{\mathrm{SU}}(N)}$	$\longmapsto$	$\mathrm{Adj}_{\mathrm{SU}(N)}$
ϵ	$\longmapsto$	$\mathbf{1}$

Table 2.7: Summary of branching rules for $\widetilde{\mathrm{SU}}(N)$ groups.

For a less trivial example, let us look at the adjoint representation in type II. We take $N\geq 4$ even and consider the branching rules for the embedding $\widetilde{\mathrm{SU}}(N-2)_{II}\subset\widetilde{\mathrm{SU}}(N)_{II}$

$$\begin{aligned}
\chi_{\mathrm{Adj}_{\widetilde{\mathrm{SU}}(N)_{II}}}|_{z_{N-2},z_{N-1}\to 1} &= \left(\frac{1+\epsilon}{2}\right)\left(\chi_{\mathrm{Adj}_{\mathrm{SU}(N)}}|_{z_{N-2},z_{N-1}\to 1}\right)+(1+N)\left(\frac{1-\epsilon}{2}\right)\\
&= \left(\frac{1+\epsilon}{2}\right)\left(\chi_{\mathrm{Adj}_{\mathrm{SU}(N-2)}}+2\chi_{(F\oplus\overline{F})_{\mathrm{SU}(N-2)}}+4\right)+(1+N)\left(\frac{1-\epsilon}{2}\right)\\
&= \chi_{\mathrm{Adj}_{\widetilde{\mathrm{SU}}(N-2)_{II}}}+2\chi_{(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(N-2)_{II}}}+3+\epsilon\,.
\end{aligned}\tag{2.3.14}$$

The crucial feature here is the $3+\epsilon$ contribution. If instead we repeat the computation for type I this term becomes $1+3\epsilon$ because of the different sign in front of the $N\epsilon$ term in the character. The branching rules are summarized in Table 2.7.

2.3.2 Hasse diagrams

Higgs branches of theories with 8 supercharges are hyperKähler cones [102], or symplectic singularities [20], and as such admit a foliation [21,103] which can be conveniently described by a Hasse diagram. Each point of the diagram represents a symplectic leaf of said foliation.

The Hasse diagram represents a partial order between the symplectic leaves, defined by inclusions in their closures. For any two given leaves which can be compared in this partial order, we have a transverse slice which describes how the smaller leaf looks as a symplectic singularity inside the closure of the bigger leaf. If the two leaves are adjacent in the Hasse diagram, we have a so called elementary transverse slice, which oftentimes has a simple geometric description as the closure of a minimal nilpotent orbit of a classical group or as a Klein singularity (this will always be the case for our purposes, see [95, 104] for examples with more exotic elementary slices).

In [22] the Hasse diagram of symplectic leaves for the Higgs branch of a classical gauge theory with 8 supercharges is identified with the Hasse diagram of phases of that gauge theory under partial Higgsing. Each leaf is labeled by the unbroken gauge group in that phase, and the elementary transverse slices describe the geometry of gauge singlets. We apply this principle to the Higgs branch of gauge theories with $\widetilde{\mathrm{SU}}$ gauge groups, and in this section, we derive the Hasse diagrams of Figures 2.2, 2.4 and 2.6 by looking at the chain of possible Higgsing patterns. As a warm-up we first review that procedure by looking at the example of $\mathrm{SU}(N_c) + N_f$ SQCD.

The Higgs branch of this theory is defined classically as a hyperKähler quotient which can be written symbolically as

$$\frac{1}{2}\left(N_f F_{N_c} + N_f \overline{F}_{N_c}\right) - \mathrm{Adj}_{N_c}\,, \tag{2.3.15}$$

where the factor of 1/2 is due to the fact that when separating fundamentals and antifundamentals we are counting half-hypers. The hyperKähler quotient is denoted by the minus sign in the above equation. Replacing each representation by its dimension, the formula above yields the quaternionic dimension of the Higgs branch.

Now we apply the branching rules under the breaking $\mathrm{SU}(N_c) \to \mathrm{SU}(N_c-1)$,

$$\frac{N_f}{2}\left[F_{N_c-1} \oplus \mathbf{1}_{N_c-1}\right] + \frac{N_f}{2}\left[\overline{F}_{N_c-1} \oplus \mathbf{1}_{N_c-1}\right] - \left[\mathrm{Adj}_{N_c-1} \oplus F_{N_c-1} \oplus \overline{F}_{N_c-1} \oplus \mathbf{1}_{N_c-1}\right]\,. \tag{2.3.16}$$

Finally, we reshuffle this expression to put it in the form

$$[\text{Matter fields charged under } \mathrm{SU}(N_c-1)] - [\text{Adjoint of } \mathrm{SU}(N_c-1)] + [\text{singlets}]\,. \tag{2.3.17}$$

The first two terms identify the theory that results after the Higgsing, and the singlets cor-

Slice	$\dim_{\mathbb{H}}$	Gauge Theory	Global Symmetry
a_N	N	U(1) with $N+1$ fundamental hypermultiplets	$\mathfrak{su}(N+1)$
c_N	N	$\mathrm{O}(1) = \mathbb{Z}_2$ with $2N$ fundamental half-hypermultiplets	$\mathfrak{sp}(N)$
d_N	$2N-3$	Sp(1) with $2N$ fundamental half-hypermultiplets	$\mathfrak{so}(2N)$

Table 2.8: List of elementary slices which appear in this work. These are the closures of the minimal nilpotent orbits of their global symmetry algebra.

respond to the transverse slice according to Table 2.8. The cancellation of the fundamentals coming from Adj_{N_c} by fundamentals coming from matter fields corresponds physically to the Higgs mechanism, where some of the gauge bosons of the initial theory acquire a mass. In our example,

$$\frac{1}{2}\left((N_f-2)F_{N_c-1} + (N_f-2)\overline{F}_{N_c-1}\right) - \mathrm{Adj}_{N_c-1} \oplus \left[\frac{1}{2}(N_f+N_f)-1\right]\mathbf{1}_{N_c-1}\,, \tag{2.3.18}$$

which means that the remaining theory after Higgsing is $\mathrm{SU}(N_c-1)+(N_f-2)$ SQCD, and the transverse slice is a_{N_f-1} i.e. the (closure of the) minimal nilpotent orbit of $\mathfrak{su}(N_f)$. The slice is identified to be a_{N_f-1} as the coefficient of the singlets of the hyperKähler quotient of the U(1) gauge theory as described in Table 2.8.

In order to distinguish between different elementary slices of the same dimension, one looks at the number of singlets coming from the adjoint of the initial gauge group and matches it with the dimension of the gauge group of the theory in the third column in Table 2.8. For the slices considered in this thesis, this number is respectively 1, 0 and 3 for the three rows of the Table.

Hasse Diagram for O

Let's proceed with our first disconnected group, and consider a theory with gauge group $\mathrm{O}(N_c)$ plus $N_f \geq N_c$ fields in the fundamental representation. The Hasse diagram is already shown in [22]; we rederive it here to illustrate the method of characters for disconnected groups. We use the same procedure shown above to find the transverse slices and resulting theories after the Higgsing. In order to make sure that we get the full Hasse diagram, we need to scan over the possible subgroups of $\mathrm{O}(N_c)$ and check which symmetry breaking patterns are possible according to the branching rules of Table 2.5. Let's begin by considering the (potential) breaking $\mathrm{O}(N_c) \to \mathrm{O}(N_c-1) \times \mathrm{O}(1)$. Note that since $\mathrm{O}(2k+1)$ can be written as a direct product, but $\mathrm{O}(2k)$ cannot, there is in principle a difference between choosing N_c odd or even. We take N_c even for now, and shall soon see that this initial choice doesn't matter.

$$
\begin{aligned}
N_f F_{\mathrm{O}(N_c)} - \mathrm{Adj}_{\mathrm{O}(N_c)} \rightarrow & N_f[(F_{\mathrm{SO}(N_c-1)} \otimes \epsilon \otimes \mathbf{1}) \oplus (\mathbf{1}_{\mathrm{SO}(N_c-1)} \otimes \mathbf{1} \otimes \epsilon)] \\
& - [(\mathrm{Adj}_{\mathrm{SO}(N_c-1)} \otimes \mathbf{1} \otimes \mathbf{1}) \oplus (F_{\mathrm{SO}(N_c-1)} \otimes \epsilon \otimes \epsilon)] \,. \qquad (2.3.19)
\end{aligned}
$$

Note that the last term coming from the decomposition of the adjoint cannot be cancelled. This means that under the symmetry breaking pattern under consideration, the gauge fields have no Goldstone bosons to eat, and therefore the Higgs mechanism cannot take place. In a similar way, we can check that $\mathrm{O}(N_c)$ also can't break to the subgroup $\mathrm{O}(p) \times \mathrm{O}(q)$ (with $p + q = N_c$).

The next possible breaking to consider is then $\mathrm{O}(N_c) \rightarrow \mathrm{O}(N_c - 1)$. This is achieved taking (2.3.19) and forgetting the second $\mathbb{Z}_2$ representation in each tensor product, as that was the one corresponding to the $\mathrm{O}(1)$ factor. Thus we have

$$
\begin{aligned}
N_f F_{\mathrm{O}(N_c)} - \mathrm{Adj}_{\mathrm{O}(N_c)} \rightarrow & N_f[(F_{\mathrm{SO}(N_c-1)} \otimes \epsilon) \oplus (\mathbf{1}_{\mathrm{SO}(N_c-1)} \otimes \mathbf{1})] \\
& - [(\mathrm{Adj}_{\mathrm{SO}(N_c-1)} \otimes \mathbf{1}) \oplus (F_{\mathrm{SO}(N_c-1)} \otimes \epsilon)] \,. \qquad (2.3.20)
\end{aligned}
$$

We see that this symmetry breaking pattern is possible, and it results in

$$
N_f F_{\mathrm{O}(N_c)} - \mathrm{Adj}_{\mathrm{O}(N_c)} \rightarrow (N_f - 1) F_{\mathrm{O}(N_c-1)} - \mathrm{Adj}_{\mathrm{O}(N_c-1)} \underbrace{\oplus N_f \mathbf{1}_{\mathrm{O}(N_c-1)}}_{c_{N_f} \text{ slice}} \,. \qquad (2.3.21)
$$

From this, we conclude that the transverse slice at the bottom of the Hasse diagram is c_{N_f}, and the remaining theory on the symplectic leaf of the Higgs branch is $\mathrm{O}(N_c - 1) + (N_f - 1)F$. Recall that we had chosen N_c even, so now we have an odd number of colours, $\mathrm{O}(N_c - 1) = \mathbb{Z}_2 \times \mathrm{SO}(N_c - 1)$. We can therefore consider the potential breaking $\mathbb{Z}_2 \times \mathrm{SO}(N_c - 1) \rightarrow \mathbb{Z}_2 \times \mathrm{O}(N_c - 2)$, where the $\mathbb{Z}_2$ representations stay the same and the branching rules are those in the second part of Table 2.5. This results in

$$
\begin{aligned}
(N_f - 1)[\epsilon \otimes F_{\mathrm{SO}(N_c-1)}] - [\mathbf{1} \otimes \mathrm{Adj}_{\mathrm{SO}(N_c-1)}] \rightarrow & (N_f - 1)[\epsilon \otimes (F_{\mathrm{O}(N_c-2)} \oplus \mathbf{1}_{\mathrm{O}(N_c-2)})] \\
& - [\mathbf{1} \otimes (\mathrm{Adj}_{\mathrm{O}(N_c-2)} \oplus F_{\mathrm{O}(N_c-2)})] \qquad (2.3.22) \\
= & (N_f - 1)[(\epsilon \otimes F_{\mathrm{O}(N_c-2)}) \oplus (\epsilon \otimes \mathbf{1}_{\mathrm{O}(N_c-2)})] \\
& - [(\mathbf{1} \otimes \mathrm{Adj}_{\mathrm{O}(N_c-2)}) \oplus (\mathbf{1} \otimes F_{\mathrm{O}(N_c-2)})] \,. \qquad (2.3.23)
\end{aligned}
$$

Once again, we see that the necesary cancellations are only possible if we forget about the $\mathbb{Z}_2$, i.e. if we consider the breaking $\mathrm{O}(N_c-1) \to \mathrm{O}(N_c-2)$. With this,

$$(N_f-1)F_{\mathrm{O}(N_c-1)} - \mathrm{Adj}_{\mathrm{O}(N_c-1)} \to (N_f-2)F_{\mathrm{O}(N_c-2)} - \mathrm{Adj}_{\mathrm{O}(N_c-2)} \underbrace{\oplus (N_f-1)\mathbf{1}_{\mathrm{O}(N_c-2)}}_{c_{N_f-1}\text{ slice}} . \tag{2.3.24}$$

That is, the second slice at the bottom of the Hasse diagram is c_{N_f-1} and the remaining theory is $\mathrm{O}(N_c-2)+(N_f-2)F$. To complete the Hasse diagram, we need to repeat the process above the necessary number of times. Note that, as advertised, it doesn't matter whether at each step the number of colour is even or odd; even if the decomposition of the representations looks different, in the end we always have that the breaking is $\mathrm{O}(N_c-k) \to \mathrm{O}(N_c-k-1)$ with transverse slice c_{N_f-k}. The chain of Higgsings only stops after N_c steps, when we have $\mathrm{O}(1) \to \{1\}$ with a $c_{N_f-(N_c-1)}$ transverse slice. The resulting Higgs branch Hasse diagram is a line, and is depicted in the bottom left of Figure 2.2.

As a byproduct of this analysis, we can also obtain the Hasse diagram for a theory with $\mathrm{SO}(N_c)$ gauge group and N_f fundamentals.[14] The process is completely analogous to the one above, except there are no $\mathbb{Z}_2$ representations making any appearance. At each step the possible Higgsing is $\mathrm{SO}(N_c-k) \to \mathrm{SO}(N_c-k-1)$ with transverse slice c_{N_f-k}. The only difference comes after N_c-2 steps, when the theory we have left is $\mathrm{SO}(2)$ with N_f-N_c+2 fundamentals. In the O case, there was still one possible nontrivial subgroup and Higgsing $\mathrm{O}(2) \to \mathrm{O}(1) = \mathbb{Z}_2$. On the other hand, now we have $\mathrm{SO}(2) = \mathrm{U}(1)$, which has no nontrivial subgroups to be Higgsed to. Therefore the only Higgsing is $\mathrm{U}(1) \to \{1\}$, with transverse slice $a_{2N_f-2N_c+3}$. We show this Hasse diagram in the bottom right of Figure 2.2. The relation between the Hasse diagrams for the O and SO theories is reminiscent of the relation between U and SU [22, 87], as made clear on the figure.

Hasse diagram for $\widetilde{\mathrm{SU}}(N)_I$

We now proceed to compute the Higgs branch Hasse diagrams for theories with $\widetilde{\mathrm{SU}}(N_c)_I$ gauge group, and matter content consisting of N_f fields in the $(F \oplus \overline{F})$ representation and N_ϵ fields in the ϵ representation. We do this by considering all the possible Higgsing patterns, using the branching rules summarised in Table 2.7. We consider only the case where N_f is large enough so that we can have complete Higgsing.

[14] We are indebted to Amihay Hanany for drawing our attention to the Hasse diagram of that theory.

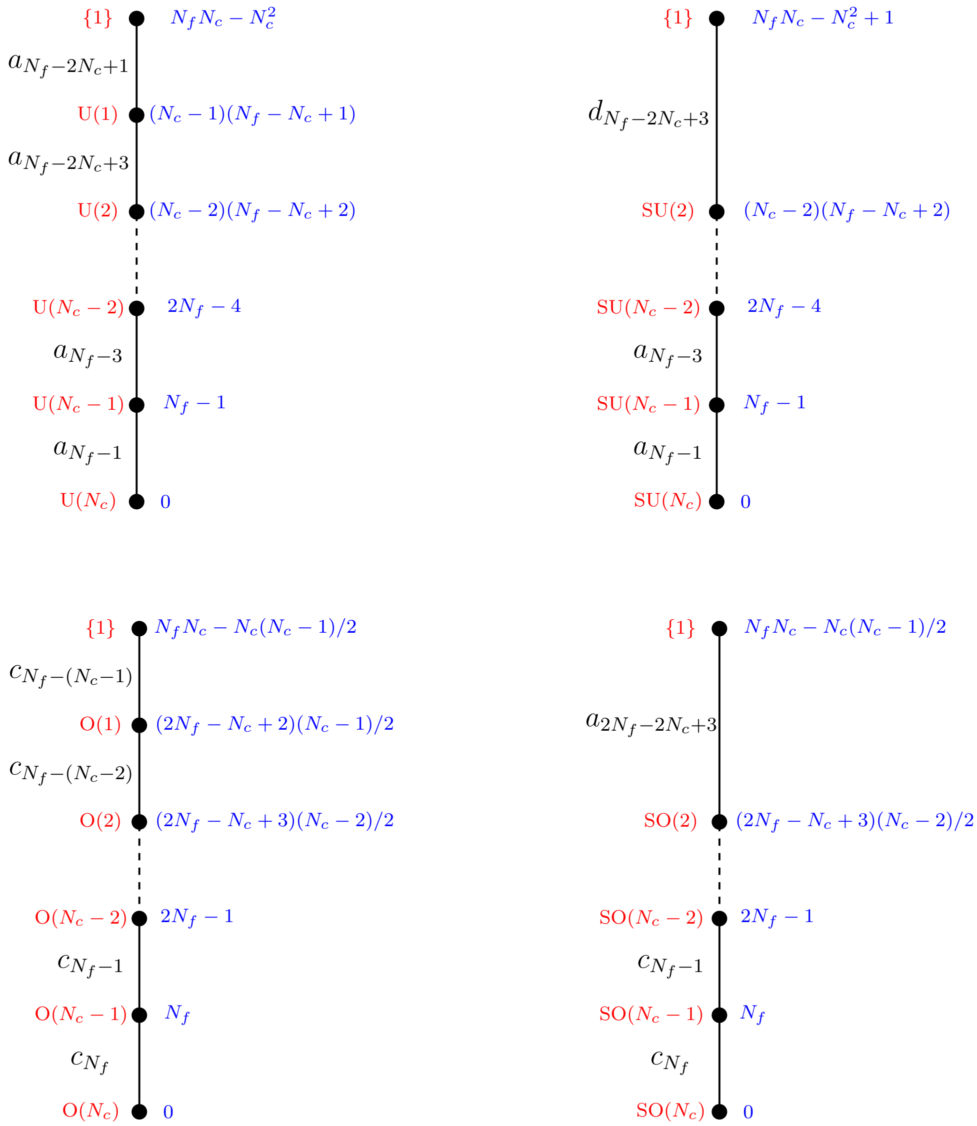


Figure 2.2: Comparison between the Higgs branch Hasse diagram for theories with $\mathrm{U}(N_c)$ (top left) and $\mathrm{SU}(N_c)$ (top right) and for theories with $\mathrm{O}(N_c)$ (bottom left) and $\mathrm{SO}(N_c)$ (bottom right) gauge groups, with the number of fundamental flavours N_f satisfying $N_f \geq N_c$. The blue numbers are the quaternionic dimensions of the leaves, and the red groups are the residual gauge groups.

Let's begin with the simple example of $\widetilde{\mathrm{SU}}(4)_I$ with 4 fundamentals as an appetizer. This representation is real, and so the theory has Sp(4) global symmetry. Computing the Higgsing to $\widetilde{\mathrm{SU}}(3)_I$, we find,

$$\begin{aligned} 4(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(4)_I} - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(4)_I} \to 4\left[(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}\oplus\epsilon\oplus\mathbf{1}\right] \\ - \left[\mathrm{Adj}_{\widetilde{\mathrm{SU}}(3)_I}\oplus(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}\oplus\epsilon\right] \end{aligned} \tag{2.3.25}$$

$$= 3(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}\oplus 3\,\epsilon - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(3)_I}\underbrace{\oplus 4\cdot\mathbf{1}}_{c_4\text{ slice}}\,, \tag{2.3.26}$$

and we see that the remaining theory is an $\widetilde{\mathrm{SU}}(3)_I$ gauge theory with 3 fundamentals and 3 fields in the ϵ. Since both the $(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}$ and the ϵ are real representations, this theory has a $\mathrm{Sp}(3)\times\mathrm{Sp}(3)$ global symmetry. On the other hand, we observe that the transverse slice at the bottom of the Hasse diagram is c_4.

We are now presented with two options regarding how to continue the chain of Higgsings. The gauge fields can either eat the fields in the ϵ, resulting on the breaking $\widetilde{\mathrm{SU}}(3)_I\to\mathrm{SU}(3)$, or fields in the $(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}$, resulting on the breaking $\widetilde{\mathrm{SU}}(3)_I\to\mathrm{SU}(2)\times\mathbb{Z}_2$. In the first case, we find

$$3(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}\oplus 3\,\epsilon - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(3)_I}\to 6\,F_{\mathrm{SU}(3)} - \mathrm{Adj}_{\mathrm{SU}(3)}\oplus\underbrace{3\cdot\mathbf{1}}_{c_3\text{ slice}}\;. \tag{2.3.27}$$

From this point onward, we have the already known Higgsing pattern and Hasse diagram of the SU groups. Let's then consider the second case,

$$\begin{aligned} 3(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(3)_I}\oplus 3\,\epsilon - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(3)_I} \to 3\left[(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(2)_I}\oplus\epsilon\oplus\mathbf{1}\right] \\ + 3\,\epsilon - \left[\mathrm{Adj}_{\widetilde{\mathrm{SU}}(2)_I}\oplus(F\oplus\overline{F})_{\widetilde{\mathrm{SU}}(2)_I}\oplus\epsilon\right]\;. \end{aligned} \tag{2.3.28}$$

Here we are making an abuse of notation: since the principal extension of SU(2) is trivial, we have $\widetilde{\mathrm{SU}}(2) = \mathrm{SU}(2)\times\mathbb{Z}_2$, and the $(F\oplus\overline{F})$ representation is in fact reducible and equal to $2\cdot F_{\mathrm{SU}(2)}$. Using this,

$$(2.3.28) = 4\cdot F_{\mathrm{SU}(2)}\;\oplus\;5\cdot\epsilon\;-\;\mathrm{Adj}_{\mathrm{SU}(2)}\;\oplus\;\underbrace{3\cdot\mathbf{1}}_{c_3\text{ slice}}\;. \tag{2.3.29}$$

We conclude that the theory after this last Higgsing splits into two decoupled theories, one consisting of SU(2) gauge group with 4 flavours and the other of a $\mathbb{Z}_2$ gauge group with

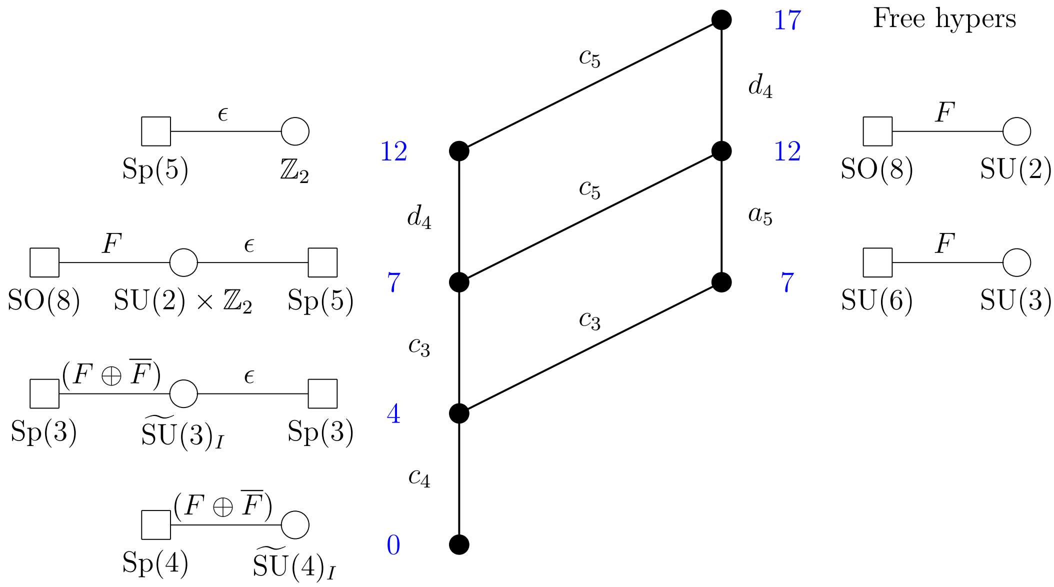


Figure 2.3: Higgs branch Hasse diagram of $\widetilde{\mathrm{SU}}(4)_I + 4\,(F \oplus \overline{F})$. Next to each symplectic leaf, we write its dimension (in blue) and the quiver of the effective theory.

5 fields in the ϵ. The overall global symmetry is $\mathrm{SO}(8) \times \mathrm{Sp}(5)$.

At this point, the possible Higgsings are trivial, since there are no more nontrivial subgroups. We can either Higgs $\mathrm{SU}(2) \to \mathbf{1}$ with the fundamental flavours, leaving the $\mathbb{Z}_2 + 5\epsilon$ alone (this produces a d_4 slice), or Higgs $\mathbb{Z}_2 \to \mathbf{1}$ with the ϵ fields (this produces a c_5 slice). Note that the $\mathrm{SU}(2) + 4F$ remaining in this transition can also be reached from the Higgsing of $\mathrm{SU}(3) + 6F$ that we obtained in the previous steps. All in all, the Hasse diagram for the Higgs branch of this theory is depicted in Figure 2.3.

Generalizing to an arbitrary number of fundamentals $N_f \geq N_c$ and fields in the ϵ is now straightforward.[15] As before, we begin by writing down the hyperKähler quotient for the Higgs branch of $\widetilde{\mathrm{SU}}(N_c)_I + N_f(F \oplus \overline{F}) + N_\epsilon \epsilon$,

$$N_f(F \oplus \overline{F})_{\widetilde{\mathrm{SU}}(N_c)} \oplus N_\epsilon \epsilon - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(N_c)_I}\,. \tag{2.3.30}$$

This theory has $\mathrm{Sp}(N_f) \times \mathrm{Sp}(N_\epsilon)$ global symmetry. Similarly to the intermediate step of the previous example, there are two possible Higgsings, with the ϵ's or with the fundamentals.

[15] The case $N_f < N_c$ is more involved, as is already visible in SQCD with SU gauge groups. One difficulty comes from the fact that the Higgs branch may be a union of several hyperKähler cones with non-trivial intersections. In addition, non complete Higgsing gives rise in some cases to nilpotent operators in the Higgs chiral ring which makes it difficult to match with a 3d $\mathcal{N} = 4$ Coulomb branch ring, as discussed in detail in [87].

Applying the branching rules of Table 2.7 in either case results in

$$\widetilde{\mathrm{SU}}(N_c)_I \to \mathrm{SU}(N_c): \quad (2.3.30) \to N_f F_{\mathrm{SU}(N_c)} \oplus N_f \overline{F}_{\mathrm{SU}(N_c)} - \mathrm{Adj}_{\mathrm{SU}(N_c)} \underbrace{\oplus N_\epsilon \cdot \mathbf{1}}_{c_{N_\epsilon}\text{ slice}} \tag{2.3.31}$$

$$\widetilde{\mathrm{SU}}(N_c)_I \to \widetilde{\mathrm{SU}}(N_c-1)_I: \quad (2.3.30) \to (N_f-1)(F \oplus \overline{F})_{\widetilde{\mathrm{SU}}(N_c-1)_I} \oplus (N_\epsilon + N_f - 1)\epsilon - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(N_c-1)} \underbrace{\oplus N_f \cdot \mathbf{1}}_{c_{N_f}\text{ slice}} \tag{2.3.32}$$

The former leads to an effective theory with $\mathrm{SU}(N_c)$ gauge group and $\mathrm{SU}(2N_f)$ global symmetry, whose Hasse diagram is already known. This is the right part of the Hasse diagram of Figure 2.4. The latter leads to a theory with $\widetilde{\mathrm{SU}}(N_c-1)_I$ gauge group, $N_f - 1$ fields in the $(F \oplus \overline{F})$, and $N_\epsilon + N_f - 1$ fields in the ϵ; thus a $\mathrm{Sp}(N_f-1) \times \mathrm{Sp}(N_\epsilon + N_f - 1)$ global symmetry. In order to continue the computation of the Hasse diagram, we are once again presented with two possibilities: Higgsing with the ϵ fields –this produces a $c_{N_\epsilon+N_f-1}$ slice that merges with the right part of the Hasse diagram corresponding to the connected gauge groups– or with the fundamentals –this produces a c_{N_f-1} slice that continues on the left side of the Hasse diagram corresponding to the disconnected gauge groups with both fundamental and ϵ matter fields–.

The Hasse diagram of Figure 2.4 is the result of iterating this procedure $N_c - 2$ times, until we reach $\widetilde{\mathrm{SU}}(2)_I = \mathrm{SU}(2) \times \mathbb{Z}_2$. At this point, as happened with the previous example, the theory will decouple into an SU(2) gauge theory with $N_f - N_c + 2$ fundamentals, and a $\mathbb{Z}_2$ gauge theory with $N_\epsilon + (N_c-2)(2N_f - N_c + 1)/2$ ϵ's. We can Higgs each of these two gauge groups separetly, resulting in the "rectangle" at the top of the Hasse diagram.

Hasse diagram for $\widetilde{\mathrm{SU}}(N)_{II}$

The computation of the Hasse diagram of the Higgs branch for theories with $\mathrm{SU}(2N_c)_{II}$ gauge group is very similar to the one we just described in detail for the type I case. There are only a few key differences that we need to take into account. The first is that $\widetilde{\mathrm{SU}}(2N-1)_I$ is not a subgroup of $\widetilde{\mathrm{SU}}(2N)_{II}$. This implies that the smallest step we can take in the chain of Higgsings is $\widetilde{\mathrm{SU}}(2N)_{II} \to \widetilde{\mathrm{SU}}(2N-2)_{II}$. The second is that the fundamental representation of the type II groups is pseudo-real rather than real, and therefore these fields will give rise to an SO global symmetry.

As in the type I case, before considering the fully general case, we begin by looking at

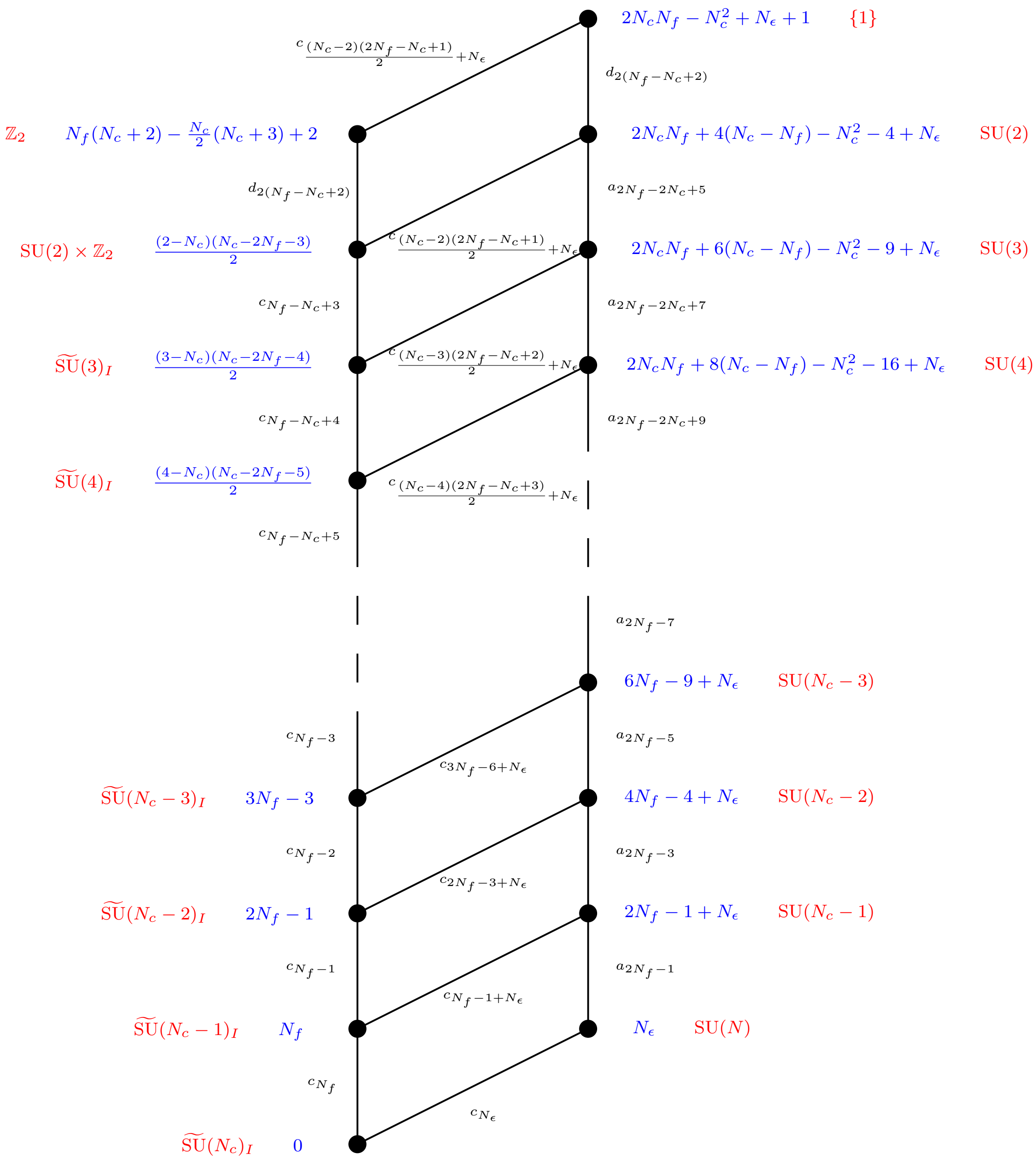


Figure 2.4: Higgs branch Hasse diagram of $\widetilde{\mathrm{SU}}(N_c)_I + N_f\,(F\oplus\overline{F}) + N_\epsilon\,\epsilon$ for $N_f \geq N_c$. The blue numbers are the quaternionic dimensions of the leaves, and the red groups are the residual gauge groups.

a concrete example, $\widetilde{\mathrm{SU}}(6)_{II}$ with 6 fields in the $(F \oplus \overline{F})$; the resulting Hasse diagram is depicted in Figure 2.5. The procedure is the same as before: we begin by writing down

$$6(F \oplus \overline{F})_{\widetilde{\mathrm{SU}}(6)_{II}} - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(6)_{II}}\,, \tag{2.3.33}$$

and apply the branching rules of Table 2.7 under the breaking $\widetilde{\mathrm{SU}}(6)_{II} \to \widetilde{\mathrm{SU}}(4)_{II}$. After some cancellations, this results in

$$4(F \oplus F)_{\widetilde{\mathrm{SU}}(4)_{II}} \oplus 11\epsilon - \mathrm{Adj}_{\widetilde{\mathrm{SU}}(4)_{II}} \underbrace{\oplus (12-3)\cdot \mathbf{1}}_{d_6 \text{ slice}}\,. \tag{2.3.34}$$

We see that the remaining effective theory has $\mathrm{SO}(8) \times \mathrm{Sp}(11)$ global symmetry, and the transverse slice, according to Table 2.8, is the minimal nilpotent orbit of $\mathfrak{so}(12)$. Again we are at a stage where we can proceed with the chain of Higgsings in two ways: either Higgs with the ϵ fields –this results in a c_{11} slice that goes to the right side of the Hasse diagram corresponding to the SU groups– or with the fundamentals –this results in a d_4 slice that continues in the left of the Hasse diagram corresponding to the disconnected groups–. Note that since in the disconnected side of the diagram the rank of the gauge group jumps by two,

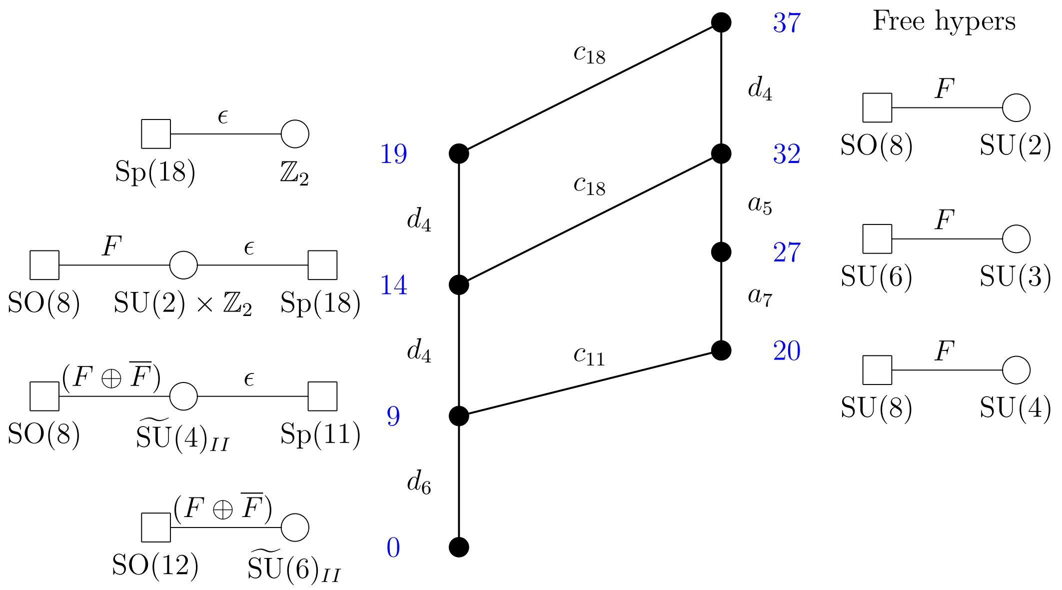


Figure 2.5: Hasse diagram of $\widetilde{\mathrm{SU}}(6)_{II} + 6(F \oplus \overline{F})$. Next to each symplectic leaf, we write its quaternionic dimension (in blue) and the quiver of the effective theory corresponding to the transverse slice from that leaf to the top leaf.

we will have extra symplectic leaves in the right side of the Hasse diagram. In our example, the extra leaf is the one of dimension 27, with gauge group SU(3) and global symmetry SU(6); meanwhile on the left side we jump directly from $\widetilde{\mathrm{SU}}(4)_{II} \to \mathrm{SU}(2) \times \mathbb{Z}_2$. As in the type I case, the disconnected version of the SU(2) group is simply a direct product, which means that at the top of the Hasse diagram we have a rectangle where the sides are the transverse slices c_{18} –corresponding to Higgsing the $\mathbb{Z}_2$ with the ϵ fields– and d_4 –corresponding to Higgsing the SU(2) with the fundamentals–.

With this in mind, generalizing to a $\widetilde{\mathrm{SU}}(2N_c)_{II} + N_f(F \oplus \overline{F}) + N_\epsilon \epsilon$ (with N_f large enough) requires no extra thinking. We only need to repeat the computation above a few times to obtain the result in Figure 2.6.

2.3.3 Magnetic quivers

In [1,85] we began the study of the Higgs branch of the 4d $\mathcal{N} = 2$ discrete gauged SQCD-like theories of type I and II. In this section we attempt to find a magnetic quiver, i.e a 3d $\mathcal{N} = 4$ theory whose Coulomb branch is equal to the Higgs branch of the $\widetilde{\mathrm{SU}}(N)$ gauge theory. We conjecture that the Higgs branch of the 4d $\mathcal{N} = 2$ $\widetilde{\mathrm{SU}}(N)_I$ gauge theory with $N_f(F \oplus \overline{F})$ hypermultiplets is the Coulomb branch of the *wreathed quivers* drawn in Figure 2.9. We check our conjecture performing the computation of the corresponding Coulomb branch Hilbert series on a selection of examples. Our main tool will be the monopole formula that was initially introduced in [19]. The generalization of this formula in the context of wreathed quivers was performed in [91].

We start with a short review of the monopole formula of [91] before applying it to the theories of interest. We work out with full details the $\widetilde{\mathrm{SU}}(3)_I$ case with $N_f = 3$ while we just report the result for $\widetilde{\mathrm{SU}}(N)_I$ with $N > 3$. The type II theories are discussed in Section 2.3.3.

Review of the Monopole formula and Wreathed quivers

We consider a 3d $\mathcal{N} = 4$ simply laced quiver with unitary nodes and only bifundamental hypermultiplets, and a finite subgroup Γ of the automorphisms of that quiver. We call V the set of vertices of the quiver; to each vertex $v \in V$ is associated a unitary gauge group $\mathrm{U}(n_v)$. We call E the set of (unoriented) edges $e = \{v, v'\}$ of the quiver, which correspond to hypermultiplets in bifundamental representations connecting the gauge nodes v and v'. The gauge group of the initial quiver is $G = \prod_{v \in V} \mathrm{U}(n_v)$.

Given such a quiver, we can construct a so-called wreathed quiver for any subgroup Γ of the automorphisms of the quiver diagram. For instance the quiver of Figure 2.7 has

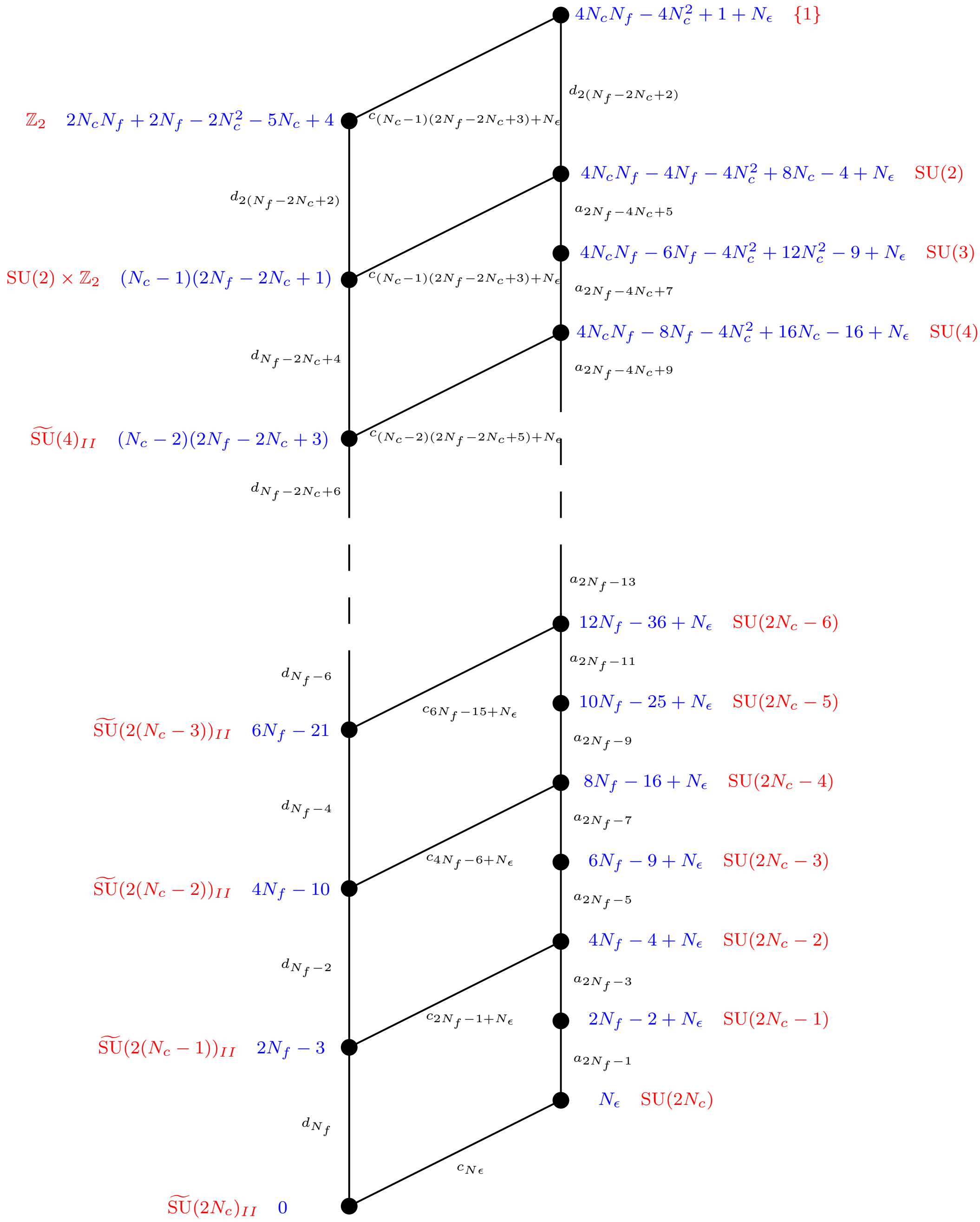


Figure 2.6: Higgs branch Hasse diagram of $\widetilde{\mathrm{SU}}(2N_c)_{II} + N_f(F \oplus \overline{F}) + N_\epsilon \epsilon$ for $N_f \geq 2N_c$. The blue numbers are the quaternionic dimensions of the leaves, and the red groups are the residual gauge groups.

a $\mathbb{Z}_2^2$ diagram automorphism generated by the permutation of the two long legs and the permutation of the two short legs. Wreathing the quiver by Γ means promoting the gauge group to the wreath product $G \wr \Gamma$. If $\Gamma \subseteq S_n$ the wreath product $G \wr \Gamma$ is defined as a set by

$$G \wr \Gamma = \left(\prod_{i=1}^{n} G_i \right) \times \Gamma \tag{2.3.35}$$

where $G_1, \dots, G_n$ are n identical copies of the gauge group G and where the product is defined by

$$(g, \sigma) \cdot (g', \sigma') = (g\sigma(g'), \sigma\sigma') \,, \text{ with } (g\sigma(g'))_i = g_i g'_{\sigma^{-1}(i)} \,. \tag{2.3.36}$$

In this notation $(g, \sigma) \in G \wr \Gamma$ is an ordered list of n elements g_i of G together with $\sigma \in \Gamma$. It should be noted that in the particular case of a symmetry permuting a bouquet of $\mathrm{U}(1)$ gauge nodes, the wreathing operation coincides with the discrete gauging of [33, 105, 106]. We refer the reader to [91] for more details about wreathed quivers.

Following [19, 91] the (unrefined) Coulomb branch Hilbert series associated to Γ takes the form

$$\mathrm{HS}_\Gamma(t) = \frac{1}{|W_\Gamma|} \sum_{m \in \mathbb{Z}^r} \sum_{\gamma \in W_\Gamma(m)} \frac{t^{2\Delta(m)}}{\det(1 - t^2 \gamma)} \,, \tag{2.3.37}$$

where $W_\Gamma := W \rtimes \Gamma \subseteq S_{r+1}$ is given by the extension of $W = \prod_{v \in V} S_{n_v}$ by the symmetry Γ of the quiver. Here $r = -1 + \sum_{v \in V} n_v$ denotes the total rank of the quiver gauge theory that we are considering, while m denotes the magnetic charge that takes value in the lattice $\mathbb{Z}^r$. For any $m \in \mathbb{Z}^r$ we call $W_\Gamma(m) = \{w \in W_\Gamma \mid w \cdot m = m\}$. Finally $\Delta(m)$ denotes the conformal dimension, defined by

$$2\Delta(m) = \sum_{\{v,v'\} \in E} \sum_{i=1}^{n_v} \sum_{j=1}^{n'_v} |m_{v,i} - m_{v',j}| - \sum_{v \in V} \sum_{i=1}^{n_v} \sum_{j=1}^{n_v} |m_{v,i} - m_{v,j}| \,. \tag{2.3.38}$$

When $\Gamma = \{1\}$ is the trivial group, (2.3.37) reproduces the standard monopole formula of [19]. Henceforth we consider quivers which possess a $\mathbb{Z}_2$ automorphism, and we set $\Gamma = \mathbb{Z}_2$.

Formula (2.3.37) can be more efficiently evaluated after exploiting the Weyl group symmetry. We introduce the *Casimir factors* P_{W_Γ} [16]

$$P_{W_\Gamma}(t;m) = \frac{1}{|W_\Gamma|} \sum_{\gamma \in W_\Gamma(m)} \frac{1}{\det(1-t^2\gamma)} \ . \tag{2.3.39}$$

This way the formula (2.3.37) can be recast in the following form

$$\mathrm{HS}_\Gamma(t) = \sum_{m \in \mathrm{Weyl}(G \wr \Gamma) \cap \mathbb{Z}^r} P_{W_\Gamma}(t;m) t^{2\Delta(m)} \ , \tag{2.3.40}$$

where $G = \prod_{v \in V} \mathrm{U}(n_v)$ is the initial gauge group and the sum is taken over the magnetic weights in the principal chamber $\mathrm{Weyl}(G \rtimes \Gamma)$.

Example: the $\widetilde{\mathrm{SU}}(3)_I$ case

We start from the magnetic quiver for the Higgs branch of 4d $\mathcal{N}=2$ SQCD with gauge group SU(3) and $N_f = 3$ flavours. The $\Gamma = \mathbb{Z}_2$ is implemented with a wreathing on the legs of the quiver as schematically shown in Figure 2.7. Note that the quiver has a full $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group, each factor exchanging two identical legs; we just wreath with the diagonal subgroup Γ. This is justified by the generalization to $N_f > N$, see Figure 2.9.

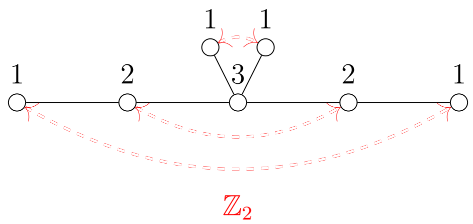
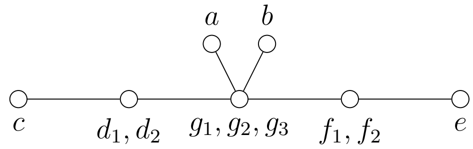


Figure 2.7: Magnetic quiver for SQCD with gauge group $\widetilde{\mathrm{SU}}(3)_I$ and 3 $(F \oplus \overline{F})$. The red dashed lines show the $\mathbb{Z}_2$ action on the legs of the quiver.

In order to check this conjecture we compute the Coulomb branch Hilbert series using formula (2.3.37). We believe it is useful to provide the full details of the computation for that example as this is the first time (2.3.37) is evaluated on a non-trivial wreathed quiver.

[16]Note that for $W_\Gamma = S_N$ this definition coincides with the definition of the Casimir factors P_U associated to unitary gauge groups, that were introduced in [19].

- To each gauge node of the quiver we associate the magnetic weights as follows:



(2.3.41)

The total rank r of the gauge group is $11 - 1 = 10$, and the sum over the magnetic charges is over elements of the form

$$m \cong (a, b, c, d_1, d_2, e, f_1, f_2, g_1, g_2, g_3 = 0) \in \mathbb{Z}^{r+1} \tag{2.3.42}$$

so this is indeed a sum over $\mathbb{Z}^r$ (see Section 2.4.3 of [107] for detailed explanation about the choice of lattice).

- The Weyl group W is the product of the Weyl groups of the simple gauge groups, namely

$$W = S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2 \times S_3 \subset S_{11} \tag{2.3.43}$$

- The wreathing group is $\Gamma = \mathbb{Z}_2$. It is generated by the permutations that exchange simultaneously $a \leftrightarrow b$, $c \leftrightarrow e$ and $d_i \leftrightarrow f_i$ $(i = 1, 2)$. Then the group $W_\Gamma = W \rtimes \Gamma \subset S_{11}$ has order $|W_\Gamma| = 48$.

- The expression (2.3.38) gives the following conformal dimension for the case at hand

$$\begin{aligned} 2\Delta(m) &= \sum_{i=1}^{3} (|a - g_i| + |b - g_i|) + \sum_{i=1}^{2} (|c - d_i| + |e - f_i|) + \sum_{i=1}^{2}\sum_{j=1}^{3} (|d_i - g_j| + |f_i - g_j|) \\ &\quad - \sum_{i,j=1}^{2} (|d_i - d_j| + |f_i - f_j|) - \sum_{i,j=1}^{3} |g_i - g_j| \,. \end{aligned} \tag{2.3.44}$$

We now work out formula (2.3.40), splitting it into six contributions, one for each generalized wall of the Weyl chamber.

- The interior of the chamber is defined by the inequality $a < b$. In that case, for any m satisfying this inequality, $W_\Gamma(m) = W(m)$ so the Casimir factors correspond to those of W, and we get the contribution

$$\mathrm{H}_1(t) = \frac{(1-t^2)}{(1-t^2)^4} \sum_{a<b} \sum_{c} \sum_{d_1 \leq d_2} \sum_{e} \sum_{f_1 \leq f_2} \sum_{g_1 \leq g_2 \leq 0} P_{\mathrm{U}}(d) P_{\mathrm{U}}(f) P_{\mathrm{U}}(g) t^{2\Delta(a,b,c,d_1,d_2,e,f_1,f_2,g_1,g_2,0)} \,. \tag{2.3.45}$$

Note that we factored out the Casimir terms for the four U(1) nodes, giving $(1-t^2)^{-4}$ in the denominator, and we include a $(1-t^2)$ in the numerator to account for the decoupled U(1). In (2.3.45) and all similar equations below, all sums run over the integers $\mathbb{Z}$.

- Then we go on the wall of the chamber defined by $a = b$. Now to avoid over counting we have to be in the interior of that wall, which we define by the inequality $c < e$. In that case clearly $W_\Gamma(m) = W(m)$, so the contribution is

$$\mathrm{H}_2(t) = \frac{(1-t^2)}{(1-t^2)^4} \sum_{a} \sum_{c<e} \sum_{d_1 \leq d_2} \sum_{f_1 \leq f_2} \sum_{g_1 \leq g_2 \leq 0} P_{\mathrm{U}}(d) P_{\mathrm{U}}(f) P_{\mathrm{U}}(g) t^{2\Delta(a,a,c,d_1,d_2,e,f_1,f_2,g_1,g_2,0)} \,. \tag{2.3.46}$$

- The third (respectively the fourth) contributions are defined by $a = b$, $c = e$ and $d_2 < f_2$ (respectively $d_2 = f_2$ and $d_1 < f_1$). This uses a lexicographic order to find a fundamental chamber relative to the fugacities of the non-abelian groups U(2). Again these constraints guarantee that $(a, b, c, d_1, d_2, e, f_1, f_2) \neq (b, a, e, f_1, f_2, c, d_1, d_2)$ so $W_\Gamma(m) = W(m)$ and the contributions are

$$\mathrm{H}_3(t) = \frac{(1-t^2)}{(1-t^2)^4} \sum_{a} \sum_{c} \sum_{f_1 \leq f_2} \sum_{d_1 \leq d_2 < f_2} \sum_{g_1 \leq g_2 \leq 0} P_{\mathrm{U}}(d) P_{\mathrm{U}}(f) P_{\mathrm{U}}(g) t^{2\Delta(a,a,c,d_1,d_2,c,f_1,f_2,g_1,g_2,0)} \,, \tag{2.3.47}$$

$$\mathrm{H}_4(t) = \frac{(1-t^2)}{(1-t^2)^4} \sum_{a} \sum_{c} \sum_{f_1 \leq f_2} \sum_{d_1 < f_1} \sum_{g_1 \leq g_2 \leq 0} P_{\mathrm{U}}(d) P_{\mathrm{U}}(f) P_{\mathrm{U}}(g) t^{2\Delta(a,a,c,d_1,f_2,c,f_1,f_2,g_1,g_2,0)} \,. \tag{2.3.48}$$

- We now reach the regions where $(a, b, c, d_1, d_2, e, f_1, f_2) = (b, a, e, f_1, f_2, c, d_1, d_2)$. In that case we can no longer use the standard Casimir factors P_{U} for the U(2) nodes. Consider first the fifth region, defined by

$$a = b \qquad c = e \qquad d_1 = f_1 \qquad d_2 = f_2 \qquad f_1 < f_2 \,. \tag{2.3.49}$$

As the factor S_3 in W is unaffected, we keep the P_{U} Casimir term for it. Let us denote $W' = W/S_3 = S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2 \subset S_8$. For a weight m satisfying (2.3.49),

$W'_\Gamma(m)$ does not depend on m, so we can factor out from (2.3.37) a prefactor

$$\frac{1}{|W'_\Gamma|}\sum_{\gamma\in W_\Gamma(m)}\frac{1}{\det(1-t^2\gamma)}=\frac{1+6t^4+t^8}{(1-t^2)^8\,(1+t^2)^4}\,. \tag{2.3.50}$$

Therefore the fifth contribution is

$$\mathrm{H}_5(t)=(1-t^2)\frac{1+6t^4+t^8}{(1-t^2)^8\,(1+t^2)^4}\sum_a\sum_c\sum_{f_1<f_2}\sum_{g_1\leq g_2\leq 0}P_{\mathrm{U}}(g)t^{2\Delta(a,a,c,f_1,f_2,c,f_1,f_2,g_1,g_2,0)}\,. \tag{2.3.51}$$

In that expression the four U(1) gauge nodes Casimirs are accounted for in (2.3.50).

- Finally the last region is defined by

$$a=b \qquad c=e \qquad d_1=f_1 \qquad d_2=f_2 \qquad f_1=f_2\,. \tag{2.3.52}$$

and for such an m we get

$$\frac{1}{|W'_\Gamma|}\sum_{\gamma\in W_\Gamma(m)}\frac{1}{\det(1-t^2\gamma)}=\frac{1-t^2+4t^4-t^6+t^8}{(1-t^2)^8(1+t^2)^4(1+t^4)}\,. \tag{2.3.53}$$

This gives the contribution

$$\mathrm{H}_6(t)=(1-t^2)\frac{1-t^2+4t^4-t^6+t^8}{(1-t^2)^8(1+t^2)^4(1+t^4)}\sum_a\sum_c\sum_{f_1}\sum_{g_1\leq g_2\leq 0}P_{\mathrm{U}}(g)t^{2\Delta(a,a,c,f_1,f_1,c,f_1,f_1,g_1,g_2,0)}\,. \tag{2.3.54}$$

The Hilbert series (2.3.37) for the case at hand is the sum of these six contributions. Evaluating each of them perturbatively, we find

$$\begin{aligned}\mathrm{HS}(t) \;=\;& 1+21t^2+20t^3+336t^4+560t^5+3850t^6+7812t^7+34643t^8+73900t^9\\ &+252132t^{10}+535920t^{11}+1533810t^{12}+3177876t^{13}+8011642t^{14}+16049712t^{15}\\ &+36748014t^{16}+O\left(t^{17}\right)\,. \end{aligned}\tag{2.3.55}$$

The Higgs branch Hilbert series for this theory has been evaluated exactly in [1] using the Molien-Weyl integration formula for disconnected groups [40], giving the result

$$\frac{1}{(1-t)^{20}(1+t)^{16}(1+t^2)^8(1+t+t^2)^{10}}\Big(1+6t+34t^2+144t^3+647t^4+2588t^5+$$
$$9663t^6+31988t^7+97058t^8+268350t^9+687264t^{10}+1628374t^{11}+3598201t^{12}+$$

$$7421198t^{13} + 14364220t^{14} + 26130494t^{15} + 44837750t^{16} + 72656468t^{17} + 111456702t^{18} + 162010222t^{19} + 223544610t^{20} + 292994926t^{21} + 365233973t^{22} + 433158422t^{23} + 489154949t^{24} + 526027956t^{25} + 538960928t^{26} + \ \dots\ + \text{palindrome}\ + \dots + t^{52}\Big) \,.$$

All computed orders in (2.3.55) agree with the above expression, giving a strong evidence that the wreathed quiver of Figure 2.7 can be considered to be a magnetic quiver for the $\widetilde{\mathrm{SU}}(3)_I$ gauge theory with 3 $(F \oplus \overline{F})$.

We note that the evaluation of the corresponding refined Hilbert series does not present any conceptual obstruction: it suffices to introduce one fugacity for each U(1) factor in the gauge group in the summand of equation (2.3.40). Concretely in the example at hand, one introduces fugacities z_1, z_2, z_3, z_4 and weight the summands of the various sums above with the term

$$z_1^{c+e} z_2^{d_1+d_2+f_1+f_2} z_3^{g_1+g_2+g_3} z_4^{a+b} \tag{2.3.56}$$

where the four fugacities satisfy the condition

$$z_1^2 z_2^4 z_3^3 z_4^2 = 1 \tag{2.3.57}$$

which corresponds to the diagonal U(1) factor. Solving (2.3.57) for z_4 one obtains the Hilbert series

$$\begin{aligned} \mathrm{HS}(t; z_1, z_2, z_3) \ = \ & 1 + \Big(z_1^2 z_3 z_2^2 + z_1 z_3 z_2^2 + z_3 z_2^2 + z_1 z_2 + z_1 z_3 z_2 + z_3 z_2 \\ & + z_2 + z_1 + z_3 + \frac{1}{z_1} + \frac{1}{z_3} + \frac{1}{z_1 z_2} + \frac{1}{z_1 z_3 z_2} + \frac{1}{z_3 z_2} \\ & + \frac{1}{z_2} + \frac{1}{z_1 z_3 z_2^2} + \frac{1}{z_1^2 z_3 z_2^2} + \frac{1}{z_3 z_2^2} + 3\Big) t^2 + O(t^3) \,, \end{aligned} \tag{2.3.58}$$

which has coefficient of t^2 equal to the character of the adjoint representation of $\mathfrak{sp}(3)$ written in the simple root basis.

Folded quiver and twisted compactification

In this paragraph, we denote by $\mathcal{C}$ the Coulomb branch of the 3d $\mathcal{N}=4$ theory defined by the wreathed quiver of Figure 2.7 and by $\mathcal{H}$ the Higgs branch of the 4d $\mathcal{N}=2$ $\widetilde{\mathrm{SU}}(3)_I$ gauge theory with $3(F \oplus \overline{F})$ matter. We now consider the following three claims:

(α) The exact Hilbert series of $\mathcal{C}$ and $\mathcal{H}$ are equal.

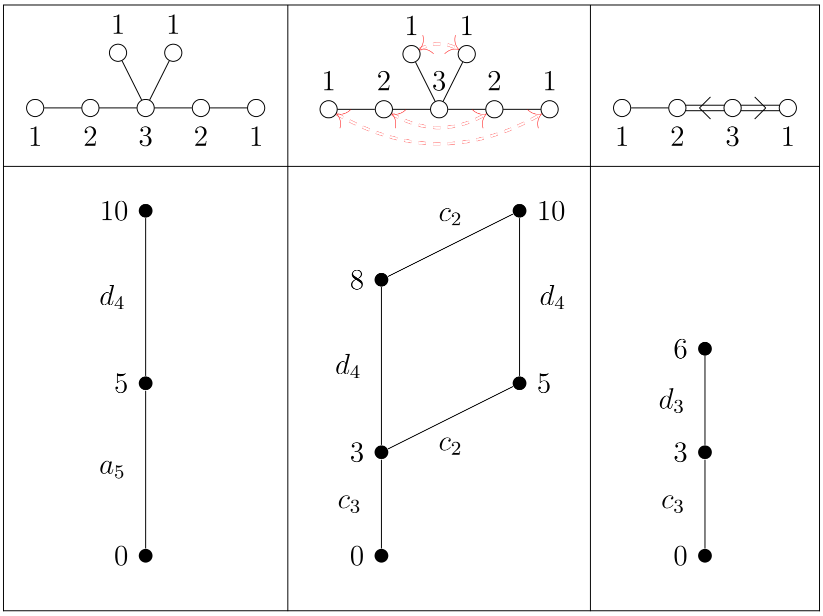
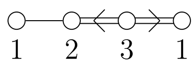


Table 2.9: Comparison of the Hasse diagrams for a quiver with a $\mathbb{Z}_2$ symmetry and the corresponding wreathed and folded quivers.

(β) The Hasse diagrams of $\mathcal{C}$ and $\mathcal{H}$ as symplectic singularities agree.

(γ) The symplectic singularities $\mathcal{C}$ and $\mathcal{H}$ are isomorphic.

The logical implications between these statements is $(\gamma) \Longrightarrow (\beta) \Longrightarrow (\alpha)$. The computation performed above strongly suggests that (α) holds. Based on that result, and on physical intuition regarding charge conjugation,[17] we conjecture that (γ) holds as well. If this is correct, then (β) should also be correct, and in combination with the results of Section 2.3.2, it means that we have identified the Hasse diagram for $\mathcal{C}$, see the middle column of Table 2.9. It is interesting to compare this Hasse diagram with the Hasse diagram of a third quiver, namely the non simply laced quiver

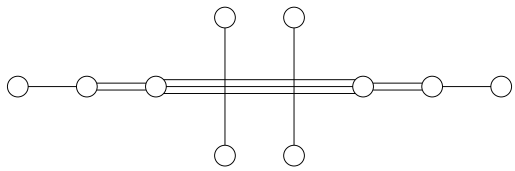


(2.3.59)

obtained by folding, presented in the last column of Table 2.9. The Hasse diagram is obtained from the quiver subtraction algorithm (see [92] for a similar computation).

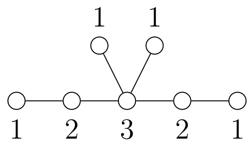
[17] It is known that the quiver of Figure 2.7 before wreathing is 3d mirror to the S^1 compactification of the 4d $\mathcal{N}=2$ SU(3) theory with 6 flavors. The $\widetilde{\mathrm{SU}}(3)$ theory is obtained by gauging charge conjugation, which is identified in [75] with the $\mathbb{Z}_2$ used for wreathing in Figure 2.7.

We note that the quiver (2.3.59) arises naturally as follows. Consider the following brane web, where vertical lines represent NS5 branes, horizontal lines represent D5 branes and circles represent (p, q)-seven-branes with appropriate charges:



(2.3.60)

This represents the 5d $\mathcal{N} = 1$ theory SU(3) with 6 fundamental hypers, with masses set to zero, and finite gauge coupling. This brane web has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (the first factor being the reflection with respect to a vertical axis, and the second factor a reflection with respect to a horizontal axis). In particular, the diagonal $\mathbb{Z}_2$, which is a rotation of angle π in the plane of the brane web, should correspond to charge conjugation in the SU(3) theory [75]. The magnetic quiver associated to this brane web is



(2.3.61)

It has a $\mathrm{SU}(6) \times \mathrm{U}(1)$ global symmetry. We can compactify this 5d theory on a circle with a $\mathbb{Z}_2$ twist, corresponding to charge conjugation, to obtain a $\mathcal{N} = 2$ theory in 4d, following [75]. Then the SU(6) factor in the global symmetry is broken to Sp(3), and the U(1) factor is completely broken. The magnetic quiver, which is derived using the rules of Appendix B of [108], is (2.3.59).

This construction sheds light on the difference between the wreathed and the folded quivers from the 4d perspective. In the first case, charge conjugation is gauged, which means that inequivalent configurations in the original theory are declared to be equivalent. Mathematically, the operation on the Higgs branch is a quotient, and the dimension is unchanged. In the second case, charge conjugation is involved in twisted compactification: mathematically, the operation on the Higgs branch is a reduction to fixed points of the discrete action, and accordingly the dimension is changed. We conclude this section with an observation of an apparent conflict with a conjecture of [91], which states that the Hasse diagram of a folded quiver should be a subdiagram of the Hasse diagram of any corresponding wreathed quiver. The diagrams of Table 2.9 contradict this conjecture (which was based on observation of a few examples), and it would be interesting to study this point further.

Type I – general case

Based on the computation performed for the $\widetilde{\mathrm{SU}}(3)_I$ case we can infer the general form of the wreathed magnetic quiver for theories of type I, see Figure 2.9. When $N_f = N$ this reduces to the quiver of Figure 2.8, where the $\mathbb{Z}_2$ action is picked from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group of the quiver by continuation from the $N_f > N$ case.

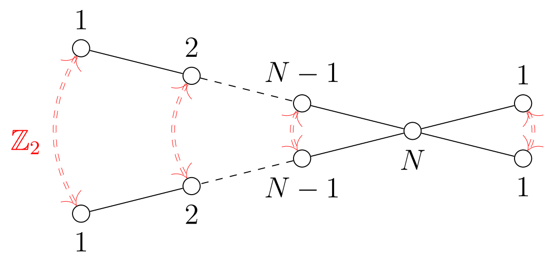


Figure 2.8: Wreathed quiver for SQCD-like theories with gauge group $\widetilde{\mathrm{SU}}(N)_I$ and N flavours. The automorphism group of this quiver is $\mathbb{Z}_2 \times \mathbb{Z}_2$, but we wreath only a $\mathbb{Z}_2$ subgroup, as made clear by the generalization to higher number of flavors in Figure 2.9.

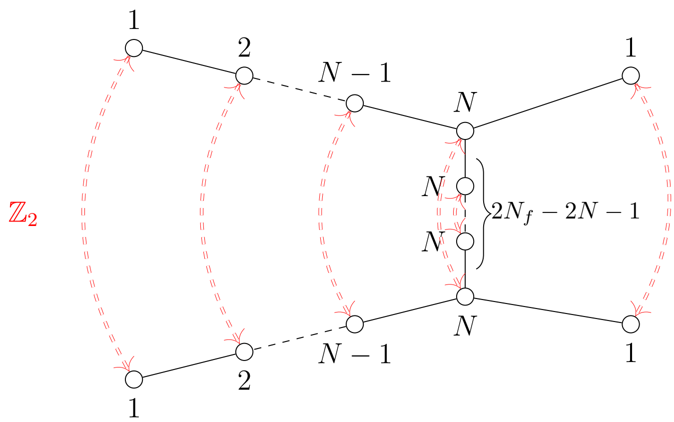


Figure 2.9: Wreathed quiver for SQCD-like theories with gauge group $\widetilde{\mathrm{SU}}(N)_I$ and $N_f > N$ flavours. There is a single $\mathbb{Z}_2$ action which flips the whole quiver about the horizontal axis. The global symmetry of this wreathed quiver is $\mathrm{Sp}(N_f)$.

For $N \geq 4$ the explicit evaluation of the Coulomb branch Hilbert series for $N_f = N$, associated to the conjectured wreathed quiver, turns out to be computationally quite involved. Due to this obstruction we checked our conjecture only for the case $N = 4$, where the application of the formula (2.3.37) gives

$$1 + 36t^2 + 1114t^4 + 24717t^6 + 417276t^8 + O(t^{10}) \ . \tag{2.3.62}$$

We observe that this expression perfectly matches with the first orders of the expansion of the Higgs branch Hilbert series for SQCD with gauge group $\widetilde{\mathrm{SU}}(4)_I$ and eight flavours, that was computed in [1].

Comment on 3d mirror symmetry We have argued that the wreathed quivers of Figure 2.9 are magnetic quivers for the Higgs branch of theories with $\widetilde{\mathrm{SU}}$ gauge groups. One could be tempted to further conjecture that this provides a 3d mirror pair. However, one would need to study $\widetilde{\mathrm{SU}}$ gauge theories in 3d $\mathcal{N} = 4$ and then examine how monopole operators are affected by the $\mathbb{Z}_2$ factor in the gauge group in order to characterize the 3d Coulomb branch of $\widetilde{\mathrm{SU}}$ gauge theories and match it with the Higgs branch of wreathed quivers. This is left for future work.

Type II

In the previous sections, we have provided the magnetic quivers for theories with one of the types of disconnected gauge groups that we have discussed in this thesis, $\widetilde{\mathrm{SU}}(N)_I$. Currently we have no candidate for a possible magnetic quiver of a theory with $\widetilde{\mathrm{SU}}(N)_{II}$ gauge group. To understand why the type II groups pose a much bigger problem than the type I groups, let's look into the logic that led us to the wreathed quiver in Figure 2.7.

Two of the main characteristics of the Higgs branch of SQCD-like theories with $\widetilde{\mathrm{SU}}$ gauge groups groups are that, on the one hand, its dimension is the same as for their connected cousins SU, while on the other hand the global symmetry is modified due to the reality properties of the fundamental representation. In particular, for $N_c = 4$ with 4 $F \oplus \overline{F}$ the quaternionic dimension of the Higgs branch is 17, and the global symmetry is SU, Sp or SO in the connected case, type I and type II respectively.

When looking for a magnetic quiver, a natural starting point is the known magnetic quiver for the SU groups, which in the $N_c = 3$ case is depicted in (2.3.61). This has the correct dimension, but the wrong global symmetry for our purposes. We also have its folded version, the non-simply laced quiver in (2.3.59); this has the correct global symmetry Sp(3), but the incorrect dimension. With this in mind, the introduction of wreathed quivers in [91] quickly leads to a potential candidate for the magnetic quiver of $\widetilde{\mathrm{SU}}(3)_I$, since the wreathing construction preserves the dimension, while modifying the global symmetry in the same way as the folding. This candidate is the one in Figure 2.7, and it turned out to be the correct one.

However, for the type II groups the puzzle is significantly more complicated. Our analysis shows that starting from the magnetic quiver of SU(4), none of the possible ways to wreath a $\mathbb{Z}_2$ gives rise to the expected global symmetry. This has been confirmed by Hilbert series computations. Thus, as stated above, we have no candidate for the magnetic quiver of $\widetilde{\mathrm{SU}}(N_c)_{II}$. It is of course possible that such a magnetic quiver may be found as a wreathing of a completely different quiver, perhaps including not only unitary nodes; or from an altogether different route.

2.3.4 Conclusions

In this section we analyzed several aspects of 4d $\mathcal{N}=2$ theories with disconnected gauge groups. In particular we studied how the global structure of these groups affects the Hasse diagrams for the Higgs branch of supersymmetric gauge theories. The main difference with respect the connected case is that these diagrams are characterized by the presence of bifurcations, physically corresponding to scalar fields transforming in different representations of the gauge group getting a VEV.

Moreover, in the second part of the section, we moved a further step towards the understanding of the Higgs branch of the 4d $\mathcal{N}=2$ SQCD like-theories with $\widetilde{\mathrm{SU}}(N)_I$ gauge group providing a candidate for a magnetic quiver that turns out to be a wreathed quiver. Our analysis also suggests that a magnetic quiver for type II theories is not a wreathed quiver of type discussed in [91] or, to the best of our knowledge, any other type 3d $\mathcal{N}=4$ quiver appearing in the literature. We leave the identification of this quiver for future investigation.

This naturally leads to a wealth of open problems, the most prominent of which being the connection between the two parts of this work, namely the Hasse diagrams and the magnetic quivers. To the best of our knowledge, the algorithms of quiver subtraction leading to Hasse diagrams has not been extended to wreathed quivers. The present work thus offers an infinite family of data points that could serve as a basis to understand how quiver subtraction applies to those quivers. In particular, it should be noted that the Hasse diagram for a wreathed quiver seems not to contain in general the Hasse diagram of the associated folded quiver, as shown in Table 2.9. This point needs to be investigated further. A brane realization of theories with $\widetilde{\mathrm{SU}}(N)$ gauge groups, possibly along the lines of [44, 45, 109, 110] would be an important step forward.

2.4 Non-invertible symmetries from discrete gauging and completeness of the spectrum

Global symmetries play a central role in Quantum Field Theory (QFT). They are used as an organizing principle to systematically construct the possible operators, their breaking pattern allows to characterize the phases of a system and their possible anomalies provide exact constraints on the dynamics. However, in recent times it has been made clear that the notion of symmetry has to be generalized from the traditional textbook definition typically in terms of Noether currents. The central idea, pioneered in [70], is that symmetries are associated to symmetry operators $T_g(M^{d-(p+1)})$ depending on a transformation g and defined on codimension $p+1$ manifolds $M^{d-(p+1)}$.[18] The crucial point is that the dependence on $M^{d-(p+1)}$ is topological: the properties of $T_g(M^{d-(p+1)})$ –for instance their correlation functions– do not change under small changes of $M^{d-(p+1)}$ as long as these do not cross any charged operator.

The textbook examples of global symmetries naturally fit in this framework. Indeed, for a continuous symmetry there is a Noether current, whose integral on M^{d-1} manifolds gives a charge Q. Clearly, slight changes of M^{d-1} do not change Q as long as these do not cross charged operators. Moreover, the exponential of Q gives a an element of the symmetry group, and thus corresponds to the $T_g(M^{d-1})$.[19] The point of view above naturally generalizes this in two directions. On one hand it allows for more generic symmetries supported on codimension $p+1$ manifolds whose charged objects are supported on p-dimensional submanifolds. These are often referred to as *higher form symmetries* or *p-form symmetries*. On the other hand, it allows to consider more generic categorical symmetries not arising from a group. This is reflected into a more generic fusion rule for symmetry operators, which in particular do not need to have an inverse (as opposed to what should happen for a fusion rule of elements in a group). These cases are often dubbed *non-invertible symmetries*.

The existence of non-invertible symmetries is well-known in lower-dimensional QFT's. In particular, in 2d there is a whole body of work studying these (see *e.g.* [111–116] for early references). Their status in higher dimensions is however a bit less clear. The case of $O(2)$ has been argued to give rise to non-invertible symmetries in [117] (see also [118], and [119] for further developments), and more exotic examples have been constructed in [120, 121]. More recently, it has been argued in [122] that indeed non-invertible symmetries are common in higher dimensions.

[18] We will mostly assume the QFT defined on a dimension d euclidean signature space, and thus talk about operator insertions.

[19] In this case, as e^{iQ} is a group element, it is often denoted by $U_g(M^{d-1})$.

Very recently, [118] (see also [123]) provided a criterion to compute the symmetry operators in a gauge theory with compact gauge group, including both invertible (*i.e.* usual symmetries associated to groups) as well as non-invertible symmetries. From the analysis in [118], it follows that if a gauge theory has local operators in all possible representations, no possible non-trivial topological operator candidate for electric 1-form symmetry operator can exist.[20] Therefore, the absence of global electric 1-form symmetries, whether group-like or non-invertible, is equivalent to the completeness of the spectrum of the QFT.[21] In turn, this has interesting implications for the Swampland Program (in short, the study of the restrictions imposed in the low energy Physics which can be consistently coupled to Quantum Gravity. See *e.g.* [124, 125] for introductions and further references), where the Absence of Global Symmetries and the Completeness of the Spectrum are two central conjectures which indeed have been long suspected to be deeply related.

In this section we study in detail (certain) higher-form global symmetries of gauge theories which include, as an element of the gauge group, charge conjugation in generic d dimensions. More precisely, we will consider gauge theories based on the gauge groups constructed in [1, 85] dubbed $\widetilde{SU}(N)$. These are principal extensions of $SU(N)$ by the $\mathbb{Z}_2$ outer automorphism corresponding to flipping the Dynkin diagram, which, in particular, exchanges the fundamental representation with the antifundamental, and thus corresponds to charge conjugation (the construction can be extended to $U(N)$, giving rise to $\widetilde{U}(N)$). Concentrating on pure gauge theories, we will study the 1-form electric symmetry, which turns out to be non-invertible (in a sense, generalizing the $O(2)$ example). Moreover, as the gauge groups are disconnected, there is a $(d-2)$-form symmetry associated to the non-trivial $\pi_0(G)$ for $G = \widetilde{SU}(N)$, $\widetilde{U}(N)$.[22] We also introduce String Theory constructions of these theories. Amusingly, these automatically all come with configurations of extended objects which break the $(d-2)$-form symmetry. From this perspective, they may be regarded as Swampland examples in the sense that when the gauge theory with gauge group G is embedded into a consistent theory of Quantum Gravity, the otherwise present $(d-2)$-form symmetry is broken by the presence of charged "matter" (in this case extended objects).

The remainder of the section is structured as follows. In section 2.4.1 we review basic

[20] By electric 1-form symmetry we mean the one that is always present in a gauge theory, associated to the field strenght. In the free Maxwell case, the Noether current is simply F.

[21] Completeness of the spectrum is defined as the existence of operators in every possible representation of the gauge group.

[22] There may be other higher form symmetries other than those we consider. In particular, there may be a $(d-3)$-form magnetic symmetry associated to the dual of the gauge field. However, its study requires the knowledge of the GNO dual group, which is not known at present for the theories at hand.

facts in the topic of higher form global symmetries, mainly from [70] and recent progress in [118]. In section 2.4.2 give a lightning review of the groups $\widetilde{SU}(N)$, $\widetilde{U}(N)$ and study the electric 1-form symmetries of pure gauge theories based on them. In section 2.4.3 we study the $(d-2)$-form symmetry coming from the fact that the groups are disconnected. We discuss the would-be charged objects, which are the so-called *Alice strings* [44] (or *twist vortices* in the nomenclature of [118]). As it is well-known, in the presence of twist vortices, only a subgroup of the gauge group is globally well-defined [126]. We also introduce a stringy construction for gauge theories based on $\widetilde{U}(N)$, which, as advertised above, automatically come with Alice strings which break the $(d-2)$-global symmetry.

2.4.1 Higher form symmetries and topological operators

In the quest to generalize the notion of symmetry to higher-form global symmetries [70], one quickly realizes that the usual textbook formulation, based on a Lagrangian and an explicit transformation of the fields, is not appropriate. Instead, the focus should be on the symmetry generators $U_g(M^{d-1})$ depending on a symmetry transformation g and associated to a manifold M^{d-1}. In the continous case, these are given by the exponentiation of the charge computed as the integral of the Noether current,

$$Q(M^{d-1}) = \int_{M^{d-1}} \star J \,, \tag{2.4.1}$$

The key is that the dependence of U_g on the manifold M^{d-1} in which it is supported is topological: U_g doesn't change under deformations of M^{d-1} unless the deformation crosses an operator charged under the symmetry.

This point of view can be easily generalized to higher-form symmetries. The symmetry operators now live on a codimension $p+1$ manifold (on whom they depend only topologically), and the charged objects are extended on p spatial dimensions.

Usually, the symmetry transformations form a group when fusing[23] the topological operators,

$$U_{g_1}(M^{d-p-1}) \cdot U_{g_2}(M^{d-p-1}) = U_{g_1 g_2}(M^{d-p-1}) \,, \tag{2.4.2}$$

and the transformation has an inverse $U_g^{-1}(M^{d-p-1}) = U_{g^{-1}}(M^{d-p-1})$. However, this require-

[23]Intuitively, and since the topological nature of the operators is best understood when considering them inside correlation functions, *fusing* them can be seen as taking their OPE: note that this operation makes sense even for non-conformal field theories due to the topological nature of said operators.

ment can be relaxed, by demanding instead that the topological operators fuse according to (we now denote the operators by T to stress that they may not come from a group)

$$T_a(M^{d-p-1}) \cdot T_b(M^{d-p-1}) = \sum_i N_{ab}^i \, T_i(M^{d-p-1}) \,, \tag{2.4.3}$$

and need not have an inverse; this structure is that of a fusion algebra. In this case we have what is called a *categorical symmetry* or *non-invertible symmetry.*[24] In dimensions three or higher, a further generalization is possible, whereby operators of different dimensions can start or end at a given topological junction, and the integer coefficients N_{ab}^i ought to be understood as the partition function of the topological theories living on said junctions [122]. The underlying structure is that of an $n-$category, and so this case is referred to as a *higher categorical symmetry.*

The action of the topological operators on the charged objects $O(\mathcal{C}^p)$ can be understood by introducing the symmetry operator on a sphere S^{d-p-1} that surrounds $\mathcal{C}^p$, and then shrinking that sphere to a point, finding[25]

$$T_a(S^{d-p-1})O(\mathcal{C}^p) = B_O(a)O(\mathcal{C}^p) \,, \tag{2.4.4}$$

where $B_O(a)$ is called the linking coefficient. As an example, we can consider the electric 1-form symmetry of a gauge theory. The charged operators are the Wilson lines $W_\rho(\gamma^1)$, with ρ a representation of the gauge group; and the symmetry operators, which we denote $T_a(M^{d-2})$, are the so called Gukov-Witten operators [129, 130], which are labelled by a conjugacy class a of the gauge group. The linking coefficient in this case is obtained from the Aharonov-Bohm interaction between the line and the codimension 2 operator [118, 131],

$$B_{W_\rho}(a) = \frac{\chi_\rho(a)}{\dim\, \rho} \mathrm{sz}(a) \,, \tag{2.4.5}$$

[24]The name categorical symmetry comes from the 2-dimensional case, where the structure that captures the features of these symmetries is that of a modular tensor category (see [127] for a more precise review and references): the objects are the topological line defects, the morphisms are the local operators where the lines can begin, end, or change; and the integer coefficients N_{ab}^i correspond to the different possible local operators in a given topological junction. See [128] for recent progress extending this to dimensions higher than 2,3.

[25]One may wonder whether more generic actions rather than the "canonical linking" (2.4.4) are possible. As an example, it may happen that symmetry operators act trivially on the would-be canonically linked operators, but non-trivially on other extended operators (an example of this being charge conjugation in $U(1)_{2N}$ Chern-Simons theory: there is no gauge-invariant local operator to begin with. Yet, charge conjugation exchanges a line with its conjugate). However, for our purposes we can assume that (2.4.4) holds [118], as we will be interested on 1-form symmetries in pure gauge theories where the O's are Wilson lines.

where $\chi_\rho(a)$ is the character of the representation ρ evaluated in the conjugacy class of a, $\mathrm{sz}(a)$ is the number of elements of the group inside said conjugacy class and $\dim\rho$ is the dimension of the representation.

In [118], the question was addressed of whether or not a Gukov-Witten operator can be topological (i.e. if it generates a, possibly non-invertible, 1-form global symmetry) if it links with an endable Wilson line. The argument is as follows: consider a gauge theory with matter fields in a representation R. Then the Wilson lines corresponding to the representation R and tensor products thereof can end and break into segments. Suppose that a GW operator 1) is topological and 2) links non-trivially with the Wilson line (i.e. the linking coefficient is different from its linking with the identity operator, $B_W(a) \neq B_1(a)$). Then, given the topological nature of the GW operator, we can consider either shrinking it on top of the Wilson line, which produces the linking coefficient $B_W(a)$; or breaking the Wilson line into segments and shrinking the GW on top of a point where there is no Wilson line, which produces a trivial linking $B_1(a)$. By comparison, it follows that if a GW operator links non-trivially with an endable Wilson line, it cannot be topological. Equivalently, a necessary condition for GW operators to be topological is to link trivially with endable Wilson lines.

In fact, in [118] it was also argued that for gauge theories based on a compact gauge group this necessary condition is also sufficient. The argument starts by enlarging the original theory $\mathcal{T}$ with the addition of an adjoint scalar into an auxiliary theory $\mathcal{T}'$. Since the gauge sector of $\mathcal{T}'$ is identical to that of $\mathcal{T}$, the possible set of Gukov-Witten operators as well as Wilson lines is the same as in the original theory. In addition, since no matter in representations not already present in $\mathcal{T}$ is introduced, the endability/non-endability of Gukov-Witten/Wilson lines is not changed from $\mathcal{T}$ to $\mathcal{T}'$. As a consequence the topological sector of $\mathcal{T}'$ and $\mathcal{T}$ is the same (of course, generic observables such as generic correlation functions in $\mathcal{T}'$ are different than those in $\mathcal{T}$), and hence so is their 1-form symmetry. A simpler version of this statement is the following: if one has $U(1)$ with a charge n electron, the 1-form symmetry is $\mathbb{Z}_n$. If one adds to the theory arbitrarily many more charge n electrons, obviously generic correlation functions will change, yet the 1-form symmetry remains $\mathbb{Z}_n$. This is true as long as one adds matter in representations already present –in this case, being $U(1)$, electrons with the same charge. Hence, all in all, one can take advantage of this to study the structure of the 1-form symmetry in $\mathcal{T}$ from that of $\mathcal{T}'$[26]: one can give a VEV to the adjoint scalar and move on the "Coulomb branch" where the gauge group is

[26]Note that in general even the dynamical behavior (*e.g.* spontaneous symmetry breaking) may be different. However we are only interested in the structure of the UV 1-form symmetry, which must be identical in $\mathcal{T}$ to the one in $\mathcal{T}'$.

$U(1)^r$ (and possibly some discrete factor). In that case, it is known that all GW operators that link trivially with a Wilson line are topological. In fact, they can be seen to precisely coincide with those selected by the necessary condition above. By continuity, at least these operators must exist at the origin of the Coulomb branch, where at most we could expect some symmetry enhancement. However, they already exhaust all the *a priori* possible ones selected by the necessary condition of the previous paragraph, and therefore one concludes that the trivial linking criterion is actually sufficient.

Let us consider pure gauge theories with a gauge group G that is disconnected. The endable Wilson lines will correspond to the adjoint representation and its tensor products. In this case, the previous argument, together with (2.4.5), leads to a very simple criterium to find the 1-form symmetry. Instead of the center of the group (as is the case in the more usual examples of connected and simply connected groups like $SU(N)$), the topological Gukov-Witten operators will correspond to the conjugacy classes of the elements in the centralizer of the identity component G^0 of G,

$$\{\text{topological GW}\} \equiv \{G^{-1}hG\,,\, h \in C_G\left(G^0\right)\}\,, \tag{2.4.6}$$

Once the topological operators have been identified, we need to determine whether they generate a group or a non-invertible symmetry. A possible way to do it is by using the so called *quantum dimension* of the operator, which is defined [118] as the linking coefficient with the identity operator, $\dim(U_a) = B_1(a)$. As an example, if we are concerned with the one-form symmetry, the topological operators are Gukov-Witten operators and their quantum dimension is (2.4.5)

$$\dim(T_a) = B_1(a) = \mathrm{sz}(a)\,. \tag{2.4.7}$$

The quantum dimensions have the property that they get multiplied under the fusion of topological operators, and summed under their sum,

$$\dim(T_a \cdot T_b) = \dim(T_a)\dim(T_b)\,, \tag{2.4.8}$$

$$\dim(T_a + T_b) = \dim(T_a) + \dim(T_b)\,, \tag{2.4.9}$$

and since the topological operator corresponding to the identity always has quantum dimension equal to 1, this allows us to infer when a symmetry has to be non-invertible from the presence of topological operators with quantum dimension ≥ 1.

In the same way that we can study the electric one-form symmetry from the topological GW operators and under which Wilson lines are charged, we can also look at the topological Wilson lines to find out about the dual $(d-2)$-form symmetry under which GW operators are charged. It turns out that this problem has a more straight-forward solution [118]. Since the gauge holonomy along a contractible loop always belongs to the identity component of the gauge group, two Wilson lines along homotopic paths differ at most by an element of G^0. Therefore, only Wilson lines corresponding to representations that map G^0 to the identity are topological. In other words,

$$\{\text{topological WL}\} \equiv \{\text{representations of } \pi_0(G)\}\,, \tag{2.4.10}$$

where $\pi_0(G)$ is the group of connected components of G. Note that this discussion is actually unchanged in the presence of matter fields in any representation of G.

While the main focus of this work lies in pure gauge theories, one can consider more general theories adding matter fields.[27] If the matter is in a representation smaller than the adjoint, the corresponding Wilson lines will become endable, and the GW operators with whom they link can no longer be topological. As a consequence, the 1-form symmetry will be reduced. Similarly, for the dual $(d-2)$-form symmetry, we can make the GW operators endable, albeit in this case by adding suitable codimension-3 objets. These were called *twist vortices* in [118] and are defined by a monodromy in G/G^0 when going around them. For the disconnected gauge groups under study in this work, these are also known as *Alice vortices*, or *Alice strings* in 4 dimensions [44].

It is worth pointing that this discussion, which is mostly “kinematical", holds irrespective of the dimensionality of the space for pure gauge theories with YM action. In particular, the existence and invertibility properties of the generators of the electric 1-form symmetry do not depend on d. However, depending on the dimensionality of the space, there can be certain additional ingredients entering the full 1-form symmetry: for instance, in $d=3$ both GW and Wilson lines are line operators; or in $d=4$ there can be a magnetic 1-form symmetry acting on Wilson lines. In both cases, the full 1-form symmetry may be more complicated than the subset of topological operators studied in this work, which is only the electric part. Moreover, depending on the spacetime dimension, one may have different topological terms (such as θ terms in 4d, Chern-Simons terms in odd d) which could as well enrich the discussion. We leave these aspects for future studies.

[27] Depending on the matter content, there may be gauge anomalies, as recently studied in [132].

Examples: SO, Sp, O

In this section, we review the higher form symmetries of the orthogonal group $O(N)$ that result from using the formalism discussed above. We also list the results for $SO(N)$ and $Sp(N)$, as we will need them on later sections.

Special orthogonal and symplectic groups: These groups are connected, therefore the 1-form symmetry of the corresponding pure gauge theory is simply given by the center (see Table 2.10). In all these cases, the 1-form symmetry is invertible.

Gauge group	Topological GW operators	Quantum dimension	1-form symmetry
$Sp(N)$	$T_0^{Sp} = \mathrm{Id}$ T_π^{Sp}	1 1	$\mathbb{Z}_2$
$SO(2)$	$T_0^{SO(2)} = \mathrm{Id}$ $T_\theta^{SO(2)}$, $\theta \in (0, 2\pi)$	1 1	$SO(2)$
$SO(2k)$, $k \geq 2$	$T_0^{SO} = \mathrm{Id}$ T_π^{SO}	1 1	$\mathbb{Z}_2$
$SO(2k-1)$, $k \geq 2$	$T_0^{SO} = \mathrm{Id}$	1	Trivial

Table 2.10: Summary of topological Gukov-Witten operators for theories with $Sp(N)$ and $SO(N)$ gauge group.

The magnetic $(d-2)$-form symmetry, which is given by the group of connected components, is trivial in all these cases.

Orthogonal groups: This is the first instance of disconnected gauge group that we encounter. Instead of the center, the 1-form symmetry is obtained from the centralizer of the identity component of the group. We need to distinguish three possible cases: $N = 2$, N even and bigger than 2, or N odd.

The case of $O(2)$ was studied in detail in [117, 118], and it is special because the identity component, $SO(2)$, is abelian. The full group $O(2)$ can be written as a semidirect product $SO(2) \rtimes \mathbb{Z}_2$. By definition the generator of the $\mathbb{Z}_2$ does not commute with the $SO(2)$, therefore the centralizer is

$$C_{O(2)}(SO(2)) = SO(2)\,. \tag{2.4.11}$$

The topological GW operators are labelled by the conjugacy classes of elements in this centralizer. Here the global structure of the group becomes relevant, as we can also conjugate by the nontrivial element in the $\mathbb{Z}_2$. If we denote this element as P, the action on an element of the centralizer is

$$P^{-1} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \tag{2.4.12}$$

i.e. it maps $\theta \mapsto -\theta$. This means that we don't have one GW operator for each $\theta \in [0, 2\pi]$, but rather one for each $\theta \in [0, \pi]$ and the quantum dimension of the operators labelled by $\theta \in (0, \pi)$ is equal to two. Therefore, the 1-form symmetry in the $O(2)$ case is non-invertible. The fusion algebra of the topological operators was reported in [118],

$$\begin{aligned} &T^{O(2)}_\theta \cdot T^{O(2)}_\varphi = T^{O(2)}_{\theta+\varphi} + T^{O(2)}_{\theta-\varphi} \,, \\ &T^{O(2)}_\theta \cdot T^{O(2)}_\pi = T^{O(2)}_{\theta+\pi} \,, \\ &T^{O(2)}_\pi \cdot T^{O(2)}_\pi = 1 \,, \\ &T^{O(2)}_\theta \cdot T^{O(2)}_\theta = 1 + W^{O(2)}_{\text{sign}} + T^{O(2)}_{2\theta} \,, \\ &T^{O(2)}_\theta \cdot T^{O(2)}_{\pi-\theta} = T^{O(2)}_\pi + W^{O(2)}_{\text{sign}} T^{O(2)}_\pi + T^{O(2)}_{2\theta-\pi} \,, \end{aligned} \tag{2.4.13}$$

where $\theta \neq \varphi$ and $W^{O(2)}_{\text{sign}}$ is the Wilson line in the sign representation of $O(2)$ ($W^{O(2)}_{\text{sign}}$ alone stands for the Wilson line in the sign representation on a trivial surface). The appearance of the Wilson line in the fusion of two GW operators is the hallmark of a higher-group global symmetry structure [133], that can also be seen from the fact that Wilson lines of the $SO(2)$ theory (namely before gauging the $\mathbb{Z}_2$) are charged under the (0-form) charge conjugation symmetry as well as the electric 1-form symmetry. In more detail, the fourth equation in (2.4.13) can be understood as follows[28]. First, consider the fusion of two GW operators corresponding to different angles θ and φ and take the limit $\varphi \to \theta$. We obtain

$$T^{O(2)}_\theta \cdot T^{O(2)}_\theta = T^{O(2)}_{2\theta} + \text{“ } T^{O(2)}_0 \text{ ”} \,. \tag{2.4.14}$$

Naively, one would say that “ $T^{O(2)}_0$ ” is equal to two copies of the identity. However, to properly investigate this one should consider the fusion inside correlation functions. When all other operators in the correlator belong to the connected component (that is, they are operators just like those in the $SO(2)$ theory), indeed $T^{O(2)}_0$ looks like twice the identity. However, if one of the inserted operators belongs to the disconnected component there may be subtleties. Indeed, suppose including in our correlator the GW operator corresponding to the $\mathbb{Z}_2 \subseteq O(2)$, which we denote $T^{O(2)}_{\text{disc}}$. This corresponds to the insertion of an Alice string

[28]We thank Miguel Montero for explaining this argument to us.

defined by a gauge connection which picks a sign upon going around the string. Suppose now $\langle T^{O(2)}_{\text{disc}} T^{O(2)}_{\theta} \cdots \rangle$. Since $T^{O(2)}_{\theta}$ shifts the gauge connection by a constant, which is clearly incompatible with the action of $T^{O(2)}_{\text{disc}}$, it follows that $T^{O(2)}_{\text{disc}} T^{O(2)}_{\theta} = 0$ for any θ. Thus, inserting (2.4.14) in the correlator with $T^{O(2)}_{\text{disc}}$ leads to the requirement "$T^{O(2)}_0$"$T^{O(2)}_{\text{disc}} = 0$, which shows that "$T^{O(2)}_0$" cannot simply be two copies of the identity. In fact, the only operator we can construct that satisfies these conditions is

$$\text{“}\, T^{O(2)}_0 \,\text{”} = 1 + W^{O(2)}_{\text{sign}} \,, \tag{2.4.15}$$

leading to the fusion rule in (2.4.13) (a similar argument would hold for the fifth equation).

The cases of $O(N)$ where $N \geq 3$ are simpler, since the centralizer is always finite. If N is odd, then $SO(N)$ has trivial center. However, in this case $O(N)$ is the direct product $SO(N) \times \mathbb{Z}_2$ and the nontrivial element of the $\mathbb{Z}_2$ (which is $-\mathbb{1}$) will appear in the centralizer. Naturally, both $\pm\mathbb{1}$ are mapped to themselves by conjugation in $O(N)$, thus, we have a $\mathbb{Z}_2$ invertible 1-form symmetry. On the other hand, if N is even, $SO(N)$ has a center isomorphic to $\mathbb{Z}_2$, where the non-trivial element is precisely $-\mathbb{1}$. In this case the extension to the orthogonal group is a semidirect product $SO(N) \rtimes \mathbb{Z}_2$, and no new elements will appear in the centralizer. We conclude that also in this case we have an invertible $\mathbb{Z}_2$ 1-form symmetry. We summarize these results in Table 2.11.

Gauge group	Topological GW operators	Quantum dimension	1-form symmetry
$O(2)$	Id $T^{O(2)}_{\theta}$, $\theta \in (0, \pi)$ $T^{O(2)}_{\pi}$	1 2 1	(2.4.13)
$O(N)$, $N \geq 3$	Id $T^{O(N)}_{\pi}$	1 1	$\mathbb{Z}_2$

Table 2.11: Summary of topological Gukov-Witten operators for theories with $O(N)$ gauge group.

The dual $(d-2)$-form symmetry is obtained from the representations of the group of connected components (2.4.10). For all the $O(N)$ groups, $\pi_0(O(N)) = \mathbb{Z}_2$, which has two representations. These two representations of $\mathbb{Z}_2$ lift to the full $O(N)$ group giving rise to the trivial and sign representation. Therefore, the topological Wilson lines are precisely $W^{O(N)}_{\mathbf{1}}(\gamma)$ and $W^{O(N)}_{\text{sign}}(\gamma)$. Note that in this case the fusion of the Wilson lines (which in general is obtained from the decomposition of the tensor product of the two initial representations) reduces to a group operation,

$$\begin{aligned} W_{\mathbf{1}}^{O(N)} \cdot W_{\mathbf{1}}^{O(N)} &= W_{\mathbf{1}}^{O(N)}\,, & W_{\mathbf{1}}^{O(N)} \cdot W_{\text{sign}}^{O(N)} &= W_{\text{sign}}^{O(N)}\,, \\ W_{\text{sign}}^{O(N)} \cdot W_{\mathbf{1}}^{O(N)} &= W_{\text{sign}}^{O(N)}\,, & W_{\text{sign}}^{O(N)} \cdot W_{\text{sign}}^{O(N)} &= W_{\mathbf{1}}^{O(N)}\,, \end{aligned} \tag{2.4.16}$$

and so the $(d-2)$-form symmetry is an invertible $\mathbb{Z}_2$.

2.4.2 Electric 1-form symmetry

In this subsection, we look at the electric 1-form symmetry of disconnected groups built as $\mathbb{Z}_2$ extensions of $SU(N)$ or $U(N)$. We derive the topological Gukov-Witten operators from the generic arguments presented in section 2.4.1, namely from the computation of the centralizer of the identity component of these groups. An alternative derivation of which are the topological GW operators, with equal results, is presented in appendix 5.C.1.

We begin by recalling some basic definitions and propierties of $\widetilde{SU}(N)$ groups. These are the principal extensions of $SU(N)$ groups, i.e. semidirect products $SU(N) \rtimes_\Theta \mathbb{Z}_2$ where $\Theta : \mathbb{Z}_2 \to \mathrm{Aut}(SU(N))$ is a lift to the group of the automorphism of the Dynking diagram of $SU(N)$. If N is odd, there is only one such possible lift up to isomorphism,

$$\Theta^I(1)(g) = g\,, \quad \Theta^I(-1)(g) = (g^{-1})^T = \overline{g}\,, \tag{2.4.17}$$

where $g \in SU(N)$ and the bar denotes complex conjugation. If N is even, there are two distinct choices of Θ that give rise to two different groups [1]. One is given by (2.4.17) and the other by

$$\Theta^{II}(1)(g) = g\,, \quad \Theta^{II}(-1)(g) = -J_N(g^{-1})^T J_N = -J_N\overline{g}J_N\,, \tag{2.4.18}$$

where

$$J_{2k} := \begin{pmatrix} 0 & -\mathbb{1}_{k\times k} \\ \mathbb{1}_{k\times k} & 0 \end{pmatrix} \ . \tag{2.4.19}$$

We denote these two different groups as $\widetilde{SU}(N)_I$ and $\widetilde{SU}(N)_{II}$ respectively, and their elements are pairs (g,η) with $g \in SU(N)$ and $\eta \in \mathbb{Z}_2$. According to the definition of semidirect product, multiplication of elements is given by

$$(g_1,\eta_1)(g_2,\eta_2) = (g_1\Theta(\eta_1)(g_2), \eta_1\eta_2)\,. \tag{2.4.20}$$

It is possible to give a matrix construction of the groups explicitly exhibiting these properties [1].

Note that we can apply the same construction beginning with $g_i \in U(N)$, although in this case we cannot call them principal extensions. We will denote these groups as $\widetilde{U}(N)_I$ and $\widetilde{U}(N)_{II}$. In many cases, it can be useful to write the elements of these groups directly in their fundamental representation. This is a $2N$ dimensional representation where an element (g, η) is represented by

$$\mathrm{fund}((g,1)) = \begin{pmatrix} g & 0 \\ 0 & \Theta(-1)(g) \end{pmatrix}, \quad \mathrm{fund}((g,-1)) = \begin{pmatrix} 0 & g \\ \Theta(-1)(g) & 0 \end{pmatrix} \tag{2.4.21}$$

With these definitions, it is easy to compute both the center as well as the centralizer of the identity component of these groups. First, recall that the center of $U(N)$ and $SU(N)$ is $U(1)$ or $\mathbb{Z}_N$ respectively, with elements $e^{i\theta}\mathbb{1}$ or $e^{i2k\pi/N}\mathbb{1}$. Second, note that elements of the disconnected component don't commute with generic elements in the connected component. Therefore, to find the center, we need to find the elements $(h, 1)$ with $h \in Z(G)$ ($G = U(N)$ or $SU(N)$) that commute with $(g, -1)$ with $g \in G$. From the definition of the semidirect product, we find

$$(h,1)(g,-1) = (hg,-1)\,, \tag{2.4.22}$$

$$(g,-1)(h,1) = (g\Theta(-1)(h),-1)\,. \tag{2.4.23}$$

Since $h \in Z(G)$, this leads to the condition

$$h = \Theta(-1)(h) = \overline{h}\,, \tag{2.4.24}$$

which doesn't depend on whether Θ corresponds to (2.4.17) or (2.4.18). This condition is only satisfied for $h = \pm\mathbb{1}$; however, note that for the case of $SU(2k-1)$ only $+\mathbb{1}$ belongs to the group. All in all, the center of these groups is given by

$$Z(\widetilde{U}(N)) = \mathbb{Z}_2\,, \tag{2.4.25}$$

$$Z(\widetilde{SU}(2n)) = \mathbb{Z}_2\,, \tag{2.4.26}$$

$$Z(\widetilde{SU}(2n+1)) = \{\mathbb{1}\}\,. \tag{2.4.27}$$

The topological GW operators can be found from the centralizers of the identity components. The computation of these centralizers is very similar to that of the centers above. The only difference is that, since these elements don't need to commute with the disconnected component, we don't need to impose (2.4.24). Therefore,

$$C_{\widetilde{U}(N)}(U(N)) = \left\{\left(e^{i\theta}\mathbb{1}, 1\right)\ ,\ \theta \in [0, 2\pi]\right\}\ , \tag{2.4.28}$$
$$C_{\widetilde{SU}(2n)}(SU(2n)) = \left\{\left(e^{i\frac{k\pi}{n}}\mathbb{1}, 1\right)\ ,\ k = 0, 1, \ldots, 2n-1\right\}\ , \tag{2.4.29}$$
$$C_{\widetilde{SU}(2n+1)}(SU(2n+1)) = \left\{\left(e^{i\frac{2k\pi}{2n+1}}\mathbb{1}, 1\right)\ ,\ k = 0, 1, \ldots, 2n\right\}\ , \tag{2.4.30}$$

Gauge group	Topological GW operators	Quantum dimension	1-form symmetry
$\widetilde{U}(N)$	$T_0^{\widetilde{U}} = \mathrm{Id}$	1	(2.4.40)
	$T_\theta^{\widetilde{U}}\ ,\ \theta \in (0, \pi)$	2	
	$T_\pi^{\widetilde{U}}$	1	
$\widetilde{SU}(2n+1)$	$T_0^{\widetilde{SU}} = \mathrm{Id}$	1	(2.4.37)
	$T_k^{\widetilde{SU}}\ ,\ k = 1, \ldots, n$	2	
$\widetilde{SU}(2n)$	$T_0^{\widetilde{SU}} = \mathrm{Id}$	1	(2.4.37)
	$T_k^{\widetilde{SU}}\ ,\ k = 1, \ldots, n-1$	2	
	$T_n^{\widetilde{SU}} = T_\pi^{\widetilde{SU}}$	1	

Table 2.12: Summary of topological Gukov-Witten operators for theories with $\widetilde{U}(N)$ and $\widetilde{SU}(N)$ gauge group.

Finally, in order to find the topological GW operators, what we need to do is compute the conjugacy classes of the elements above in $\widetilde{U}(N)$ and $\widetilde{SU}(N)$. Working directly in the fundamental representation,

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & \Theta(-1)(g) \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} \Theta(-1)(g) & 0 \\ 0 & g \end{pmatrix}\ . \tag{2.4.31}$$

From the definition of $\Theta(-1)$ we see that conjugating with an element of the disconnected component leads to mapping the phase $\theta \mapsto -\theta$, or $k \mapsto -k$. All in all, for the different gauge groups, we have topological GW operators labelled by

$$\widetilde{U}(N):\quad T_g^{\widetilde{U}} = T_{g(\theta)}^{\widetilde{U}}\,,\quad g(\theta) = e^{i\theta}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\ ,\ \theta \in [0, \pi]\,, \tag{2.4.32}$$

$$\widetilde{SU}(2n):\quad T_g^{\widetilde{SU}} = T_{g(k)}^{\widetilde{SU}}\,,\quad g(k) = e^{i\frac{k\pi}{n}}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\ ,\ k = 0, 1, \ldots, n\,, \tag{2.4.33}$$

$$\widetilde{SU}(2n+1): \quad T_g^{\widetilde{SU}} = T_{g(k)}^{\widetilde{SU}}\,, \quad g(k) = e^{i\frac{2k\pi}{2n+1}} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \, k = 0, 1, \ldots, n\,. \tag{2.4.34}$$

Note that the conjugacy classes corresponding to $\theta \in (0, \pi)$ have two elements belonging to them, and likewise for $k = 1, \ldots, \lceil N/2 \rceil - 1$. Therefore the corresponding GW operators have quantum dimension two: this means that the symmetry is non-invertible, as there are no operators that we can fuse with e.g. $T_1^{\widetilde{SU}}$ and produce the identity operator, according to (2.4.8). This can also be checked by directly computing the fusion rule, which we proceed to illustrate in the example of $\widetilde{SU}(N)$. Note that the GW operator $T_k^{\widetilde{SU}}$ can be written as a sum of two GW operators of SU gauge theory,

$$T_k^{\widetilde{SU}}(M^{d-2}) = T_k^{SU}(M^{d-2}) + T_{-k}^{SU}(M^{d-2})\,. \tag{2.4.35}$$

If we fuse two operators with $k \neq k'$, we find

$$\begin{aligned} T_k^{\widetilde{SU}} \cdot T_{k'}^{\widetilde{SU}} &= \left(T_k^{SU} + T_{-k}^{SU}\right) \cdot \left(T_{k'}^{SU} + T_{-k'}^{SU}\right) \\ &= T_k^{SU} \cdot T_{k'}^{SU} + T_k^{SU} \cdot T_{-k'}^{SU} + T_{-k}^{SU} \cdot T_{k'}^{SU} + T_{-k}^{SU} \cdot T_{-k'}^{SU}\,. \end{aligned} \tag{2.4.36}$$

The 1-form symmetry of $SU(N)$ gauge theory is invertible, i.e. the fusion of the T_k^{SU} GW operators obeys (2.4.2). Thus,

$$\begin{aligned} T_k^{\widetilde{SU}} \cdot T_{k'}^{\widetilde{SU}} &= T_{k+k'}^{SU} + T_{k-k'}^{SU} + T_{k'-k}^{SU} + T_{-(k+k')}^{SU} \\ &= T_{k+k'}^{\widetilde{SU}} + T_{|k-k'|}^{\widetilde{SU}}\,, \end{aligned} \tag{2.4.37}$$

The fusion rule in the case when $k = k'$ is more subtle and we cannot compute it directly. However, since the behaviour of the 1-form symmetry of $\widetilde{SU}$ and $\widetilde{U}$ groups seems in many aspects a natural generalization of the $O(2)$ case described above, we expect that the same argument to that between equations (2.4.14)–(2.4.15) should also apply; i.e. whenever the naive substitution $k' \to k$ gives rise to $T_0^{\widetilde{SU}}$ we should identify it with the condensation defect $1 + W_{\text{sign}}^{\widetilde{SU}}$ (see also [128] for a related discussion in the language of categories). That is, the fusion rule should be

$$T_k^{\widetilde{SU}} \cdot T_k^{\widetilde{SU}} = 1 + W_{\text{sign}}^{\widetilde{SU}} + T_{2k}^{\widetilde{SU}}\,. \tag{2.4.38}$$

If N is even, the case $k' = \frac{N}{2} - k$ also has to be studied independently, and again similarly to the $O(2)$ case,

$$T_k^{\widetilde{SU}} \cdot T_{\frac{N}{2}-k}^{\widetilde{SU}} = T_{\frac{N}{2}}^{\widetilde{SU}} + W_{\text{sign}}^{\widetilde{SU}} T_{\frac{N}{2}}^{\widetilde{SU}} + T_{2k-\frac{N}{2}}^{\widetilde{SU}} \,. \tag{2.4.39}$$

For the gauge group $\widetilde{U}$, the computation is completely analogous, and the only difference is that instead of the discrete parameters k, k' we have continious parameters θ, θ',

$$T_\theta^{\widetilde{U}} \cdot T_{\theta'}^{\widetilde{U}} = T_{\theta+\theta'}^{\widetilde{U}} + T_{|\theta-\theta'|}^{\widetilde{U}} \,. \tag{2.4.40}$$

We can gain further intuition on the appearance of non-invertible symmetries if we consider the theory before gauging the outer automorphism (i.e. just $U(N)$ or $SU(N)$ gauge theory), when the topological Gukov-Witten operators have quantum dimension one. However, the (now global) zero-form symmetry is not independent of the center one-form symmetry; a feature that can be seen from the fact that the fundamental Wilson line is acted upon by charge conjugation $\mathcal{C}$ as

$$\mathcal{C}\,:\, W_{\mathbf{fund}} \mapsto W_{\overline{\mathbf{fund}}} \,, \tag{2.4.41}$$

This implies that the total global symmetry of the theory is not a direct product of both, but rather a higher categorical object known as a *2-group symmetry* (see *e.g.* [133–138]).

This observation provides further support to the identification of the symmetry operators for the $\widetilde{SU}(N)/\widetilde{U}(N)$ theories above (along lines similar to [139]). The GW operators of the $U(N)/SU(N)$ theory are acted in a similar way to (2.4.41) by $\mathcal{C}$. It is then clear that the $\mathcal{C}$-invariant combinations are precisely (2.4.35) (and the converse for $U(N)$), which are then the leftover GW after gauging $\mathcal{C}$. Note that the appearance of non-invertible symmetries is due to the “folded structure" in (2.4.35), which from this point of view is inherited from the fact that the 1-form generators of the $SU(N)/U(N)$ theory are acted by the 0-form symmetry $\mathcal{C}$.

2.4.3 Dual $(d-2)$-form symmetry

In this subsection, the aim is twofold: on the one hand, we study the $(d-2)$-form symmetry generated by topological Wilson lines; and on the other hand, we look for brane constructions of $\widetilde{SU}(N)$ theories. It seems that the key to achieving the second is to include defects so that the $(d-2)$-form symmetry is broken.

Wilson lines and Alice strings

In the previous pages we looked for topological GW operators that generate the electric 1-form symmetry. The charged objects were Wilson lines, and the symmetry can be broken if we include particles that make the Wilson lines endable. In this section, we look for topological Wilson lines: the charged objects are precisely the Gukov-Witten operators. As we have discussed above, the Wilson line along a contractible path always belongs to the connected component of the group, and this makes it so that the topological ones, which generate the $(d-2)$-form symmetry, are given by the representations of the gauge group that map the whole identity component G^0 to 1 (2.4.10).

In the case of $\widetilde{SU}(N)$ and $\widetilde{U}(N)$, the group of connected components is $\mathbb{Z}_2$. This group has two representations, the trivial and the fundamental. At the level of the group, they correspond to the trivial representation and the sign representation respectively, where the sign representation is defined as

$$\mathrm{sign}((g,1)) = 1\,, \quad \mathrm{sign}((g,-1)) = -1\,, \tag{2.4.42}$$

for g in $\widetilde{SU}(N)$ or $\widetilde{U}(N)$. Therefore, the topological Wilson lines are $W^{\widetilde{G}}_{\mathbf{1}}$ and $W^{\widetilde{G}}_{\text{sign}}$. Similarly to the case of the $O(N)$ groups (2.4.16), the topological Wilson lines fuse according to the $\mathbb{Z}_2$ product, and therefore we have a $\mathbb{Z}_2$ invertible $(d-2)$-form symmetry.

In a similar fashion to the 1-form symmetry, this $(d-2)$-form symmetry can be broken by including defects on which the charged GW operators can end. These were dubbed *twist vortices* in [118], and in the context of the charge conjugation symmetry have been usually called *Alice strings* [44]. These defects always have a transverse $\mathbb{R}^2$ at each point, and are defined by the fact that when going around them, operators undergo a monodromy corresponding to the outer automorphism of the gauge group.

An important feature of Alice strings is that their presence reduces the globally well defined gauge group [131]. This is because the outer automorphism doesn't commute with all gauge transformations, and states that should be gauge equivalent can pick up different Aharonov-Bohm phases from the action of the monodromy. This is a contradiction: what happens is that, even if in a region that doesn't include the string the gauge group is apparently G, the presence of the string reduces it to the centraliser of the outer automorphism of G, i.e. precisely the subgroup that does commute with the monodromy.

Let's be more explicit in the case at hand of $\widetilde{SU}(N)_{I,II}$. When going around the Alice string, fields are acted upon by the element $(1,-1)$ which corresponds to the outer automorphism of $SU(N)$. Note that this action does depend on the choice of semidirect product Θ (2.4.17) or (2.4.18). The globally well defined gauge group is the centraliser $C_{\widetilde{SU}(N)_{I,II}}((1,-1))$, which can be easily found by computing

$$(g,\eta)(1,-1) = (g\Theta(\eta)(1),-\eta) = (g,-\eta)\,, \tag{2.4.43}$$

$$(1,-1)(g,\eta) = (\Theta(-1)(g),-\eta)\,. \tag{2.4.44}$$

We see that g needs to satisfy

$$g = \Theta(-1)(g)\,. \tag{2.4.45}$$

If $\Theta = \Theta^I$ (2.4.17), this implies that $g \in SO(N)$. On the other hand, if $\Theta = \Theta^{II}$ (2.4.18), which can only happen if N is even, we have $g \in Sp(N/2)$. Note also that the element $(1,-1)$ will always belong in the centraliser, which therefore will take the form of a direct product.

In summary, if we add an Alice string to break the $(d-2)$-form symmetry of $\widetilde{SU}(N)$ theory, the gauge group becomes $SO(N) \times \mathbb{Z}_2$ or $Sp(N/2) \times \mathbb{Z}_2$. It is interesting to look back to the electric 1-form symmetry of these theories. From Table 2.10, we see that the topological Gukov-Witten operators, after the reduction of the well defined gauge group, correspond to $+\mathbb{1}$ or $-\mathbb{1}$ (the latter only when it belongs to the group). Comparing with the results of Table 2.12, these are precisely the GW operators with quantum dimension 1 in $\widetilde{SU}(N)$ theory. Therefore, it seems that breaking the $(d-2)$-form symmetry ultimately results in the disappearance of the non-invertible 1-from symmetries. We believe that this is a phenomenon deserving of further exploration.

Brane constructions

Brane setups engineering Alice strings and $\widetilde{U}(N)$ gauge groups can be achieved by intersecting branes with orientifolds. Following [130], consider

	0	1	2	3	4	5	6	7	8	9
N D3	×	×	×	×						
O3	×	×			×	×				
r D3'	×	×			×	×				

(2.4.46)

The argument in [130] suggests that the theory on the D3-branes is $\widetilde{U}(N)$. However, from the point of view of the D3-branes, the O3 $+r$ D3' act as a codimension 2 defect with a monodromy associated to the element $(1,-1)\in\widetilde{U}(N)$, that is, an Alice string. As a result, the globally well defined gauge group is $O(N)$ (in the $\widetilde{U}(N)_I$ case) or $Sp(N/2)$ (in the $\widetilde{U}(N)_{II}$ case), and the full $\widetilde{U}(N)_{I,II}$ is only manifest on top of the defect. The type of semidirect product extension Θ^I or Θ^{II} depends on the choice of orientifold plane. For O3$^+$, $\widetilde{\mathrm{O3}}^+$ we find $\widetilde{U}(N)_I$, leading to a globally well-defined $O(N)$ in the presence of the twist vortex, and for an O3$^-$ we find $\widetilde{U}(N)_{II}$ leading to a globally well-defined $Sp(N/2)$ in the presence of the twist vortex.[29]

This arrangement of branes can be straightforwardly generalised to N Dp-branes along the $x^{0,\ldots,p}$ directions and Op $+k$ Dp' along the $x^{0,\ldots,p-2,p+1,p+2}$. In this way, we can engineer $\widetilde{U}(N)$ theories in a different number of dimensions.

Besides (2.4.46), there are two more setups where we can engineer a $\widetilde{U}(N)$ theory in a similar fashion. These are the type IIB configuration

	0	1	2	3	4	5	6	7	8	9
N D3	$\times$	$\times$	$\times$	$\times$						
k D7'	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
O7	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

(2.4.47)

The IIB configuration with an $O7^-$ and 4 extra flavor $D7$ branes was studied in detail in [45, 140], where it was argued that the 2d intersection acts as an Alice string for the $\widetilde{U}(N)_I$ theory on the D3, out of which only a $O(N)$ is globally well-defined.

In addition, we also have the T-dual in type IIA,

	0	1	2	3	4	5	6	7	8	9
N D2	$\times$		$\times$	$\times$						
k D6'	$\times$				$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
O6	$\times$				$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

(2.4.48)

In both these cases, the theory on the D3 (or D2) is a $\widetilde{U}(N)$ gauge theory, and the orientifold intersection appears as an Alice string which reduces the well defined gauge group to an orthogonal or symplectic subgroup. However, as opposed to (2.4.46), the identification

[29] It would be interesting to clarify the distiction bewteen O3$^-$ and $\widetilde{\mathrm{O3}}^-$.

of the type of orientifold with the semidirect product extension is reversed: an O7$^+$ will give rise to $\widetilde{U}(N)_{II}$ on the D3, and an O7$^-$ to $\widetilde{U}(N)_I$.

An important observation is that all the brane constructions that we have discussed have one thing in common: when seeking to engineer a disconnected gauge group, the Alice string appears automatically, and it seems impossible to the best of our knowledge to find the former without the latter. As a consequence, as we have discussed in the previous section, the presence of the Alice string also serves the purpose of breaking the $(d-2)$-form global symmetry of these theories. This strongly resonates with the conjectured absence of global symmetries in quantum gravity.

2.4.4 Conclusions and outlook

In this note we have studied the electric 1-form and $(d-2)$-form symmetries of gauge theories based on the gauge groups which include charge conjugation as part of the gauge symmetry introduced in [1, 85]. As for the electric 1-form symmetry, concentrating on pure gauge theories, we have found that these QFT's provide very simple and explicit examples of non-invertible symmetries using the technology developed in [118], supporting the claim in [122] that indeed non-invertible symmetries are ubiquitous also in dimensions higher than three. In this case, the emergence of non-invertible symmetries can be heuristically understood considering a copy of the same theory but with gauge group $SU(N)$ (an analogous discussion holds for the $U(N)$ case). In that case, the symmetry operators associated to the electric 1-form symmetry are permuted by the 0-form charge conjugation symmetry (forming actually a 2-group). The operators which carry over to the version of the theory with gauged charge conjugation are the combinations which are $\mathcal{C}$-invariant, and this "folds" the GW operators as in (2.4.35) leading to non-invertible symmetries in very much the same way as in the $O(2)$ case discussed in [118]. The non-invertible character of the symmetries can be read-off from the fact that their quantum dimension is 2 (instead of 1). This manifests itself also in the fusion rules, which mimic the $O(2)$ case. Even though we have provided arguments in support of the fusion rules in section 2.4.2, it would be very interesting to further study this aspect to put them on firmer grounds.

More generally, the existence of non-invertible symmetries has been shown to be closely related to mixed anomalies (see e.g. [121, 141]). The fact that, when considering a theory which includes charge conjugation as part of the gauge group, one immediately finds a non-invertible 1-form symmetry, may signal that such a mixed anomaly between the gauge group and its outer automorphism should be present (see [142] for a related discussion). It would be very interesting to investigate this point further, complementing the studies of [132].

Since the gauge groups we are considering are disconnected ($\pi_0(G) = \mathbb{Z}_2$), QFT's based on them automatically exhibit a $(d-2)$-form symmetry [118]. The objects charged under this symmetry are twist vortices (Alice strings in the $d = 4$ case). As it is well-known, in the presence of twist vortices only a subgroup of the full gauge group is well defined. In the case at hand it is $SO(N)$ for $\widetilde{SU}(N)_I$ and $Sp(N/2)$ for $\widetilde{SU}(N)_{II}$ –recall that this latter version is only available for even N. Elaborating on [130] and [45, 140] we have suggested String Theory embeddings for these theories. Amusingly, they automatically come with twist vortices, thus breaking the $(d-2)$-symmetry. This is very much consistent with the Swampland criteria that any global symmetry should be broken. These String Theory constructions involve intersecting orientifolds. Roughly speaking, the type of orientifold ($Op^{\pm}$ and their tilde versions) matches the possible theories. However, it would be very interesting to study these constructions in more detail (in particular including the relation between the two proposed constructions). Moreover, the String Theory construction may be used to study the duality properties of these theories. As a consequence, this would allow to study magnetic $(d-3)$-form symmetries as well as possible 't Hooft anomalies. Note that some of these aspects may depend on the dimension d. We leave these very interesting aspects for future studies.

2.5 Concluding remarks

In this chapter, we have discussed a number of novel features of Quantum Field Theories with gauge groups that are disconnected. These represent a region of the landscape of QFTs that has remained so far largely unexplored, but that as we have seen present many rich traits even at a kinematical level.

Let us briefly recap the contents of this chapter. The first item of business, discussed in the first part of section 2.2, was to identify the possible principal extensions of SU(N), of which we saw rigorously that there are two, by establishing a correspondence between said extensions and the known classification of symmetric spaces. The reason to restrict to these kind of disconnected group is that that they are better studied in the mathematical literature, and they have properties that make them more amenable to e.g. Hilbert series computations via Molien integration. Intuitively, this is because they come with a map, specifying the semidirect product, that morally speaking allows us to jump from one connected component to another. Of course, it is legitimate to build a gauge theory based on a more generic disconnected group that is not a principal extension, but we can't say anything about those.

In the context of 4d $\mathcal{N}=2$ theories, the first physical consequence of the disconnected gauge group is that it modifies the global symmetry of the theory. This is because the reality properties of the various representations (in particular the fundamental) are modified: instead of complex they are real or pseudo-real. At the lagrangian level, we saw that we can build a superpotential with half-hypermultiplets of the same chirality, which results in an additional exchange symmetry requirement for the matrix of couplings, therefore making the global symmetry either $\mathrm{Sp}(N_f)$ or $\mathrm{SO}(N_f)$. We verified this modification of the global symmetry with Hilbert series calculations.

Regarding their moduli spaces, one of the main observations is that the Coulomb branch is not freely generated; i.e. there are relations in its coordinate ring. This is relevant, for example, in the program of bottom-up classification of 4d $\mathcal{N}=2$ SCFTs, since very often one restricts the possible Coulomb branches to freely generated ones as a simplification (see e.g. [143–145]). As for the Higgs branch, apart from having a different isometry than in the connected case (an almost tautological consequence of the different global symmetry), its most interesting property is that the possible Higgs branch flows, encoded in the Hasse diagram, can be considerably more intrincate than for $\mathrm{SU}(N)$. This is because upon higgsing, there will appear matter fields in the pseudo-scalar representation, even if the initial theory has none; this pseudo-scalars are then available to higgs the discrete factor of the gauge group. We also checked, using Hilbert series, that we can find magnetic quivers for these Higgs branches using wreathed quivers, a recently discovered operation in the context of SUSY moduli spaces that precisely implements the desired quotient.

Another extremely interesting feature of theories with disconnected gauge groups are their generalized symmetries. Wilson lines of the usual SU gauge theory are charged both under the center 1-form symmetry as well as the 0-form symmetry of charge conjugation, which together form a (split) 2-group. Upon gauging the later, it turns out that the electric 1-form symmetry becomes non-invertible. Remarkably, this fact doesn't depend on the dimensionality of the theory; this is a very unusual fact, as the dimension of spacetime often plays a very important role in the discussion of generalized global symmetries. This non-invertible 1-form symmetry exists for supersymmetric theories with only vector multiplets (the presence of hypermultiplets explicitly breaks it), but also for non-supersymmetric theories as long as the matter fields live only in the adjoint or products thereof.

When trying to build brane systems that engineer theories with disconnected gauge groups, we found that intersections of orientifold planes with D-branes tend to make an appearance. In the field theory these intersections are Alice strings, which have the consequence of breaking the globally well defined gauge group. Sadly, what this means is that, even if naively the brane system would realise the disconnected gauge group with a non-invertible symmetry, in fact only a direct product gauge group with an invertible symmetry remains globally well defined.

There are many questions that remain unanswered regarding disconnected gauge groups. Possibly the most interesting have to do with their construction from higher dimensions, as this would shed light on many other problems. For example, their realization in class $\mathcal{S}$ can be a very important step towards extending the classification of 4d $\mathcal{N} = 2$ SCFTs beyond Coulomb branches that are freely generated. In this direction, a detailed study of the Seiberg-Witten curve of the superconformal theory with gauge group $\widetilde{\mathrm{SU}}(3)$ looks very promising: it is a particular case whose Coulomb branch is freely generated, and it has low enough rank that it should belong to the already classified SCFTs. Moreover, the conformal dimensions of the Coulomb branch generators, a key data point in this regard, has been computed in this work via the Coulomb branch Hilbert series. One may have the hope that a generalization to higher ranks then becomes visible. Another desirable possible outcome of this study would be to understand the realization of the non-invertible 1-form symmetry from the 6d point of view.

In a similar spirit, it would be interesting to be able to construct the $\widetilde{\mathrm{SU}}$ theories in String Theory without Alice strings making an appearance, in order to study said 1-form symmetry from a holographic point of view. In fact, given that the local dynamics of a $\widetilde{\mathrm{SU}}$ theory are the same as that of the connected gauge group SU, it should be possible to realize the holographic dual directly as $AdS_5 \times S^5$, and the boundary conditions of some discrete $\mathbb{Z}_2$ field should encode the change in the global form. The puzzle remains of what should this discrete field correspond to in type IIB, as a first analysis indicates it's not either $(-1)^{F_L}$ nor Ω; and instead should be related to the -1 in the center of $\mathrm{SL}(2, \mathbb{Z})$. In modern language, all this information is encoded on the symmetry topological field theory (SymTFT). This is an auxiliary 5d theory to our theory of interest in 4 dimensions, and from the string theory realization it should be obtained by integrating the topological terms in type IIB on S^5.

The SymTFT has many applications of its own. One of the most compelling is that it also contains information about anomalies. In the case at hand, these would be global 't Hooft anomalies mixing the outer automorphism $\mathbb{Z}_2$ with various other fields (one would expect the background fields for the 1-form symmetry, at least). One important subtlety in this story is the fact that there are two different extensions giving rise to two distinct gauge groups $\widetilde{\mathrm{SU}}(N)_{I,II}$, in the case that N is even. It is a mystery if and how this distinction is realized at the level of the SymTFT, whether it is as different choices of boundary conditions, different choice of discrete $\mathbb{Z}_2$ field (if this were the case, then one may ask how would it arise in the reduction from type IIB), or some sort of discrete theta angle.

Before wrapping up this chapter, let us conclude with some philosophical musings. In this work, what has been done is to consider one family of particularly manageable disconnected gauge groups, adding a $\mathbb{Z}_2$ (the smallest group) to $\mathrm{SU}(N)$ in –almost– the simplest possible way (a semidirect product). Yet, we have seen that this innocent gauging of a $\mathbb{Z}_2$ has very wild consequences, ranging from the change of the generalized global symmetries to the basic geometric properties of their moduli spaces. It is a sobering thought to realize that there are many more disconnected groups than there are connected ones, and that most of those have more complicated structures than the principal extensions studied here. This entails a vast enlargement of the landscape of possible QFTs, that remain virtually unexplored, and of which this work is just a first look onto the most controlled corner.

Chapter 3

Non-supersymmetric theories at large charge

When one drops the assumption of supersymmetry, many of the computational tools that allow for control over non-perturbative phenomena, rooted on holomorphy and the non-renormalization theorems, loose their capacity to accurately describe the behaviour of Quantum Field Theories. At the classical level, there might still exist e.g. moduli spaces, and in principle one can still study their geometry with the same tools; but none of those aspects are likely to survive quantum corrections, as they are not protected.

In these situations, when exact results are beyond reach, the best one can hope for in most situations is to find a reliable approximation such that computations are feasible and the error is under control. Textbook perturbation theory at weak coupling is one such approximation. However, rather tautologically, this most well known approach is not one that can provide information about the behaviour of QFTs at strong coupling. An alternative method, and the one we shall focus on in this chapter, is the large quantum number expansion, where the inverse of said quantum number is the small parameter that one can use to expand.

The main idea is, once we are given a QFT, to look at a subsector of the Hilbert space that has a fixed and large charge under some global symmetry. By virtue of being a global symmetry, we have that the symmetry generator Q commutes with the hamiltonian of the system,

$$[H, Q] = 0\,. \tag{3.0.1}$$

On the other hand, since we are looking at a sector of fixed charge, we can find the lowest energy state inside that sector and denote it by $|0_Q\rangle$ (note that it is not the vacuum of the complete theory) which will satisfy

$$Q|0_Q\rangle \neq 0\,. \tag{3.0.2}$$

Equations (3.0.1) and (3.0.2) are the definition of spontaneous symmetry breaking, even if it is caused not by some scalar VEV but by restricting to a subsector of fixed charge. One then may use the standard techniques to write down an Effective Field Theory (EFT) for a spontaneous symmetry broken phase, which can be used to describe the subsector of our initial theory with fixed charge. Moreover, among the in principle infinite terms of the EFT, only a handful of them (very few in fact) survive the limit when the fixed charge is large. In this way, one ends with a EFT for the large charge sector that includes only a finite amount of terms. This EFT can then be used to compute correlators, anomalous dimensions, and various other observables.

The goal of this chapter is to understand the simplifications brought about by the large charge limit directly in terms of the UV theory and fields. This is less generic than the EFT approach, and in particular we will take not only the large charge limit, but a double scaling limit involving the coupling of the theory, g, such that

$$Q \to \infty\,, \qquad g \to 0\,, \qquad gQ^2 = \text{ fixed}\,, \tag{3.0.3}$$

in order to have control when flowing to the IR. The advantages are that the way the large charge limit enters in the computation becomes very intuitive, both in the path integral as well as directly in terms of Feynman diagrams.

The rest of this chapter is organized as follows. In section 3.1 we give a brief review of the large charge EFT; in a way, this serves as a motivation that we should look for simplifications at large charge already in the UV. In section 3.2 we discuss the existence of the aforementioned double scaling limit in the simple example of the O(2) model, showing in detail how the different Feynman diagrams are suppressed, and one can resum the complete series of remaining ones. In section 3.3 we utilize the limit to compute more general correlation functions in the theory. Another application is shown in section 3.4, where we study the relation between various fixed points of scalar theories in 6d. We finish in 3.5 with some concluding remarks.

3.1 Lightning review of the large charge expansion

In this section, we briefly review the generic arguments leading to the existence of a large quantum number expansion in QFT (see e.g. [146] for a more thorough review). As mentioned above, the key idea is that restricting to a sector of fixed charge is the same as looking at a theory with a spontaneously broken symmetry. However, Goldstone's theorem doesn't apply in the same way as in *bona fide* SSB of relativistic field theories. Reviewing this is the goal of subsection 3.1.1. In 3.1.2 we discuss how to then write down the corresponding low energy theory and how the large charge approximation leaves us with only finitely many terms.

3.1.1 Spontaneous symmetry breaking by fixed charge

In a relativistic QFT with the usual Spontaneous Symmetry Breaking caused by a scalar gaining a VEV, Goldstone's theorem tells us that the low energy theory describing the IR dynamics of the system consists of a massless Goldstone boson which transforms non-linearly under the broken symmetry. The counting of Goldstones is straightforward: there will be one for each broken generator of the symmetry. If the VEVs break the global symmetry G to a subgroup H, the EFT is a NLSM with target space G/H. In the simplest example of a complex scalar with a U(1) global symmetry, one splits the complex field into its modulus and phase: the VEV implies the modulus is fixed to the given value, and the angle becomes the massless Goldstone boson, which indeed has values in U(1) as expected.

If the field theory is not relativistic, or if there is spontaneous breaking of spacetime symmetries, things are not so simple. In general there are two types of Goldstone bosons, those which are massless and with relativistic dispersion relation (type I) and those with masses and cuadratic dispersion relation (type II). Their counting also doesn't immediately follow from the number of broken generators, as it can happen that the same field serves as the Goldstone for two different broken symmetries.

Our interest lies in CFTs, as they describe the fixed points of generic RG flows. When looking at a subsector with fixed charge, we are artificially introducing one scale into the problem, namely the charge density $\rho = Q/V$; therefore, conformal symmetry is also broken. We see that we are in the second situation regarding Goldstone's theorem, were we also have spontaneous breaking of a spacetime symmetry. However, since the underlying theory we want to describe is still a CFT, we'll want to restore scale invariance into our EFT of massless Goldstone's. This can be achieved by adding a dilaton with a particular non-linear transformation under scalings.

As an example, consider the effective theory for one potential Goldstone boson χ for a U(1) broken symmetry in 4 dimensions,

$$\mathcal{L}[\chi] = \frac{f_\pi^2}{2}\partial_\mu\chi\partial^\mu\chi - C^4\,. \tag{3.1.1}$$

The constants f_π and C are dimensionfull and break scale invariance. Then, we add a second field σ such that under a scale transformation $x \to e^\alpha x$, σ transforms as

$$\sigma \to \sigma - \frac{\alpha}{f}\,, \tag{3.1.2}$$

with f a new constant of dimension $[f] = -1$. Now, for any operator $\mathcal{O}_\Delta$ of dimension Δ, we can dress it as

$$\mathcal{O}_\Delta \to e^{(\Delta-2)f\sigma}\mathcal{O}_\Delta\,, \tag{3.1.3}$$

which is now scale invariant. Applying this recipe to (3.1.1), and adding a kinetic term for the dilaton σ leads to

$$\mathcal{L}[\chi,\sigma] = \frac{1}{2}g^{\mu\nu}f_\pi^2 e^{-2f\sigma}\partial_\mu\chi\partial_\nu\chi - C^4e^{-4f\sigma} + \frac{1}{2}g^{\mu\nu}\left(g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \xi\frac{R}{f^2}\right)\,. \tag{3.1.4}$$

Even though the dimensionful constants still appear, they conspire to produce a conformal invariant action. This can be made manifest by the following transformation of the fields,

$$(\sigma,\chi) \to \Sigma = \sigma + if_\pi\chi \to \phi = \frac{1}{\sqrt{2}f}e^{-f\Sigma}\,, \tag{3.1.5}$$

upon which the lagrangian becomes

$$\mathcal{L}[\phi] = \partial_\mu\phi^*\partial^\mu\phi - \xi R\phi^*\phi - 4g(\phi^*\phi)^2\,, \tag{3.1.6}$$

where

$$g = (Cf)^4\,, \tag{3.1.7}$$

is a dimensionless constant. The conclusion is that the low energy EFT resulting from a broken U(1) symmetry, plus the restoration of conformal symmetry, will look like the theory of a complex scalar (3.1.6), possibly including higher derivative corrections.

3.1.2 Effective Field Theory at large charge

We are now in a position to construct the EFT that will describe the large charge sector of our theory. For concreteness, we can focus on the O(2) model, which as we have just seen, can be described in the IR with a complex scalar field. Note that the significance of the analysis of the previous section is that this conclusion remains true even after the RG flow, i.e. the complex scalar in the IR ϕ doesn't need to be the same, or even have any simple relation, to the complex scalar in the UV denoted as φ.

With this in mind, we can rewrite (3.1.6) with yet another useful parametrization of the complex scalar ϕ in terms of a modulus a and an angle χ,

$$\phi = \frac{a}{\sqrt{2}} e^{ib\chi} , \tag{3.1.8}$$

where b is a constant. The result is

$$\mathcal{L}[a, \chi] = \frac{1}{2}\partial_\mu a \partial^\mu a + \frac{1}{2} b^2 a^2 \partial_\mu \chi \partial^\mu \chi - \frac{\xi R}{2} a^2 - \frac{1}{4} g a^4 + \dots , \tag{3.1.9}$$

where the dots stand for the higher derivative corrections to the EFT. It is now time to exploit the fact that the fixed charge is large, as so far we have only used it to argue that there is SSB of the global and scaling symmetry. The next steps are

1. Find the ground state in the fixed charge sector.
2. Study the perturbations around said ground state.
3. Integrate out the massive modes of said perturbations, keeping only the terms that scale with positive powers of Q, i.e. neglect $O(1/Q)$.

The first step can be easily achieved by solving the Euler-Lagrange equations. The equation of motion for χ is the equation of charge conservation, so it is as this point that the value of the charge Q, or more precisely the charge density $\rho = Q/V$, enters the game. Schematically, the solution looks like

$$\chi = \mu t , \qquad a = \nu , \tag{3.1.10}$$

where μ and ν are constants that depend on ρ. Importantly, if $\rho \neq 0$ then also $\nu \neq 0$.

The second step is to study the fluctuations around this ground state. This can be achieved by parametrising the complex scalar field in (3.1.6) as

$$\phi(t,x) = e^{ib\mu t}\left(\frac{\nu}{\sqrt{2}} + \pi(t,x)\right), \tag{3.1.11}$$

and studying the resulting lagrangian in terms of the fluctuation π. There are two different modes, corresponding to the real and imaginary parts of π, or in a different basis to its radial and angular parts. Their dispersion relation can be studied by looking at their propagator; it turns out that one of the modes, the angular one, is massless, with $\omega(k) = k/\sqrt{3}$ (where 3 arises as the number of space directions) and the other is massive, with $m \propto \rho^{1/3} = \Lambda_Q$.

If the charge is large, and we are at energies lower than the scale set by the charge density $\Lambda \ll \Lambda_Q$, we can proceed with the third step, namely integrating out the massive radial component a in (3.1.9) and ending up with a EFT for χ only. Since a^2 appears in the kinetic term for χ, this computation is very complicated; and when proceeding in Wilsonian fashion –writing all terms of correct dimension compatible with the symmetries– it has the consequence that the combination $(\partial_\mu\chi\partial^\mu\chi)$ can also appear with negative powers. Moreover, also the curvature R and other geometric invariants can make an appearance, which further complicates the possible structure of the possible terms.

At this point, one would end with a EFT with infinitely many terms, depending on infinitely many Wilson coefficients. However, we can exploit the large charge limit one last time. We notice that, through the dependence of the constant μ on ρ, we have that $\partial_0\chi \propto \Lambda_Q \propto Q^{1/3}$. This can be used to keep only the terms that are not suppressed in the large charge limit, of which there are only finitely many. In 4 dimensions, the effective action schematically looks like

$$\mathcal{L}[\chi] = k_0\,(\partial_\mu\chi\partial^\mu\chi)^2 + k_1\,R\,\partial_\mu\chi\partial^\mu\chi + \text{geometry}\,, \tag{3.1.12}$$

where we have omitted terms having to do with geometric invariants such as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$; these terms become important when exploiting conformal symmetry by considering the theory on different manifolds, such as the sphere, the cylinder, etc.

Once we have this result, one can now begin to compute observables of the low energy theory in the large charge sector, such as correlators, anomalous dimensions, etc. Indeed, also corrections of order $1/Q$ and higher can be computed, by keeping the corresponding terms in the EFT description. Despite the fact that we have only discussed a particularly simple example, the conclusion that observables in CFT can be computed in the large charge regime from as an aproximation in terms of $1/Q$ remains true for more complicated systems.

An alternative viewpoint to the fixed charge EFT is to consider the $O(2)$ model at the Wilson-Fisher fixed point and exploit conformal symmetry to map the theory in flat space to the cylinder [147]. In this way, the state-operator correspondence of CFT provides a powerful tool to compute the scaling dimensions of operators with fixed charge (in flat space) from the energies of states with fixed energy (in the cylinder). It turns out that in so doing, the natural parameter making an appearance is the 't Hooft-like coupling $\bar{\lambda} = g_* Q$, where g_* is the value of the coupling g at the fixed point. The large charge limit can then be taken while keeping $\bar{\lambda}$ fixed. In the case $\bar{\lambda} \gg 1$ one recovers the same results as in the large charge EFT, as expected. In what follows, we shall study the opposite regime of $\bar{\lambda} \ll 1$.

3.2 The large charge limit of scalar field theories and the Wilson-Fisher fixed point at $\epsilon = 0$

Despite decades of huge research efforts, Quantum Field Theory (QFT) is far from analytic reach beyond perturbative approaches which, in practice, typically comprise the computation of certain observables to a few loop accuracy. It thus comes as a very welcome surprise that, in certain cases, it is possible to identify special limits which lead to drastic simplifications and sometimes to a reorganization of perturbation theory. A prototypical example is the large N limit discovered by 't Hooft. In a gauge theory, one takes the rank N of the gauge algebra to infinity at the same time that the Yang-Mills coupling $g_{\rm YM}$ is sent to zero in such a way that the 't Hooft coupling $g_{\rm YM}^2 N$ is fixed. This limit selects planar diagrams in the perturbative expansion of the theory, which naturally organize themselves into a genus expansion very reminiscent of a string theory, a connection that has been intensively studied over the last two decades.

A different approach is to explore asymptotic regimes in the space of operators in a certain QFT, in particular focusing on those with large charge n under a global symmetry of the theory. This remarkable suggestion was made in [148] and it was further explored in many relevant papers including [149–162]. A new perturbation expansion emerges in terms of a small effective coupling represented by the inverse of the charge, $1/n$ (see also [163, 164] for other interesting physical applications).

Recently, a new "double-scaling" large charge limit was introduced in [165] in the context of $\mathcal{N} = 2$ 4d superconformal field theories. In this case, supersymmetric localization provides an efficient method to compute "extremal" correlators of chiral primary operators $({\rm Tr}\phi^2)^n$ [64] (being ϕ the scalar field in the vector multiplet). In the $\mathcal{N} = 2$ SCFT context, the double scaling limit of [165] corresponds to taking $g_{\rm YM} \to 0$, $n \to \infty$ keeping $\lambda = g_{\rm YM}^2 n$ fixed. This limit systematically isolates, at each loop order in the perturbative expansion of SQCD, a certain contribution. Its existence requires that at k loops, any extremal correlator has a leading behavior n^k, which remarkably turns out to be the case to all loop orders. Detailed aspects of this limit were discussed in the relevant articles [166,167]. It was recently understood in an important paper [168] that this limit can be viewed as the standard 't Hooft limit of an associated random matrix model. In particular, this explains why the limit exists, at least in this theory. In addition, the matrix model interpretation of [168] allows one to obtain the exact λ dependence in correlators in closed form by employing standard matrix model techniques.

An obvious question is whether the existence of the double-scaling, large charge limit is a peculiarity of highly supersymmetric theories such as $\mathcal{N} = 2$ SCFTs. In this note we find that an analogous limit exists for a familiar non-supersymmetric theory, namely scalar field theory with quartic potential. We will show that the very familiar Wilson-Fisher (WF) fixed point for the $O(2)$ theory provides perhaps the simplest example where one can study non-trivial correlation functions in said limit, by means of a complete resummation of Feynman diagrams.

One may more generally consider the $O(N)$ model in $4-\epsilon$ dimensions, but for simplicity we shall restrict the discussion to the $N = 2$ case. This can be recast as the theory for a complex scalar ϕ with a quartic interaction controlled by a coupling g. Appropriately tuning the mass parameter, there is a renormalization group flow to the Wilson-Fisher fixed point where $g \sim \epsilon$. One of the remarkable applications of the ϵ expansion is to extrapolate the results to $\epsilon = 1$, where the model describes the ferromagnetic transition of the 3d Ising model. Although this limit is far from the perturbative regime, the analytical results nevertheless remarkably agree with the numerical values for various critical exponents. On the other hand, in taking the limit $\epsilon \to 0$ the theory is simply led to the gaussian fixed point in $d = 4$. However, the limit of [165–168] suggests that one can consider sectors of large global charge which might have non-trivial dynamics. Specifically, we consider operators $\mathcal{O}_n \equiv \phi^n$ of $U(1)$ charge n and engineering dimension $n\,(1-\frac{\epsilon}{2})$. It turns out that in the limit $g \to 0$, the sector of operators with $n \to \infty$ such that $\lambda = g\,n^2$ is fixed, have non-trivial correlators, which can be exactly computed through a resummation of the surviving Feynman diagrams. We also provide an alternative derivation from the path integral: in the double scaling limit, it is dominated by a saddle-point, giving rise to the same correlation function previously obtained diagrammatically. The saddle-point calculation suggests that a similar limit may exist in other theories. In particular, we also consider the $O(2)$ theory in three dimensions for a potential $(\bar{\phi}\phi)^3$, where we identify the relevant limit and compute the exact two-point correlation function for the operators ϕ^n, $\bar{\phi}^n$.

Outlook Let us comment on some interesting open problems. It would be interesting to consider higher point functions in detail. A preliminary observation is as follows. Consider, for instance, a 3-point function $\langle \mathcal{O}_n(x)\, \mathcal{O}_n(y)\, \bar{\mathcal{O}_{2n}}(0)\rangle$ in the simplest context of the $O(2)$ model studied in this note. One can show that, to next-to-leading order, there are diagrams surviving the limit, yielding a result consistent with the structure dictated by conformal symmetry. Clearly, it would be of interest to extend this study to all orders and to arbitrary k-point functions. It would also be very interesting to systematically study the structure of 2-point functions following [154]. This might lead to universal relations involving the central charges of the conformal algebra. A challenging problem is to see if, as suggested in [168], the double scaling limit of the $O(2)$ theory can be understood as a 't Hooft limit of a "dual" random matrix model. One may also study large R charge correlators in ABJM theory in the same limit, which could be compared against results from supersymmetric localization.

3.2.1 The Wilson-Fisher fixed point for a complex scalar field

Let us consider the $O(N)$ model in $d = 4 - \epsilon$ dimensions. This model is both of pedagogical interest –as the historic laboratory for QFT and RG– as well as of practical interest: for different values of N it is known to describe various phase transitions of relevant physical systems (for instance, for $N = 1$, at $\epsilon = 1$, it describes the 3d ferromagnetic transition). The action reads

$$S = \int d^{4-\epsilon}x \left(\frac{1}{2}\,(\partial\vec{\varphi})^2 - \frac{1}{2}\,m^2\,\vec{\varphi}^2 - \frac{g}{16}\,(\vec{\varphi}^2)^2 \right)\,; \tag{3.2.1}$$

where $\vec{\varphi}$ is the N component field rotated by the $O(N)$ symmetry. As it is well-known, upon tuning the mass to zero this flows to the Wilson-Fisher fixed point at the critical value

$$g_{\rm WF} = \frac{32\,\pi^2}{N+8}\,\epsilon\,. \tag{3.2.2}$$

We will be interested in theories with a global $U(1)$ charge, for which the simplest example is $N = 2$. In that case the theory can be re-written as the theory for a complex scalar field in $4-\epsilon$ dimensions with action

$$S=\int d^{4-\epsilon}x\,\left(\partial\bar{\phi}\,\partial\phi-m^2\,\bar{\phi}\,\phi-\frac{g}{4}\,(\bar{\phi}\phi)^2\right)\,. \tag{3.2.3}$$

With these conventions, the Feynman rule for the vertex is just $-i\,g$. We will be interested in the critical case where $m^2=0$.

Note that this construction allows one to take the $g\to 0$ limit along a family of Conformal Field Theories. Nevertheless, since we are ultimately interested in the extreme weak coupling limit, we may alternatively simply consider the $g\,(\bar{\phi}\,\phi)^2$ theory in $d=4$.

It is easy to compute the anomalous dimension of scalar operators of the form $\mathcal{O}_n=\phi^n$ to $O(g)$. One finds $\gamma_{\mathcal{O}_n}\sim g\,n^2\sim\epsilon\,n^2$ (see *e.g.* [169]). The emergence of the combination $\lambda=g\,n^2\sim\epsilon\,n^2$ suggests the existence of a double scaling limit:

$$g\to 0\,,\qquad n\to\infty\,,\qquad \lambda=g\,n^2\ \text{fixed}\,. \tag{3.2.4}$$

The existence of the limit may also be suggested by earlier investigations on the exponentiation property of multiparticle amplitudes [170–172].

3.2.2 The double scaling limit on correlation functions

We will now investigate the limit (3.2.4) in the exact two-point correlation function $\langle\mathcal{O}_n(x)\bar{\mathcal{O}}_{n'}(0)\rangle$, with $\mathcal{O}_n=\phi^n$, $\bar{\mathcal{O}}_n=\bar{\phi}^n$ in $d=4-\epsilon$ dimensions. These operators have a definite $U(1)$ charge n and hence they are automatically orthogonal for different charges.[1]

Diagrammatic computation

Let us first compute the 2-point functions by evaluation of the relevant Feynman diagrams. As a preliminary step, let us consider the bubble diagram in fig.3.1, which is ubiquitous in the perturbative expansion of such correlators.

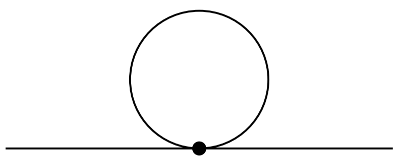


Figure 3.1: Bubble diagram.

[1]It should be noted that, at fixed charge n, the most general operators are of the form $\mathcal{O}_{n,k}=(\bar{\phi}\phi)^k\,\mathcal{O}_n$. We will restrict to the lowest tower with $k=0$.

This diagram has no dependence on the external momenta; therefore it can only be proportional to the mass. Since we will be interested in the critical theory, this diagram vanishes. Thus, when computing $\mathcal{O}_n$ correlators, we shall only consider diagrams that do not contain any bubble.

Let us now consider the systematics of the perturbative expansion of the correlation function $\langle \mathcal{O}_n(x)\, \bar{\mathcal{O}}_n(0) \rangle$. As usual, at each order in the perturbative expansion in g there are several topologically different diagrams, each one coming with a certain dependence on n. We are going to be interested in taking n to infinity, and inspection of all topologies shows that in this limit a class of them dominates over the rest. As shown below, the dominant topology can be viewed as an iteration of the one-loop diagram of fig. 3.2, that we will call *Kermit the frog*'s diagram.

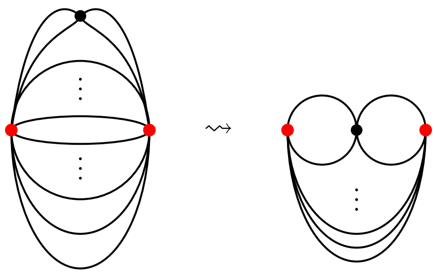


Figure 3.2: The relevant one-loop diagram and its *Kermit the frog* representation.

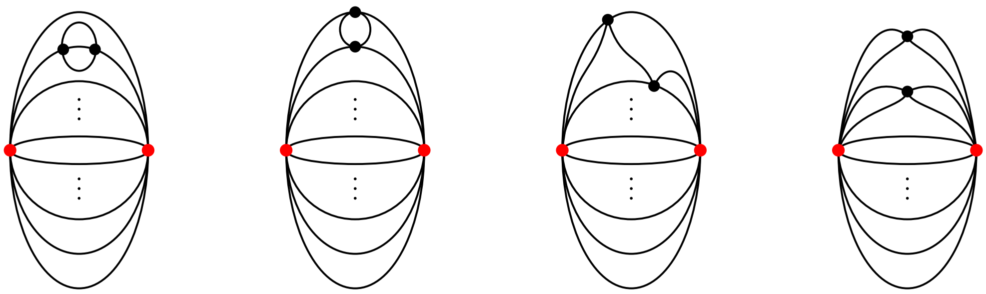


Figure 3.3: Four topologies contributing at order $O(g^2)$.

At order g^2, we have the four diagrams of fig. 3.3 above. The key point to identify the dominant diagram at large n, at any given loop order, is the n dependence, which comes from the combinatorial factor. This is given by $n!n!/k!$, where k is the number of lines that do not undergo interactions. Therefore, the diagram that has the highest power of n is the one with the smallest k. Using this formula, we thus find that the combinatorial factor of the fourth diagram in fig. 3.3 is $n!\, n\,(n-1)\,(n-2)\,(n-3)$. On the other hand, the formula

$n!n!/k!$ implies that the combinatorial factors of the first, second and third diagrams are $n!\,n$, $n!\,n\,(n-1)$ and $n!\,n(n-1)(n-2)$ respectively (we omit numerical coefficients standing from symmetrization, which do not affect the n dependence). Thus, for large n, the diagram on the right dominates. We will call this one the *two-loop Kermit diagram*.

Now consider the general m-loop diagram with m vertices. The lines in each vertex can go either to another vertex or join some of the n lines of the operators ϕ^n or $\bar{\phi}^n$. The diagram which has a smaller number of lines that do not undergo interactions is when two lines of each vertex join two of the n lines of the operator ϕ^n and the other two lines join two of the n lines of the operator $\bar{\phi}^n$ (it is not possible to have three lines of the vertex joining three lines of the operators ϕ^n because of charge conservation; vertices are of the form $\phi\phi\bar{\phi}\bar{\phi}$). This corresponds to the iteration of the Kermit diagram and has a combinatorial factor $n!^2/(n-2m)!$ which has the highest power of n (see (3.2.5) and below for the derivation of the combinatorial factor at m loop order including other symmetry factors).

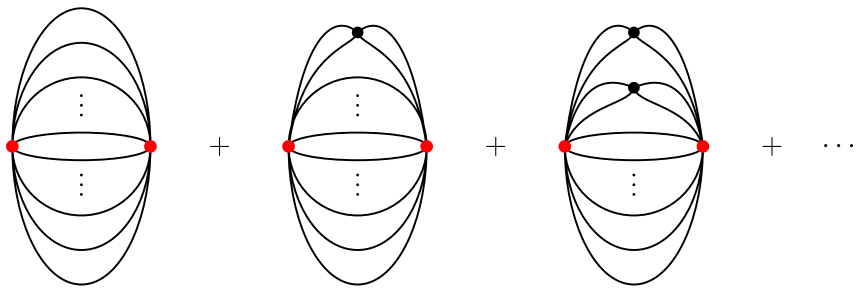


Figure 3.4: Diagrams contributing to $\langle \mathcal{O}_n(x)\, \bar{\mathcal{O}}_n(x)\rangle$ at large n.

Thus, we conclude that a class of diagrams dominate the correlation function, the m-loop Kermit diagrams of fig. 3.9. Denoting by K_m the contribution from the Kermit diagram with m interaction vertices, the correlator is

$$\langle \mathcal{O}_n(x)\, \bar{\mathcal{O}}_n(0)\rangle = n! \sum_{m=0} (-ig)^m\, K_m\, \frac{1}{4^m}\, \frac{n!}{(n-2m)!}\, \frac{1}{m!}\,. \tag{3.2.5}$$

The combinatorial factor can be understood from fig. 3.9. There are n lines on each side, giving rise to a factor $(n!)^2$ obtained by permutations. Then one must divide over the number of permutations that lead to equivalent configurations. There is a factor $1/(n-2m)!$ associated with the permutations of the $n-2m$ lines that do not undergo interaction. There is also a factor $1/2^m$ on each side associated with the permutations of the pair of lines in the m loops. The factor $1/m!$ originates from the expansion of the exponential of the interaction term.

Using the de Moivre-Stirling formula, for $n \gg 1$ one obtains

$$\frac{n!}{(n-2m)!} \approx n^{2m}\,, \qquad n \gg 1\,. \tag{3.2.6}$$

Therefore we can define the limit

$$n \to \infty\,, \qquad g \to 0\,, \qquad \lambda = g\,n^2 = \text{fixed}\,. \tag{3.2.7}$$

The correlator then becomes

$$\langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = n!\sum_{m=0} K_m \left(\frac{-i\,\lambda}{4}\right)^m \frac{1}{m!}\,. \tag{3.2.8}$$

To further proceed, note that, in position space, the Kermit diagram K_m is

$$\begin{aligned} K_m &= G(0,\,x)^{n-2m}\prod_{i=1}^{m}\int d^4z_i\,G(0,\,z_i)^2\,G(z_i,\,x)^2 \\ &= G(0,\,x)^n\left(\frac{1}{G(0,\,x)^2}\int d^4z\,G(0,\,z)^2\,G(z,\,x)^2\right)^m\,, \end{aligned} \tag{3.2.9}$$

where $G(x,\,y)$ is the propagator of the ϕ field. Thus

$$\langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = n!\,G(0,\,x)^n\sum_{m=0}\left(\frac{-i\,\lambda\,\mathcal{K}}{4}\right)^m\frac{1}{m!}\,, \tag{3.2.10}$$

with

$$\mathcal{K} = \frac{1}{G(0,\,x)^2}\int d^4z\,G(0,\,z)^2\,G(z,\,x)^2\,. \tag{3.2.11}$$

Since $n!\,G(0,\,x)^n = \langle \mathcal{O}_n(x)\,\mathcal{O}_n(0)\rangle_0$ is the correlation function in the free theory, and the sum can be trivially resumed, we find

$$\langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = \langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle_0\,e^{-i\,\frac{\lambda\mathcal{K}}{4}}\,. \tag{3.2.12}$$

Next, consider the computation of $\mathcal{K}$, which is carried out in the appendix. Note that $\mathcal{K}$ represents the $O(g)$ correction to the $\mathcal{O}_2$ correlator. We have

$$\mathcal{K} = -\frac{i}{8\,\pi^2}\,\log(\Lambda^2x^2)\,. \tag{3.2.13}$$

As a cross-check of this result, one can see that in the $N=1$ case, and upon appropriately taking into account numerical conventions, this yields the correct $O(g)$ anomalous dimension of the $\mathcal{O}_n$ operator (*cf.* for example [169]).

Thus

$$\langle \mathcal{O}_n(x)\, \bar{\mathcal{O}}_n(0) \rangle = \langle \mathcal{O}_n(x)\, \bar{\mathcal{O}}_n(0) \rangle_0 \, \frac{1}{|x|^{\frac{\lambda}{16\,\pi^2}}} \,. \tag{3.2.14}$$

Since in position space

$$G(0,\, x) = \frac{1}{4\,\pi^2}\, \frac{1}{|x|^2} \,, \tag{3.2.15}$$

we finally find

$$\langle \mathcal{O}_n(x)\, \bar{\mathcal{O}}_n(0) \rangle = \frac{n!}{(4\,\pi^2)^n\, |x|^{2\,(n+\frac{\lambda}{32\,\pi^2})}} \,. \tag{3.2.16}$$

In particular, this gives the following formula for the dimension of the $\mathcal{O}_n$ operator in the double scaling limit

$$\Delta_{\mathcal{O}_n} = n + \frac{\lambda}{32\,\pi^2} \,. \tag{3.2.17}$$

Saddle-point derivation

The underlying reason behind the existence of a large charge limit can be understood from a saddle-point calculation. It is convenient to rescale the scalar field and define new variables

$$\sigma = g^{\frac{1}{4}}\, \phi \,, \qquad \bar{\sigma} = g^{\frac{1}{4}}\, \bar{\phi} \,. \tag{3.2.18}$$

The correlator is then given by

$$\langle \mathcal{O}_n(x_1)\, \bar{\mathcal{O}}_n(x_2) \rangle = \frac{1}{g^{\frac{n}{2}} Z} \int D\sigma D\bar{\sigma}\; e^{-S} \,, \tag{3.2.19}$$

where the Euclidean action, including source terms, is given by

$$S = \int d^4x \left(g^{-\frac{1}{2}} \partial\bar{\sigma}\, \partial\sigma + \frac{1}{4}\, (\bar{\sigma}\sigma)^2 - n\delta(x - x_1) \log\sigma - n\delta(x - x_2) \log\bar{\sigma} \right) . \tag{3.2.20}$$

In the large n limit, this integral is dominated by a saddle-point. Indeed, the saddle-point analysis is very similar to the one carried out in section 2.3 of [154]. The saddle-point equations are given by

$$\partial^2\sigma = -ng^{\frac{1}{2}}\delta(x-x_2)\frac{1}{\bar\sigma} + \frac{1}{2}g^{\frac{1}{2}}\bar\sigma\sigma^2\ , \qquad \partial^2\bar\sigma = -ng^{\frac{1}{2}}\delta(x-x_1)\frac{1}{\sigma} + \frac{1}{2}g^{\frac{1}{2}}\bar\sigma^2\sigma\ . \tag{3.2.21}$$

The crucial point is that, in the limit $g\to 0$, $n\to\infty$, where $\lambda = n^2 g =$ fixed, the interaction term can be ignored. The resulting equations become

$$\bar\sigma\partial^2\sigma = -\lambda^{\frac{1}{2}}\delta(x-x_2)\ , \qquad \sigma\partial^2\bar\sigma = -\lambda^{\frac{1}{2}}\delta(x-x_1)\ . \tag{3.2.22}$$

These equations are now equivalent to those discussed in [154]. The solution is given by

$$\sigma(x) = \lambda^{1/4}\,\frac{e^{i\beta_0}|x_1-x_2|}{2\pi(x-x_2)^2}\ , \qquad \bar\sigma(x) = \lambda^{1/4}\,\frac{e^{-i\beta_0}|x_1-x_2|}{2\pi(x-x_1)^2}\ . \tag{3.2.23}$$

Let us now substitute this solution into the action. Consider first the interaction term. This is absent in [154] and it is indeed the interesting part in our case. We have

$$\int d^4x\frac{1}{4}\,(\bar\sigma\sigma)^2 = \frac{\lambda}{4(2\pi)^4}\int d^4x\frac{|x_1-x_2|^4}{(x-x_1)^4(x-x_2)^4}\ . \tag{3.2.24}$$

The integral can be computed by using (5.D.3), (5.D.5), upon shifting $x\to x+x_1$. We get

$$\int d^4x\frac{1}{x^4\,(x-(x_2-x_1))^4} = \frac{4\pi^2}{(x_2-x_1)^4}\,\log|x_2-x_1|\,. \tag{3.2.25}$$

Thus

$$\int d^4x\frac{1}{4}\,(\bar\sigma\sigma)^2 = \frac{\lambda}{32\,\pi^2}\log(x_1-x_2)^2\,. \tag{3.2.26}$$

Let us now consider the remaining terms in the action. Following [154], we have

$$\begin{aligned} g^{-\frac{1}{2}}\int d^4x\,\partial\bar\sigma\,\partial\sigma - n\log\left(\sigma(x_1)\bar\sigma(x_2)\right) &= -n\log\left(\sigma(x_1)\bar\sigma(x_2)\right) + n \\ &= -\frac{n}{2}\log\lambda + n + n\log\left(2\pi(x_1-x_2)\right)^2\ . \end{aligned}$$

Putting all pieces together, we find

$$\langle\mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = \frac{n!}{(4\,\pi^2)^n\,|x_1-x_2|^{2\,(n+\frac{\lambda}{32\,\pi^2})}} \qquad n!\sim(2\pi)^{1/2}n^{n+\frac{1}{2}}e^{-n}\ . \tag{3.2.27}$$

which is precisely the result (3.2.16) found from the perturbative calculation based on resumming Feynman diagrams.

It is worth noting that (3.2.27) can also be written as

$$\langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = \frac{n!}{(4\,\pi^2)^n\,|x|^{2\,n\,(1+\frac{\widehat{\lambda}}{32\,\pi^2})}}\ , \tag{3.2.28}$$

where $\widehat{\lambda} = \frac{\lambda}{n} = g\,n$. This makes contact with the limit of [170–172], recently discussed in [147, 173], where $\widehat{\lambda}$ is kept fixed. More precisely, on general grounds, correlation functions for large charge operators admit a double, 't Hooft-like, expansion in n, $\widehat{\lambda}$, so that $\Delta = \sum_{i=0}^{\infty} n^{1-i} F_n(\widehat{\lambda})$. At the same time, for weak $\widehat{\lambda}$ coupling, F_0 must admit a perturbative expansion $F_0 = 1 + a\,\widehat{\lambda} + \cdots$, where, by explicit computation, $a = \frac{1}{32\pi^2}$. Fixed λ implies $\hat{\lambda} \ll 1$ when $n \gg 1$, which corresponds to the weak coupling regime in the $\widehat{\lambda}$ expansion. In this way our formula above is recovered as the $n \to \infty$ (akin to the planar limit in the familiar 't Hooft $\frac{1}{N}$ expansion) at weak $\widehat{\lambda}$ coupling. From this perspective, the $\frac{1}{n}$ corrections to the saddle-point approximation reconstruct the double expansion described above.

3.2.3 The double-scaling limit in $d = 3$

We can similarly consider $(\bar{\phi}\phi)^3$ theory in $d = 3$, defined by the action

$$S = \int d^{3-\epsilon}x\,\left(\partial\bar{\phi}\,\partial\phi - m^2\,\bar{\phi}\,\phi - \frac{g}{3!}\,(\bar{\phi}\phi)^3\right)\,. \tag{3.2.29}$$

It should be noted that this model cannot describe the $\epsilon \to 1$ limit of the WF fixed point discussed in the previous section, as it contains a sextic (as opposed to quartic) interaction. Note that in both cases the interaction term is classically marginal in their respective dimensions and that the fixed points lie in the perturbative regime. In fact, just as in the WF case above, our strategy in this $d = 3$ model will be to consider large charge operators in the extreme weak coupling regime.

Let us consider the saddle-point calculation for the same correlator $\langle \mathcal{O}_n(x_1)\,\bar{\mathcal{O}}_n(x_2)\rangle$. After scaling $\sigma = g^{\frac{1}{6}}\,\phi\ ,\quad \bar{\sigma} = g^{\frac{1}{6}}\,\bar{\phi}$. the action becomes

$$S = \int d^3x\,\left(g^{-\frac{1}{3}}\partial\bar{\sigma}\,\partial\sigma + \frac{1}{3!^2}\,(\bar{\sigma}\sigma)^3 - n\delta(x-x_1)\log\sigma - n\delta(x-x_2)\log\bar{\sigma}\right)\,. \tag{3.2.30}$$

A similar saddle-point analysis now leads to the equations

$$\partial^2\sigma = -ng^{\frac{1}{3}}\delta(x-x_2)\frac{1}{\bar{\sigma}} + \frac{1}{12}g^{\frac{1}{3}}\bar{\sigma}^2\sigma^3\ ,\qquad \partial^2\bar{\sigma} = -ng^{\frac{1}{3}}\delta(x-x_1)\frac{1}{\sigma} + \frac{1}{12}g^{\frac{1}{3}}\bar{\sigma}^3\sigma^2\ . \tag{3.2.31}$$

We now take the limit

$$n \to \infty \,, \quad g \to 0 \,, \quad \text{with} \;\; \lambda = n^3 g = \text{fixed} \,. \tag{3.2.32}$$

As in $d = 4$, the interaction term vanishes in the limit. The solutions of the saddle-point equations are obtained just like in the $d = 4$ case, finding now

$$\sigma(x) = \lambda^{1/6} \frac{e^{i\beta_0} |x_1 - x_2|^{\frac{1}{2}}}{\sqrt{4\pi} |x - x_2|} \,, \qquad \bar{\sigma}(x) = \lambda^{1/6} \frac{e^{-i\beta_0} |x_1 - x_2|^{\frac{1}{2}}}{\sqrt{4\pi} |x - x_1|} \,. \tag{3.2.33}$$

The anomalous dimension now comes from the contribution

$$\int d^3x \frac{1}{3!^2} \left(\bar{\sigma}\sigma\right)^3 = \frac{\lambda}{3!^2 (4\pi)^3} \int d^3x \frac{|x_1 - x_2|^3}{|x - x_1|^3 \, |x - x_2|^3} \,. \tag{3.2.34}$$

This integral represents the Feynman diagram of fig. 3.5, the "sleeping Kermit".

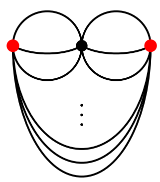


Figure 3.5: Sleeping Kermit. The diagram represents the leading non-trivial contribution to the two-point correlation function of the $d = 3$ theory in the double-scaling limit.

This integral can be done using the results in appendix, leading to

$$\int d^3x \frac{1}{3!^2} \left(\bar{\sigma}\sigma\right)^3 = \frac{\lambda}{(24\pi)^2} \, \log |x_1 - x_2|^2 \Lambda^2 \,. \tag{3.2.35}$$

The saddle-point calculation implies that, as in the $d = 4$ case, this contribution exponentiates, leading to a correlator

$$\langle \mathcal{O}_n(x) \, \bar{\mathcal{O}}_n(0) \rangle = \frac{n!}{(4\pi)^n \, |x_1 - x_2|^{2\left(n + \frac{\lambda}{(24\pi)^2}\right)}} \,, \qquad n! \sim (2\pi)^{1/2} n^{n+\frac{1}{2}} e^{-n} \,. \tag{3.2.36}$$

3.3 Correlation functions in scalar field theory at large charge

We have seen in previous sections that sectors with large charge under a global symmetry of a given quantum field theory enjoy remarkable simplification properties, which allow a systematic and analytic study [148–150]. A prototypical example is the $O(2)$ model in $d = 3$, where, if one is interested in the sector of operators with large charge n under the global symmetry, it is possible to write an effective field theory governing their dynamics. This allows to compute their anomalous dimensions, which are found to scale as $\Delta \sim n^{\frac{3}{2}} + \mathcal{O}(n^{\frac{1}{2}})$. This result can be understood from a "microscopic" description starting with the $U(1)$ Wilson-Fisher (WF) fixed point in $d = 4 - \epsilon$ dimensions, described by the action

$$S_0 = \int d^{4-\epsilon}x \left(\partial\bar{\phi}\,\partial\phi - \frac{g}{4}\,(\bar{\phi}\phi)^2\right) . \tag{3.3.1}$$

This approach in fact uncovers a rich structure, as the sector of operators at fixed charge n is described by an effective theory depending on both n and g (recall that at the WF fixed point, $g \sim \epsilon$), in such a way that, depending on how the large charge limit is taken, a different behavior emerges. This was first discussed in [4], where the two-point function of the operators ϕ^n, $\bar{\phi}^n$ was computed by a full resummation of the dominant Feynman diagrams –dubbed Kermit L-loop diagrams– that survive in a double scaling limit with $n \to \infty$ at fixed $g\,n^2 \equiv \lambda$. In this limit the sum over Feynman diagrams exponentiates, giving the result

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle = \frac{n!}{(4\pi^2)^n|x|^{2\Delta}} \,, \qquad \Delta = n + \frac{\lambda}{32\pi^2} \,. \tag{3.3.2}$$

This result can also be derived by a saddle-point evaluation of the two-point function, which becomes exact in the double scaling limit with fixed $\lambda = gn^2$ [4].

A general way to organize the large n expansion was then discussed in [147, 173, 174] (generalizing earlier work in [170–172]). The effective description of the large n sector naturally depends on n and $g\,n = \hat{\lambda}$, i.e. $\hat{\lambda} = \lambda/n$, in such a way that a 't Hooft-like double expansion emerges and, in particular, one has

$$\Delta = \sum_{k=-1}^{\infty} n^{-k}\,\Delta_k(\hat{\lambda}) \,. \tag{3.3.3}$$

In the strict large n limit the dominant term is $\Delta_{-1}(\hat{\lambda})$, which in turn must admit a perturbative expansion for small $\hat{\lambda}$. This gives [147, 173]

$$\Delta_{-1}(\hat{\lambda}) = 1 + \frac{\hat{\lambda}}{32\pi^2} + O\big(\hat{\lambda}^2\big)\,. \tag{3.3.4}$$

In the double-scaling limit of [4] at fixed λ, the $O\big(\hat{\lambda}^2\big)$ terms are given by Feynman diagrams which are suppressed by powers of $1/n$. Thus, from this point of view, the result (3.3.2) can be viewed as the leading term of the more general double expansion in n, $\hat{\lambda}$. In the opposite limit of large $\hat{\lambda}$ –which overlaps with the regime of validity of the large charge effective theory– one finds $\Delta \sim \hat{\lambda}^{\frac{4-\epsilon}{3-\epsilon}} + \cdots$, thus recovering the expected scaling, with the extra bonus of providing an analytic expression for the actual coefficients in the large charge expansion [147, 173, 174].

Large charge expansions also exist in general CFT's with a marginal coupling. An example of a CFT depending on an exactly marginal parameter $g_{\rm YM}$ is $\mathcal{N} = 2$ supersymmetric four-dimensional QCD with gauge group $SU(N)$ and $2N$ fundamental flavors. The large charge limit of this theory was first introduced in [165] and studied using supersymmetric localization. It was shown that the perturbative expansion of correlators of $(\mathrm{Tr}\,\phi^2)^n$ – ϕ being the adjoint scalar in the vector multiplet of unit R-charge – has a well-defined large n limit provided one takes a double-scaling limit of large n and fixed $g_{\rm YM}^2\, n$. This limit ensures that all terms in the perturbative expansion are finite and non-vanishing. Further aspects were studied in detail in [166, 167]. Subsequently, the existence of a double-scaling limit was understood in terms of a "hidden" matrix model description in [168].

The mere fact that it is possible to compute observables of a QFT in a closed form in the large charge sector is remarkable *per se*. Motivated by this, in this work, we study higher point functions in the $O(2)$ theory in the sector of operators with large charge. Focusing in the weak coupling regime in the double expansion in $1/n$, $\hat{\lambda}$, we compute "extremal" correlators (of the form $\langle \phi^{n_1} \cdots \phi^{n_r} \bar{\phi}^m \rangle$) as well as 4-point functions in the "non-extremal" case. As discussed above, in the double scaling limit, these results become exact. We shall use the saddle point method employed in [4].

3.3.1 Higher point functions in the $O(2)$ model

We will follow the approach of [4], where the two-point function was computed in a double-scaling limit, $n \to \infty$, $g \to 0$ at fixed $g\, n^2 \equiv \lambda$. This limit yields the exact exponentiation of the the leading non-trivial term in the more general n, $\hat{\lambda}$ expansion in the large n and weak $\hat{\lambda}$ regime. In the case of higher-point functions, we are interested in general correlation functions of the form

$$\langle\phi(x_1)^{n_1}\cdots\phi(x_r)^{n_r}\,\bar\phi(y_1)^{m_1}\cdots\bar\phi(y_s)^{m_s}\rangle\ ,\qquad \sum_{i=1}^{r} n_i=\sum_{j=1}^{s} m_j\ . \tag{3.3.5}$$

We will assume the following scaling

$$n_i=a_i n\ ,\qquad m_j=b_j n\ ,\qquad g\to 0,\ \ n\to\infty\ ,\quad gn^2=\text{fixed}\ ,$$

and fixed $a_i,\ b_j$. In the case of the two-point function $\langle\phi(x)^n\bar\phi(y)^n\rangle$, it was shown in the previous section that in the double-scaling limit all higher loop diagrams vanish except those with a particular topology (the "Kermit the frog" L-loop diagram), corresponding to the case where two lines of each of the L vertices join two of the n lines of the operator ϕ^n and the other two lines join two of the n lines of the operator $\bar\phi^n$. In particular, Feynman diagrams having lines joining one vertex to another one vanish in the double-scaling limit. As a result, the two-point function can be exactly computed by a complete resummation of the surviving L-loop Feynman diagrams.

Alternatively, the double-scaling limit can be understood from a saddle-point calculation. This can be easily generalized to the general correlation function (3.3.5). We first introduce the scaled scalar field

$$\sigma=g^{\frac14}\,\phi\ ,\qquad \bar\sigma=g^{\frac14}\,\bar\phi\ . \tag{3.3.6}$$

The general correlation function (3.3.5) is then given by

$$\langle\phi(x_1)^{n_1}\cdots\phi(x_r)^{n_r}\,\bar\phi(y_1)^{m_1}\cdots\bar\phi(y_s)^{m_s}\rangle=\frac{1}{g^{\frac{m}{2}}Z}\int D\sigma D\bar\sigma\ e^{-S}\ ,\qquad m\equiv\sum_{j=1}^{s} m_j\ ,$$

where the Euclidean action, including source terms, is given by

$$S=S_{\text{free}}+S_{\text{int}} \tag{3.3.7}$$

$$\begin{aligned}
S_{\text{free}} &= \int d^4x\,\left(g^{-\frac12}\partial\bar\sigma\,\partial\sigma-\sum_i n_i\delta(x-x_i)\log\sigma-\sum_j m_j\delta(x-y_j)\log\bar\sigma\right)\\
&= \int d^4x\,\left(g^{-\frac12}\partial\bar\sigma\,\partial\sigma-\log\sigma(x_1)^{n_1}\cdots\sigma(x_r)^{n_r}\bar\sigma(y_1)^{m_1}\cdots\bar\sigma(y_s)^{m_s}\right)\ , \qquad (3.3.8)\\
S_{\text{int}} &= \int d^4x\,\frac14\,(\bar\sigma\sigma)^2\ . \qquad (3.3.9)
\end{aligned}$$

The saddle-point equations are given by

$$\partial^2\sigma = -ng^{\frac{1}{2}}\sum_j b_j\delta(x-y_j)\frac{1}{\bar\sigma} + \frac{1}{2}g^{\frac{1}{2}}\bar\sigma\sigma^2\ , \qquad \partial^2\bar\sigma = -ng^{\frac{1}{2}}\sum_i a_i\delta(x-x_i)\frac{1}{\sigma} + \frac{1}{2}g^{\frac{1}{2}}\bar\sigma^2\sigma\ .$$

In the double-scaling $g \to 0$, $n \to \infty$, with fixed $\lambda = n^2 g$, the cubic term vanishes. The equations simply become

$$\bar\sigma\partial^2\sigma = -\lambda^{\frac{1}{2}}\sum_j b_j\delta(x-y_j)\ , \qquad \sigma\partial^2\bar\sigma = -\lambda^{\frac{1}{2}}\sum_i a_i\delta(x-x_i)\ . \tag{3.3.10}$$

Extremal correlators

We shall first consider a special class of correlation functions where the resulting expressions in the double-scaling limit are conspicuously simple. These are the "extremal" correlators

$$\langle\phi(x_1)^{n_1}\cdots\phi(x_r)^{n_r}\,\bar\phi(y)^m\rangle\ , \qquad \sum_{i=1}^{r} n_i = m\ . \tag{3.3.11}$$

The name "extremal" correlators was here taken from $\mathcal{N}=2$ supersymmetric gauge theories, where correlation functions of this form having r chiral primary operators and one anti-chiral primary operator enjoy special properties because of supersymmetry. In the present case, there is of course no supersymmetry. Yet, for extremal correlators of the form (3.3.11), the double-scaling limit will still single out the particular topologies generalizing the Kermit diagrams of [4] with the two lines of each vertex being distributed among the r different points. The reason of the simplicity of this correlator is more transparent in the saddle-point calculation, which for this case admits a simple solution. The solution to (3.3.10) is

$$\sigma = \frac{\lambda^{\frac{1}{2}}b}{\bar\sigma_0(y)}G(x-y)\ , \qquad \bar\sigma = \bar\sigma_0(y)\sum_{i=1}^{r}\frac{a_i G(x-x_i)}{bG(x_i-y)}\ , \tag{3.3.12}$$

where $G(x)$ is the Green function

$$G(x) = \frac{1}{4\pi x^2}\ , \qquad \partial^2 G(x) = -\delta(x)\ ,$$

Note that the factor $\bar\sigma_0(y) = \bar\sigma(y)$. cancels out in computing the action. Substituting this solution into the free part of the action, we obtain

$$S_{\text{free}} = -\log\sigma(x_1)^{n_1}\cdots\sigma(x_r)^{n_r}\bar\sigma(y)^m + m\ . \tag{3.3.13}$$

This gives

$$\langle \phi(x_1)^{n_1} \cdots \phi(x_r)^{n_r}\,\bar\phi(y)^m\rangle_{\rm free} = \frac{m^m e^{-m}}{(4\,\pi^2)^m} \prod_{i=1}^r \frac{1}{|x_i-y|^{2n_i}} \ . \tag{3.3.14}$$

The factor $m^m e^{-m}$ is the leading approximation for $m!$ (the Gaussian integration in the saddle-point approximation completes the standard form of the de Moivre-Stirling formula $m! \approx \sqrt{2\pi m}\, m^m e^{-m}$).

Next, let us consider the interaction term.

$$\begin{aligned} S_{\rm int} = \int d^4x \frac{1}{4}\,(\bar\sigma\sigma)^2 \ &= \ \frac{\lambda}{4} \int d^4x\, G(x-y)^2 \left(\sum_i \frac{a_i G(x-x_i)}{G(x_i-y)} \right)^2 \\ &= \ \frac{\lambda}{4} \left(\sum_{i=1}^r a_i^2 I(x_i,y) + 2 \sum_{i<j}^r a_i a_j I(x_i,x_j,y) \right) \ , \end{aligned}$$

where [2]

$$I(x_i,y) \equiv \frac{1}{G(x_i-y)^2} \int d^4x\, G(x-y)^2 G(x-x_i)^2 = \frac{1}{4\pi^2} \log(\mu|x_i-y|) \tag{3.3.15}$$

$$\begin{aligned} I(x_i,x_j,y) \ &\equiv \ \frac{1}{G(x_i-y)G(x_j-y)} \int d^4x\, G(x-y)^2 G(x-x_i) G(x-x_j) \\ &= \ \frac{1}{8\pi^2} \log\left(\mu\, \frac{|x_i-y|\,|x_j-y|}{|x_i-x_j|}\right) , \end{aligned} \tag{3.3.16}$$

where μ is a reference mass scale, which in what follows will be set to one (see comments on section 3.3.1).

Combining the free and the interacting part, we finally obtain

$$\langle \phi(x_1)^{n_1} \cdots \phi(x_r)^{n_r}\,\bar\phi(y)^m\rangle = \frac{m!}{(4\pi^2)^m \prod_{i=1}^r |x_i-y|^{2(n_i+\frac{\lambda\, a_i b}{32\pi^2})} \prod_{i<j}^r |x_i-x_j|^{-\frac{\lambda\, a_i\, a_j}{16\,\pi^2}}} \ . \tag{3.3.17}$$

We can now check that this structure is consistent with the expected structure dictated by conformal symmetry. Consider first the particular case of the three-point function, that is, $r=2$. With no loss of generality, we can set $y=0$. The result can be written in the equivalent form

$$\langle \phi(x_1)^{n_1} \phi(x_2)^{n_2}\,\bar\phi(y)^m\rangle = \frac{m!}{(4\pi^2)^m |x_1|^{\Delta_1+\bar\Delta-\Delta_2} |x_2|^{\Delta_2+\bar\Delta-\Delta_1} |x_1-x_2|^{\Delta_1+\Delta_2-\bar\Delta}} \ , \tag{3.3.18}$$

[2]Details on the calculation of these integrals can be found in [175] (see also [4]).

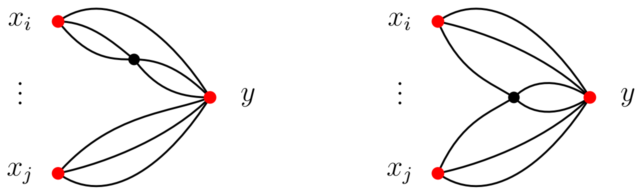


Figure 3.6: Types of diagrams that contribute to the extremal correlators.

where $m = n_1 + n_2$ and

$$\Delta_1 = n_1 + \frac{\lambda a_1^2}{32\pi^2}\,, \quad \Delta_2 = n_2 + \frac{\lambda a_2^2}{32\pi^2}\,, \quad \bar{\Delta} = (n_1+n_2) + \frac{\lambda (a_1+a_2)^2}{32\pi^2}\,.$$

Higher-point extremal correlators are given explicitly by the remarkably simple formula (3.3.17). When $r \geq 3$, the exponents in the formula (3.3.17) can no longer be expressed purely in terms of the dimensions $\{\Delta_i, \bar{\Delta}\}$ as in the three-point function (3.3.18).

Summarizing, we found the exact “extremal" correlators in the double-scaling limit where all charges go to infinity scaling in the same way. The result represents the resummation of the infinite number of L-loop Feynman diagrams that survive the limit. These are shown in figure 3.11 and generalize the “Kermit the frog" diagrams described in detail in [4]. The existence of the limit can be understood from the saddle-point analysis, which led to finite expressions that become exact at $n = \infty$. For large, but finite, charges, the double-scaling limit can be viewed as the leading result in a $1/n$ expansion. The next $O(1/n)$ terms in the expansion may be systematically derived from corrections to the saddle point approximations, obtained from the Taylor expansion of the action around the saddle-point.

Non-extremal correlators

Let us now discuss general (“non-extremal") correlation functions. The general solution to (3.3.10) is given by

$$\sigma(x) = \lambda^{\frac{1}{2}} \sum_{j=1}^{s} \frac{b_j}{\bar{\sigma}(y_j)}\, G(x,\, y_j)\,, \qquad \bar{\sigma}(x) = \lambda^{\frac{1}{2}} \sum_{i=1}^{r} \frac{a_i}{\sigma(x_i)}\, G(x,\, x_i)\,, \tag{3.3.19}$$

One can check that these equations are consistent provided $\sum_{i=1}^{r} n_i = \sum_{j=1}^{s} m_j$. General correlation functions can be obtained by substituting (3.3.19) into the action (3.3.8), (3.3.9). In what follows we shall focus on the four-point function.

Four-point non-extremal correlator

As an explicit example, let us consider the case $r = s = 2$, *i.e.* the four-point function

$$\langle \phi(x_1)^{n_1}\, \phi(x_2)^{n_2}\, \bar{\phi}(y_1)^{m_1}\, \bar{\phi}(y_2)^{m_2} \rangle\,, \qquad n_1 + n_2 = m_1 + m_2\,. \tag{3.3.20}$$

In addition, in this subsection we shall consider the particular case

$$a_1 = a_2 = b_1 = b_2 = 1\,, \tag{3.3.21}$$

so that $n_i = m_i = n$. Then

$$\sigma(x) = \sigma_0(x_2)\, \frac{\sqrt{\frac{G(x_1,\,y_2)}{G(x_2,\,y_1)}}\, G(x,\, y_1) + \sqrt{\frac{G(x_1,\,y_1)}{G(x_2,\,y_2)}}\, G(x,\, y_2)}{\sqrt{G(x_1,\, y_2)\, G(x_2,\, y_1)} + \sqrt{G(x_1,\, y_1)\, G(x_2,\, y_2)}}\ , \tag{3.3.22}$$

and

$$\bar{\sigma}(x) = \frac{\lambda^{\frac{1}{2}}}{\sigma_0(x_2)} \left(G(x,\, x_2) + \sqrt{\frac{G(x_2,\, y_1)\, G(x_2,\, y_2)}{G(x_1,\, y_1)\, G(x_1,\, y_2)}}\, G(x,\, x_1) \right). \tag{3.3.23}$$

The factor $\sigma_0(x_2) = \sigma(x_2)$ cancels out when computing the action. Substituting the solution into the free part of the action, given in (3.3.8), we obtain

$$S_{\text{free}} = 2n - n \log \lambda \left(\sqrt{G(x_1,\, y_2)\, G(x_2,\, y_1)} + \sqrt{G(x_1,\, y_1)\, G(x_2,\, y_2)} \right)^2. \tag{3.3.24}$$

It is convenient to rename $(y_1, y_2) \to (x_3, x_4)$ and define $r_{ij} \equiv |x_i - x_j|$. Thus we obtain

$$\langle \phi(x_1)^n\, \phi(x_2)^n\, \bar{\phi}(x_3)^n\, \bar{\phi}(x_4)^n \rangle_{\text{free}} = \frac{n^{2n} e^{-2n}}{(4\pi^2)^{2n}} \left(\frac{1}{r_{14} r_{23}} + \frac{1}{r_{13} r_{24}} \right)^{2n}\ . \tag{3.3.25}$$

Let us now compute the interaction term. Substituting the solutions (3.3.22), (3.3.23) for σ and $\bar{\sigma}$ into (3.3.9), we get an expression with nine integrals. Using the formulas (3.3.15), (3.3.16) and the integral computed in [176]

$$\int d^4x\, G(x,\, x_1)\, G(x,\, x_2)\, G(x,\, x_3)\, G(x,\, x_4) = \frac{H}{2^8 \pi^6\, r_{13}^2\, r_{24}^2} \tag{3.3.26}$$

where

$$H=\frac{1}{1-x-y}\Big(\log x(1-y)\,\log\frac{y}{1-x}-2\,\mathrm{Li}_2(x)+2\,\mathrm{Li}_2(1-y)\Big)\,, \tag{3.3.27}$$

being u, v the conformal ratios

$$u\equiv\frac{r_{12}r_{34}}{r_{13}r_{24}}\,,\qquad v\equiv\frac{r_{14}\,r_{23}}{r_{13}\,r_{24}}\,, \tag{3.3.28}$$

and

$$x=\frac{\rho\,u^2}{1+\rho\,u^2}\,,\quad y=\frac{\rho v^2}{1+\rho\,v^2}\,,\quad \rho=\frac{2}{1-u^2-v^2-\lambda}\,,\quad \lambda=\sqrt{(1-u^2-v^2)^2-4\,u^2\,v^2}\,.$$

one finds that

$$S_{\mathrm{int}}=\frac{\lambda}{16\pi^2}\,\log\frac{r_{13}r_{24}}{r_{12}r_{34}}+\frac{\lambda}{16\pi^2}\,\log\frac{r_{14}r_{23}}{r_{12}r_{34}}+\frac{\lambda}{16\pi^2}\,\log(r_{12}\,r_{34})+S'_{\mathrm{int}}\,, \tag{3.3.29}$$

where

$$S'_{\mathrm{int}}\equiv\frac{\lambda}{16\pi^2}\frac{1}{(r_{14}r_{23}+r_{13}r_{24})^2}\left(H\,r_{14}^2\,r_{23}^2-r_{13}^2r_{24}^2\log\frac{r_{13}r_{24}}{r_{12}r_{34}}-r_{14}^2r_{23}^2\log\frac{r_{14}r_{23}}{r_{12}r_{34}}\right)\,. \tag{3.3.30}$$

Thus, altogether, we obtain

$$\langle\phi(x_1)^n\,\phi(x_2)^n\,\bar\phi(x_3)^n\,\bar\phi(x_4)^n\rangle=\frac{(n!)^2}{(4\pi^2)^{2n}}\frac{(r_{14}r_{23}+r_{13}r_{24})^{2n}\,(r_{12}r_{34})^{\frac{\lambda}{16\pi^2}}}{(r_{14}r_{23}r_{13}r_{24})^{2\Delta}}\;e^{-S'_{\mathrm{int}}}\,. \tag{3.3.31}$$

The final expression (3.3.31) has the symmetries under the exchanges $x_1\leftrightarrow x_2$ and $x_3\leftrightarrow x_4$. These symmetries are not manifest in the term with H, but they can be shown to hold using standard properties of $\mathrm{Li}_2(x)$ (see discussion in appendix C of [176]). The four-point function (3.3.31) also has the expected singular behavior in the channels $x_1=x_3$, $x_1=x_4$, $x_2=x_3$, $x_2=x_4$, with a power governed by the full scaling dimension Δ of the operators, including the anomalous dimension. While the free part does not contain any singularity in the channels $r_{12}=0$ and $r_{34}=0$ because of charge conservation, due to the interaction, there is a behavior $(r_{12}r_{34})^{\frac{\lambda}{16\pi^2}}$. This behavior was already present in the extremal correlators. The terms with $\log r_{12}$ and $\log r_{34}$ in S'_{int} exactly cancel out with similar terms originating from H in the limit where either $r_{12}\to 0$ or $r_{34}\to 0$, so there is no extra contribution to this behavior. As a non-trivial check, we must recover the extremal three-point function (3.3.18) in the limit $x_4 \to x_3$. We obtain

$$\lim_{x_4\to x_3} \langle \phi(x_1)^n\, \phi(x_2)^n\, \bar\phi(x_3)^n\, \bar\phi(x_4)^n \rangle = \frac{(2n)!}{(4\pi^2)^{2n}} \frac{(\epsilon\, r_{12})^{\frac{\lambda}{16\pi^2}}}{r_{13}^{\bar\Delta} r_{23}^{\bar\Delta}}\ , \qquad \bar\Delta = 2n + \frac{\lambda}{8\pi^2}\ , \tag{3.3.32}$$

where $\epsilon = r_{34}$. This reproduces (3.3.18) for $n_1 = n_2 = n$, $m = 2n$, with an extra factor ϵ multiplying r_{12}. This factor is to be absorbed into the reference scale μ; see discussion in section 3.3.1.

The important prediction of the double-scaling limit is that the $O(\lambda)$ correction exponentiates. The saddle-point method exactly computes the full resummation of the surviving multiloop Feynman diagrams in the double-scaling limit. The saddle point approximation receives $1/n$ corrections that we are not computing and reorganize into a more general expansion in powers of $1/n$ and λ.

On the scale dependence of correlation functions

It is worth noting that the first two terms in (3.3.29) and $S'_{\rm int}$ are dimensionless quantities (which can in fact be written in terms of the standard conformal ratios). Upon restoring the reference mass scale μ, this appears only in the term $\log(\mu^2\, r_{12}\, r_{34})$ term. One may wonder how, in a CFT, a non-trivial dependence on a scale appeared in a correlation function. To understand this point, let us first consider the case of the two-point function written it in terms of dimensionless operators using the reference scale μ. This leads to

$$\begin{aligned}\langle \Big(\frac{\phi(x_1)}{\mu}\Big)^n \Big(\frac{\bar\phi(x_2)}{\mu}\Big)^n \rangle \;&=\; \frac{n!}{(4\pi^2)^n} \frac{e^{-S_{\rm int}}}{(\mu\, |x_1 - x_2|)^{2n}} = \frac{n!}{(4\pi^2)^n} \frac{e^{-\frac{\lambda}{16\pi^2}\log(\mu\,|x_1-x_2|)}}{(\mu\,|x_1 - x_2|)^{2n}} \\ &=\; \frac{n!}{(4\pi^2)^n} \frac{1}{(\mu\,|x_1 - x_2|)^{2\Delta}}\ , \qquad \Delta = n + \frac{\lambda}{32\,\pi^2}\ . \end{aligned} \tag{3.3.33}$$

Thus, the μ dependence in the argument of the logarithm is precisely what it is required to soak up the dimensions of x as it should be for a correlator of dimensionless operators. In other words, the μ dependence in the argument of the logarithm is reflecting the fact the operator has anomalous dimension.

Now consider the four-point function (3.3.31). Similarly, the factor of μ arising from the term $\log(\mu^2\, r_{12}\, r_{34})$ in (3.3.29) combines with the factor μ^{-4n} to give a net factor $\mu^{-4\Delta}$, which is, in this case, the expected factor given that each operator has dimension Δ. Restoring the

μ dependence, the non-extremal four-point function is given by

$$\begin{aligned}\frac{1}{\mu^{4n}}\langle\phi(x_1)^n\phi(x_2)^n\bar{\phi}(x_3)^n\bar{\phi}(x_4)^n\rangle &= \frac{(n!)^2}{(4\pi^2)^{2n}}\frac{\mu^{4n}(r_{14}r_{23}+r_{13}r_{24})^{2n}\,(\mu^2 r_{12}r_{34})^{\frac{\lambda}{16\pi^2}}}{\mu^{8\Delta}\,(r_{14}r_{23}r_{13}r_{24})^{2\Delta}}\; e^{-S'_{\rm int}}\\ &= \frac{(n!)^2}{(4\pi^2)^{2n}}\frac{(r_{14}r_{23}+r_{13}r_{24})^{2n}\,(r_{12}r_{34})^{\frac{\lambda}{16\pi^2}}}{\mu^{4\Delta}\,(r_{14}r_{23}r_{13}r_{24})^{2\Delta}}\; e^{-S'_{\rm int}} \quad (3.3.34)\end{aligned}$$

One can check that the same property holds for the general extremal correlator (3.3.17): the only μ-dependence in $\mu^{-\sum_i n_i}\mu^{-m}\langle\phi(x_1)^{n_1}\cdots\phi(x_r)^{n_r}\,\bar{\phi}(y)^m\rangle$ is in a factor $\mu^{-\sum_i\Delta_i}\mu^{-\bar{\Delta}}$ on the RHS, as expected.

Generating functional for the free part

Here we shall compute general higher-points correlation functions for the free theory by computing the generating functional. This will also serve as a cross-check of the free ($\lambda=0$) part of the previous results. We consider the following correlation function:

$$\langle\prod_{i=1}^{r}e^{\alpha_i\phi(x_i)}\prod_{j=1}^{s}e^{\beta_j\bar{\phi}(y_j)}\rangle. \quad (3.3.35)$$

The desired (free) correlator (3.3.5) is then obtained by expanding the generating functional in powers of α_i and β_j and isolating the term with the required powers $n_i,\ m_j$. Including the source terms, the action is given by

$$S_{\rm free}=\int d^4x\left(\partial\bar{\phi}\,\partial\phi-\sum_i\alpha_i\delta(x-x_i)\phi-\sum_j\beta_j\delta(x-y_j)\bar{\phi}\right). \quad (3.3.36)$$

The functional integral is Gaussian and can be computed exactly, with no need of taking any large charge limit, by solving the saddle-point equations. These are given by

$$\partial^2\phi=-\sum_{j=1}^{s}\beta_j\delta(x-y_j)\ ,\qquad \partial^2\bar{\phi}=-\sum_{i=1}^{r}\alpha_i\delta(x-x_i)\ . \quad (3.3.37)$$

The advantage of working with exponential operators is that the equations have now the straightforward solutions

$$\phi(x)=\sum_{j=1}^{s}\beta_jG(x-y_j)\ ,\qquad \bar{\phi}(x)=\sum_{i=1}^{r}\alpha_iG(x-x_i)\ . \quad (3.3.38)$$

Substituting these solutions into the action we obtain

$$S_{\text{free}} = -\sum_{j=1}^{s}\sum_{i=1}^{r} \alpha_i \beta_j G(x_i - y_j) \ . \tag{3.3.39}$$

Using this formula, we reproduce the previous results for the free part of the *extremal* correlators in a straightforward way.

Let us now consider non-extremal correlators. These have a more complicated structure involving several sums of terms, which originate from many new possible contractions arising in Feynman diagrams. As an example, here we consider the four-point correlation function

$$G_4 \equiv \langle \phi(x_1)^{n_1} \phi(x_2)^{n_2} \bar{\phi}(y_1)^{m_1} \bar{\phi}(y_2)^{m_2} \rangle \ . \tag{3.3.40}$$

We have

$$\langle e^{\alpha_1 \phi(x_1)} e^{\alpha_2 \phi(x_2)} \, e^{\beta_1 \bar{\phi}(y_1)} e^{\beta_2 \bar{\phi}(y_2)} \rangle = e^{\alpha_1 \beta_1 G(x_1 - y_1)} e^{\alpha_2 \beta_2 G(x_2 - y_2)} e^{\alpha_1 \beta_2 G(x_1 - y_2)} e^{\alpha_2 \beta_1 G(x_2 - y_1)} \ .$$

Expanding in powers of $\alpha_i, \ \beta_j$ and isolating the terms with given powers $\alpha_1^{n_1} \alpha^{n_2} \beta_1^{m_1} \beta_2^{m_2}$, we find

$$G_4 = n_1! n_2! m_1! m_2! \sum_k \frac{G(x_1 - y_1)^k G(x_2 - y_2)^{k + n_2 - m_1} G(x_1 - y_2)^{n_1 - k} G(x_2 - y_1)^{m_1 - k}}{k!(n_1 - k)!(k + n_2 - m_1)!(m_1 - k)!} \tag{3.3.41}$$

Thus far this is exact, valid for any values of n_1, n_2, m_1, m_2, with the sum over k restricted to $k \geq 0$, $k \leq m_1$, $k \geq m_1 - n_2$, $k \leq n_1$.

Obtaining the correct asymptotic large charge behavior requires some care, as the approximation $(n-k)! \approx n!\, n^{-k}$ cannot be applied in (3.3.41) because this holds for $k \ll n$ and terms with $k \sim n$ give a relevant contribution to the sum. To illustrate this, let us consider in particular the case $n_1 = n_2 = m_1 = m_2 \equiv n$. Then we get

$$\begin{aligned} G_4 &= (n!)^4 \sum_{k=0}^{n} \frac{G(x_1 - y_1)^k G(x_2 - y_2)^k G(x_1 - y_2)^{n-k} G(x_2 - y_1)^{n-k}}{k!^2 (n-k)!^2} \\ &= \frac{n!^2}{(4\pi^2)^{2n}} \frac{1}{r_{14}^{2n}\, r_{23}^{2n}} \ {}_2F_1(-n, -n, 1, v^2) \, , \end{aligned} \tag{3.3.42}$$

where we renamed $(y_1, \, y_2) \to (x_3, \, x_4)$. This formula is in agreement with the results presented in [177] for the cases $n = 1$ and $n = 2$, given by (6.17) and (6.21) in [177] (for a real scalar field). Explicitly,

$$\frac{1}{r_{14}^{2n}\, r_{23}^{2n}}\, {}_2F_1(-n,\, -n,\, 1,\, v^2) = \begin{cases} u + \frac{u}{v} & \text{if} \quad n = 1\ , \\ u^2 + \frac{u^2}{v^2} + 4\,\frac{u^2}{v} & \text{if} \quad n = 2\ . \end{cases} \tag{3.3.43}$$

The missing term "1" in (6.2) of [177] is easily understood, as it comes from the identity operator which in the present $O(2)$ case cannot be exchanged in the $\phi(x_1)\,\phi(x_2)$ fusion due to charge conservation. As a further consistency check, in the limit $x_4 \to x_3$ we find

$$\langle \phi(x_1)^n\, \phi(x_2)^n\, \bar{\phi}(x_3)^{2n} \rangle = \frac{(2n)!}{(4\pi^2)^{2n}}\, \frac{1}{r_{13}^{2n}\, r_{23}^{2n}}\ , \tag{3.3.44}$$

which is precisely the free part of the 3-point function ($c.f$ eq.(3.3.18) for $\lambda = 0$).

The exact result (3.3.42) can be used to cross-check the free part computed earlier in (3.3.25). The asymptotic large n behaviour can be obtained from the integral representation of the hypergeometric function, which at large n is dominated by a saddle-point. This gives

$${}_2F_1(-n, -n, 1, v^2) \approx \frac{1}{\sqrt{4\pi n}}\, \frac{(1+v)^{1+2n}}{v^{\frac{1}{2}}}\ .$$

Substituting this formula into (3.3.42), we obtain

$$\langle \phi(x_1)^n\, \phi(x_2)^n\, \bar{\phi}(x_3)^n\, \bar{\phi}(x_4)^n \rangle \approx \frac{n!^2}{(4\pi^2)^{2n}}\, \frac{1}{\sqrt{4\pi n}}\, \frac{1}{(r_{14} r_{23} r_{24} r_{13})^n}\, \left(\sqrt{v} + \frac{1}{\sqrt{v}} \right)^{1+2n}\ . \tag{3.3.45}$$

For large n, this coincides with (3.3.25).

3.4 On the UV completion of the $O(N)$ model in $6 - \epsilon$ dimensions: a stable large charge sector

While in the previous sections we focused in the $O(2)$ model near 4 dimensions, we shall now consider the more general $O(N)$ model for various values of d. This is one of the most extensively studied quantum field theories, made of N real scalar fields ϕ^i and a quartic interaction $g\,(\vec{\phi}^{\,2})^2$. The theory has a number of interesting applications in $d \leq 4$ as it describes universal features of critical phenomena, including, for instance, a precise description of the second-order phase transition in the three-dimensional Ising model near criticality.

Besides its importance as a description of systems of great physical interest, the $O(N)$

model is also of relevance in the context of the AdS/CFT duality. It was conjectured [178] that the Vasiliev higher spin theories on AdS_{d+1} [179–184] are dual to the singlet sector of the $O(N)$ model in d dimensions. In recent years, the duality was investigated particularly in $d < 4$. It is thus natural to inquire how the duality could work for the $d > 4$ theories.

The upper critical dimension of the quartic interaction is 4. For $d < 4$ the $O(N)$ model flows to an IR fixed point which is typically strongly coupled [185, 186]. Despite this, several techniques have been developed to study the IR fixed point and their combination gives a qualitative picture of the physics of the model. In particular, large N methods, the ϵ-expansion around the upper/lower critical dimensions and, more recently, the bootstrap; have yielded a qualitative understanding of the $O(N)$ model in dimensions $2 < d < 4$, including approximate results for certain quantities such as scaling dimensions of simple operators.

Although for $d > 4$ the theory is not renormalizable by power counting, the large N analysis as in [187] nevertheless suggests that the theory may still admit a UV fixed point, a scenario akin to Weinberg's asymptotic safety(see also [188]). This opens the very interesting possibility to construct an interacting, strongly coupled, non-SUSY CFT in $d = 5$. The existence of a UV fixed point at large N is consistent with the observation in [189], drawing on the results in [190–195], that the continuation to negative ϵ of the standard ϵ-expansion yields sensible results, at least within perturbation theory. Indeed, for certain observables such as scaling dimensions of some operators, the continuation to negative ϵ –that is, in $d = 4 + |\epsilon|$ dimensions– of the expressions for generic d leads to compelling results for the scaling dimensions, as they lie within the unitarity bounds [187, 196, 197].

Fei, Giombi and Klebanov [189] proposed a UV completion of the quartic $O(N)$ model in terms of a cubic theory with $N+1$ fields and the same $O(N)$ global symmetry. The cubic $O(N)$ model was studied in the past [198] in an ϵ-expansion about its critical dimension $d = 6 - \epsilon$. By examining the one-loop β functions, it was found that, for $N > N_{\rm cr}$, with $N_{\rm cr} \sim 1038$, the theory has an IR fixed point. This result was reproduced in [189] , where it was shown that, at the IR fixed point, physical observables, such as scaling dimensions of operators, remarkably agree with their counterparts in the quartic model. A three loop analysis [199] suggested that $N_{\rm cr}$ at $\epsilon = 1$ may dramatically decrease up to $N_{\rm cr} \sim 64$, while four loop corrections [200] seem to place the value of $N_{\rm cr}$ around $N_{\rm cr} \sim 400$.

The striking agreement between the IR fixed point of the cubic model and the UV fixed point of the quartic model is non-trivial. The quartic model is most conveniently studied upon performing a Hubbard-Stratonovich (HS) transformation [187, 201], which effectively converts it into a cubic model albeit with no (tree level) dynamics for the HS scalar field. One might think that the cubic model proposed in [189] is just the resummation of the higher-loop momentum dependence of the HS field. However, as shown in [202], this is not the case, as the cubic model is at a larger universality class and it is only upon fine-tuning to a critical manifold that one can fall in the universality class of the quartic model with the same critical exponents.

In any case, the fixed points of either theory can only be at most metastable. Indeed, it is known from long ago that the quartic theory at $d > 4$ cannot admit a true fixed point [203]. A sign of this is the fact that, in $d = 4 + \epsilon$ dimensions, the continuation of the standard perturbative fixed point lies at negative values of the coupling g, implying an inverted, unstable, potential. In turn, while the IR fixed point [189] of the cubic theory which UV-completes the quartic model lives at positive values of the couplings, the potential is cubic and it is therefore unbounded from below. An analysis of the exact RG in these models [202, 204, 205] indeed shows that there is no fixed point. The problem was further studied in [206], where instanton instabilities of the unbounded potentials in both theories were computed. In particular, it was shown how the instanton saddle points are responsible for giving exponentially small imaginary parts to the scaling dimensions of the operators in both models.

In parallel, very recently it was appreciated that, on general grounds, the sectors of large charge under a global symmetry in a CFT enjoy special properties which make them analytically tractable (see, *e.g.* [148,149,151,158,162,207,208], and [153,161] for comparison with lattice simulations). This may be regarded as a generalization of very well known particular large charge limits extensively studied in the literature, such as the *pp*-wave limit or even large spin expansions in CFT. The key observation is that the correlation functions of operators with large charge under a global symmetry can be regarded, upon mapping to the cylinder, as a finite charge density state whose energy selects the relevant correlator. The scale of the charge density (operator charge) defines a UV cut-off, while the size of the sphere sets an IR cut-off. Then, provided one considers a large charge state which parametrically gives a large separation of the UV and IR scales, one can write an effective theory from which one can compute the desired correlation function. While this construction is universal, as it relies on generic assumptions, it has recently been realized that a microscopic derivation can also be provided. Focusing on the $O(2)$ model, one may consider large charge n operators in the sector where g scales as n^{-1}, so that n acts as $\hbar$. Hence, the large charge limit acts as a “classical limit" where $\hbar \sim n^{-1}$ is sent to zero at fixed $g\,n$. Of course, in the sense of the standard perturbative expansion in Feynman diagrams, this “classical limit" digs into the quantum regime of the theory as it resums infinitely many Feynman diagrams. An efficient way to select the leading relevant Feynman diagrams is to consider a similar large n limit but with fixed $g\,n^2$. One can then explicitly compute and resum these diagrams, dubbed “Kermit the frog" in [4].

The existence of this double scaling limit was first hinted for scalar theories long ago in [170–172], and very recently reconsidered in [4, 5, 147, 173]. Even though these studies concern scalar theories (mostly the $O(2)$ model), a similar double-scaling limit was found in $\mathcal{N} = 2$ SQCD in [165] and further studied in [166–168, 209] (see also [154, 156]). It is very interesting to note that in the large n limit of $\mathcal{N} = 2$ QCD the (Yang-Mills) instanton sector is exponentially suppressed [165] and thus the perturbative series (or, if summed, its continuation to strong coupling) is exact.

In view of the simplifications in large charge sectors, it is natural to apply these techniques to the study of the $O(N)$ model in $d > 4$. We shall see that in the sector of large charge operators the agreement between the description of the fixed points from the UV of the quartic theory and from the IR of the cubic theory can be explicitly checked, including the contribution of infinitely many diagrams in the standard Feynman expansion. In addition, just as in the $\mathcal{N} = 2$ SQCD case, instanton corrections are absent. This will have the implication that the sector of large charge operators is stable. In particular, the scaling dimensions of large charge operators do not have imaginary components (they are suppressed exponentially with the charge n, even at finite N). With this motivation in mind, in this section we will study a class of large charge operators which are in the $[n,\,0,\cdots,\,0]$ representation of the $O(N)$ group.[3] One natural way to do this, for even N, is to re-write the theory so that it explicitly exhibits $U(1)\times SU(N) \subset O(2N)$. It turns out that the correlators of the $[n,\,0,\cdots,\,0]$ in $O(2N)$ are computed by correlators of n-th powers of the fundamental of $SU(N)$. Using this, we will compute the (purely real) scaling dimension of the operators in the $[n,\,0,\cdots,\,0]$ of $O(2N)$ from their 2-point functions and explicitly check the agreement between the cubic and quartic models. We will also explicitly compute a class of higher-point functions in the same double-scaling limit, in particular, obtaining results for the three-point function consistent with conformal symmetry.

In the case of even n, we will see that the same correlator can be computed as well by means of correlators of meson operators represented by n-fold symmetrized powers of the adjoint of $SU(N)$. This can be carried over to a version of the theory projected to $U(1)$-

[3] As the models contain no pseudoscalar fields, in all cases, representations transform trivially under parity.

invariant states.

In order to solve the saddle-point equations, we will resort to perturbation theory. As a consequence, strictly speaking, our results are valid in $d = 6-\epsilon$ dimensions in the limit $\epsilon \to 0$. It is possible that, by including additional corrections, the results could be extrapolated to higher coupling so as to get to the region of $\epsilon = 1$, that is, to the $d = 5$ $O(N)$ model. This would provide a sector free of instabilities which may be of relevance in the AdS/CFT context, primarily in the case of the theory projected to the $U(1)$ singlet sector.

The rest of this section is organized as follows. In section 3.4.1 we introduce the class of operators that we consider and compute their 2-point functions in order to read off their scaling dimensions. We first do this in the context of the cubic model and then describe the agreement with the quartic model. We also discuss the absence of instanton contributions in the double scaling limit which thus renders this sector stable. In section 3.4.2 we compute higher-point functions for the so-called extremal case – a terminology borrowed from the supersymmetric case that alludes to correlators with exactly one insertion of an antiholomorphic field. In section 3.4.3 we compute the scaling dimension of meson operators. Some concluding remarks will be made in section 3.4.4, which includes a discussion of open problems. Finally, in the appendices we collect some useful formulas as well as a standard derivation of the relevant correlation functions, including combinatorial factors, for the cubic interaction.

3.4.1 Large charge operators in the cubic $O(2N)$ theory in $d = 6 - \epsilon$ dimensions

Our starting point is the $d = 6 - \epsilon$ dimensional theory investigated in [189,206]. It is defined by the action

$$S = \int d^d x \left(\frac{1}{2} \left(\partial \vec{\varphi}\right)^2 + \frac{1}{2} \left(\partial \eta\right)^2 + \frac{g_1}{2}\, \eta \left(\vec{\varphi}\right)^2 + \frac{g_2}{6}\, \eta^3 \right) . \tag{3.4.1}$$

Here $\vec{\varphi}$ is a vector of $O(2N)$. As discussed in the Introduction, this theory has an IR stable fixed point for $N > N_{\rm cr}$. The critical $N_{\rm cr}$ was estimated in [189, 198] to be $2N_{\rm cr} \sim 1038$ using the one-loop β functions. However, further analysis [199,200] suggests that higher loop corrections may, at $\epsilon = 1$, dramatically reduce this value. From the one-loop β functions, one finds that the theory (3.4.1) has an IR stable fixed point which at large N sits at [189]

$$g_1^* = \sqrt{\frac{6\,(4\pi)^3\,\epsilon}{2N}}\,\left(1+\mathcal{O}\Big(\frac{1}{N}\Big)\right)\,, \qquad g_2^* = 6\,\sqrt{\frac{6\,(4\pi)^3\,\epsilon}{2N}}\,\left(1+\mathcal{O}\Big(\frac{1}{N}\Big)\right)\,. \tag{3.4.2}$$

For $N < N_{\rm cr}$, the critical couplings move to the complex plane and the fixed point cannot describe a unitary theory.

Correlation functions for a class of large charge operators

In the theory (3.4.1), the elementary fields φ^i fill a vector representation of $O(2N)$, whose Dynkin labels are $[1,\,0,\cdots,\,0]_{D_N}$. Composite operators are then formed from their products and derivatives. Let us consider the class of operators formed solely by symmetrized (as we are dealing with bosons) powers of the φ^i. One can check that

$$\mathrm{Sym}^n\Big([1,\,0,\cdots,\,0]_{D_N}\Big) = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}[n-2i,\,0,\cdots,\,0]_{D_N}\,. \tag{3.4.3}$$

It is clear that all but the $i=0$ term in the sum in (3.4.3) will contain powers of $\vec{\varphi}^{\,2}$ when constructed in terms of fields. As an illustrative example, one may consider the case of even n, when the last term in the sum in (3.4.3) is $[0,\,0,\cdots,\,0]_{D_N}$; corresponding to $(\vec{\varphi}^{\,2})^{\frac{n}{2}}$. For reasons which will become clear momentarily, let us consider precisely the $i=0$ term in the sum in (3.4.3), *i.e.* the $[n,\,0\cdots0]_{D_N}$ representation of $O(2N)$.[4] Now, $O(2N)$ has a $SU(N)\times U(1)$ subalgebra. When branched in $SU(N)\times U(1)$, the $[n,\,0\cdots0]_{D_N}$ gives

$$[n,\,0,\cdots,\,0]_{D_N} \to \sum_{k=0}^{n}[n-k,\,0\cdots0,\,k]_{(\mathbf{n-2\,k})}\,, \tag{3.4.4}$$

where the RHS is meant to refer to A_{N-1} Dynkin labels and the subscript is the $U(1)$ charge.

Let us consider the $k=0$ term in the sum on the RHS of (3.4.4). In terms of fields, this representation is easily constructed: in terms of the complex combinations $\phi^I = \frac{\varphi^I+i\,\varphi^{I+N}}{\sqrt{2}}$ with $I=1,\cdots,N$, the action of the theory is

$$S = \int d^dx\,\left(|\partial\vec{\phi}|^2 + \frac{1}{2}\big(\partial\eta\big)^2 + g_1\,\eta\,|\vec{\phi}|^2 + \frac{g_2}{6}\,\eta^3\right)\,, \tag{3.4.5}$$

so that

[4] Let us stress that our $[n,\,0\cdots0]_{D_N}$ operator is composed out of n fields and thus has –classical– scaling dimension $\Delta_{\rm cl} = n\,(2-\frac{\epsilon}{2})$ in $d=6-\epsilon$. Of course, one may construct operators in the same representation by adding arbitrary powers of the singlet $|\vec{\varphi}|^2$, increasing arbitrarily its dimension.

$$[n,\, 0, \cdots, 0]_{(\mathbf{n})} = \phi^{I_1} \cdots \phi^{I_n}\,. \tag{3.4.6}$$

Consider now the operator $\mathcal{O}_n = (\phi^1)^n$. It has n indices and it has $U(1)$ charge n. Such operator can only be an entry of the $[n,\, 0, \cdots, 0]_{(\mathbf{n})}$ representation corresponding to $k = 0$ in the sum in (3.4.4). Moreover, since this operator does not contain any power of $|\vec{\phi}|^2 = \vec{\varphi}^{\,2}$, it can only correspond to the $i = 0$ term in the sum in eq. (3.4.3), that is, to the $[n,\, 0, \cdots, 0]_{D_N}$ representation of $O(2N)$. Thus, the operator $\mathcal{O}_n$ can only be an entry of the $[n,\, 0, \cdots, 0]_{D_N}$ representation of $O(2N)$ with classical scaling dimension $\Delta_{\rm cl} = n\,(2 - \frac{\epsilon}{2})$. In particular, it follows that, by computing correlators of $\mathcal{O}_n$, we determine the correlators of the $[n,\, 0, \cdots, 0]_{D_N}$ representation of $O(2N)$.

Thus, all in all, we will be interested on correlators of $\mathcal{O}_n$, from which we will read-off the correlators (and, in particular, the anomalous dimension) of the $[n,\, 0, \cdots, 0]_{D_N}$ in $O(2N)$. Let us stress that there is no other operator to which $\mathcal{O}_n$ can correspond to, other than the $[n,\, 0, \cdots, 0]_{D_N}$ of $O(2N)$, and thus, when computing correlators, there is no mixing to take into account.

The correlators of interest can be computed by the path integral

$$\langle \mathcal{O}_n(x_1)\, \overline{\mathcal{O}_n}(x_2)\rangle = Z^{-1} \int D\phi\, e^{-S_{\rm eff}}\,, \tag{3.4.7}$$

with $Z = \int D\phi\, e^{-S}$ and

$$S_{\rm eff} = \int d^dx \left(|\partial\vec{\phi}|^2 + \frac{1}{2}\big(\partial\eta\big)^2 + g_1\,\eta\,|\vec{\phi}|^2 + \frac{g_2}{6}\,\eta^3 - n\,\log(\phi_1)\,\delta(x-x_1) - n\,\log(\phi_1^*)\,\delta(x-x_2)\right).$$

It is convenient to extract an overall factor of n and re-scale fields and couplings as

$$\phi^I = \sqrt{n}\,\Phi^I\,, \qquad \eta = \sqrt{n}\,\rho \qquad g_1 = \frac{h_1}{\sqrt{n}}\,, \qquad g_2 = \frac{h_2}{\sqrt{n}}\,; \tag{3.4.8}$$

so that, up to a constant

$$S_{\rm eff} = n \int d^dx \left(|\partial\vec{\Phi}|^2 + \frac{1}{2}\big(\partial\rho\big)^2 + h_1\,\rho\,|\vec{\Phi}|^2 + \frac{h_2}{6}\,\rho^3 - \log(\Phi_1)\,\delta(x-x_1) - \log(\Phi_1^*)\,\delta(x-x_2)\right). \tag{3.4.9}$$

Since $S_{\rm eff}$ is a function of $(n,\, h_1,\, h_2)$, when inserted in (3.4.7), the correlator of interest will be a function of these same variables, and hence must admit a double expansion in n,

$h_i = g_i\sqrt{n}$ (this is similar to the double expansion discussed in [147] for $|\phi|^4$ theory). In particular, the scaling dimension of the operator $[n,\,0,\cdots,\,0]_{D_N}$ must have the form

$$\Delta_{[n,\,0,\cdots,\,0]_{D_N}} = n\sum_{k=0} n^{-k}\,\Delta_k(h_1,\,h_2)\,. \tag{3.4.10}$$

Note that, in (3.4.9), n plays the role of $\hbar^{-1}$, and it is thus the loop counting parameter. Written in this form, it is natural to consider the "classical limit"

$$n\to\infty\,,\qquad h_{1,\,2}\equiv \text{fixed}\,. \tag{3.4.11}$$

In this limit, the saddle point approximation becomes exact, and, in particular, selects the term $k=0$ in (3.4.10). Let us stress that, even though this is formally a classical limit, what plays the role of $\hbar$ is n^{-1}. This "classical limit" still resums an infinite series of Feynman diagrams in the conventional loop expansion (see Fig. 3.9).

The saddle-point equations are given by

$$\partial^2\Phi_I = h_1\,\rho\,\Phi_I\,,\qquad \partial^2\Phi_I^* = h_1\,\rho\,\Phi_I^*\,,\qquad I=2,\,\cdots,\,N\,, \tag{3.4.12}$$

$$\partial^2\Phi_1 + \frac{1}{\Phi_1^*}\,\delta(x-x_2) = h_1\,\rho\,\Phi_1\,,\qquad \partial^2\Phi_1^* + \frac{1}{\Phi_1}\,\delta(x-x_1) = h_1\,\rho\,\Phi_1^*\,, \tag{3.4.13}$$

$$\partial^2\rho = h_1\,|\vec{\Phi}|^2 + \frac{h_2}{2}\,\rho^2\,. \tag{3.4.14}$$

In the saddle-point method, the correlator is then determined as usual by the evaluation of the integrand of (3.4.7) on the solution to the saddle-point equations.

To further proceed, let us consider the regime where $h_1,\ h_2\ll 1$, so that we can solve (3.4.12), (3.4.13), (3.4.14) in perturbation theory. To begin with, note that $\Phi_I=\Phi_I^*=0$ for $I\neq 1$ is an exact solution. To order zero, (3.4.13), and (3.4.14) are solved by

$$\Phi_1^{(0)} = \frac{G(x-x_2)}{\sqrt{G(x_1-x_2)}}\,,\qquad \Phi_1^{(0)*} = \frac{G(x-x_1)}{\sqrt{G(x_1-x_2)}}\,, \tag{3.4.15}$$

$$\rho^{(0)} = 0\,, \tag{3.4.16}$$

where $G(x-y)$ is the the Green's function for the laplacian defined as

$$\partial^2 G(x-y) = -\delta(x-y)\,. \tag{3.4.17}$$

In the solution, we have set to one an arbitrary multiplicative constant in $\Phi_1^{(0)}$, and the inverse constant in $\Phi_1^{(0)*}$, as they cancel out in the computation of the action.

To the next order, given that only $\Phi_1^{(0)}$, $\Phi_1^{(0)*}$ are non-zero, (3.4.14) gives

$$\partial^2 \rho^{(1)} - \frac{h_2}{2}\,\rho^{(1)\,2} = h_1\,\frac{G(x-x_1)\,G(x-x_2)}{G(x_1-x_2)}\,. \tag{3.4.18}$$

It obviously follows that $\rho^{(1)}$ is of order h_1 itself. Hence, the RHS of (3.4.13) will be of order $\mathcal{O}(h_1^2)$, which in turn show that both $\Phi_1^{(1)}$ and $\Phi_1^{(1)*}$ will be of order $\mathcal{O}(h_1^2)$. Thus, to leading order in $h_{1,2}$, $\Phi_1^{(1)} = \Phi_1^{(1)*} = 0$, and we only need to solve (3.4.18). To that matter, let us introduce $\rho^{(1)} = h_1\,\varrho$. Then the equation becomes

$$\partial^2 \varrho - h_1\,h_2\,\varrho^2 = \frac{G(x-x_1)\,G(x-x_2)}{G(x_1-x_2)}\,. \tag{3.4.19}$$

In perturbation theory, we can approximate this equation by

$$\partial^2 \varrho = \frac{G(x-x_1)\,G(x-x_2)}{G(x_1-x_2)}\,. \tag{3.4.20}$$

The solution to this equation is simply

$$\varrho = -\frac{1}{G(x_1-x_2)}\int d^6x_3\,G(x-x_3)\,G(x_3-x_1)\,G(x_3-x_2)\,. \tag{3.4.21}$$

Therefore, the solution in perturbation theory to order $\mathcal{O}(h_i^2)$ is

$$\Phi_1 = \frac{G(x-x_2)}{\sqrt{G(x_1-x_2)}} + \mathcal{O}(h_i^2)\,, \qquad \Phi_1^* = \frac{G(x-x_1)}{\sqrt{G(x_1-x_2)}} + \mathcal{O}(h_i^2)\,, \tag{3.4.22}$$

$$\Phi_I = \Phi_I^* = 0\,, \qquad I = 2,\cdots,N\,, \tag{3.4.23}$$

$$\rho = -\frac{h_1}{G(x_1-x_2)}\int d^6x_3\,G(x-x_3)\,G(x_3-x_1)\,G(x_3-x_2) + \mathcal{O}(h_i^2)\,. \tag{3.4.24}$$

In order to compute the correlator, it only remains to evaluate the action on the saddle point solution. Let us write $S_{\text{eff}} = S_{\text{free}} + S_{\text{int}}$, with

$$S_{\rm free} = n \int d^d x \left(|\partial \vec{\Phi}|^2 + \frac{1}{2} (\partial \rho)^2 - \log(\Phi_1)\, \delta(x - x_1) - \log(\Phi_1^*)\, \delta(x - x_2) \right) , \tag{3.4.25}$$

and

$$S_{\rm int} = n \int d^d x \left(h_1\, \rho\, |\vec{\Phi}|^2 + \frac{h_2}{6}\, \rho^3 \right) . \tag{3.4.26}$$

Computing the free part, up to multiplicative constant, we find the expected factor $e^{-S_{\rm free}} = G(x_1 - x_2)^n \sim |x_1 - x_2|^{-2\,\Delta_{\rm cl}}$. Therefore

$$\langle \mathcal{O}_n(x_1)\, \overline{\mathcal{O}}_n(x_2) \rangle \sim \frac{e^{-S_{\rm int}}}{|x_1 - x_2|^{2\,\Delta_{\rm cl}}} . \tag{3.4.27}$$

It only remains to compute the interaction piece of the action evaluated on the saddle point solution. We obtain

$$-S_{\rm int} = \frac{n\, h_1^2}{G(x_1 - x_2)^2} \int d^6 x \int d^6 y\, G(x{-}x_1)\, G(x{-}x_2)\, G(x{-}y)\, G(y{-}x_1)\, G(y{-}x_2) . \tag{3.4.28}$$

Upon shifting $x \to -x + x_1$ and $y \to -y + x_1$, this becomes ($z = x_1 - x_2$)

$$-S_{\rm int} = \frac{n\, h_1^2}{G(z)^2} \int d^6 x \int d^6 y\, G(x)\, G(x - z)\, G(x - y)\, G(y)\, G(y - z) . \tag{3.4.29}$$

$S_{\rm int}$ involves the integral $I = \int d^6 x \int d^6 y\, G(x)\, G(x - z)\, G(x - y)\, G(y)\, G(y - z)$, which can be written as

$$I = \int \frac{d^6 P}{(2\pi)^6}\, e^{-i\,P\,z}\, \tilde{I} , \qquad \tilde{I} = \prod_{i=1}^{2} \int \frac{d^6 p_i}{(2\pi)^6}\, \tilde{G}(p_1)\, \tilde{G}(p_2)\, \tilde{G}(p_1 - p_2)\, \tilde{G}(p_1 + P)\, \tilde{G}(p_2 + P) . \tag{3.4.30}$$

where $\tilde{G}(p) = 1/p^2$. Two-loop integrals of this form have been computed in [**?**]. In $d = 6 - \epsilon$ dimensions one finds

$$\tilde{I} = \frac{\pi^{6-\epsilon}}{(2\pi)^{12}}\, (P^2)^{1-\epsilon} \left[-\frac{1}{3\,\epsilon^2} - \frac{3 - \gamma_E}{3\,\epsilon} + \text{finite} \right] . \tag{3.4.31}$$

Fourier-transforming and using the explicit expression for the Green's function (see appendix A), we finally find

$$I = \frac{1}{64\,\pi^3}\, G^2(z)\; \log|z|^2\,. \tag{3.4.32}$$

Thus

$$-S_{\rm int} = \frac{1}{64\,\pi^3}\, n\, h_1^2\; \log|x_1 - x_2|^2\,. \tag{3.4.33}$$

Using this result and the value of h_1 at the fixed point coming from (3.4.2), (3.4.8), to leading order in $\frac{1}{N}$ we find

$$-S_{\rm int} = \frac{3\,\epsilon\, n^2}{N}\, \log|x_1 - x_2|^2\,. \tag{3.4.34}$$

Then (3.4.27) becomes

$$\langle \mathcal{O}_n(x_1)\, \overline{\mathcal{O}}_n(x_2)\rangle \sim \frac{1}{|x_1 - x_2|^{2\,(\Delta_{\rm cl} + \gamma_{[n,\,0,\cdots,\,0]_{D_N}})}}\,, \tag{3.4.35}$$

where the anomalous dimension is given by

$$\gamma_{[n,\,0,\cdots,\,0]_{D_N}} = -\frac{3\,\epsilon\, n^2}{N}\,. \tag{3.4.36}$$

It is important to stress that, even though this result is to first order in perturbation theory at weak coupling in the h_i and in the "classical limit" defined by the large n limit, it resums an infinite series of Feynman diagrams in the usual perturbative expansion. To see this, note that expanding $e^{-S_{\rm int}} = 1 - S_{\rm int} + \cdots$, we have

$$\langle \mathcal{O}_n(x_1)\, \overline{\mathcal{O}}_n(x_2)\rangle = G(x_1 - x_2)^n + G(x_1 - x_2)^n\,(-S_{\rm int}) + \cdots \tag{3.4.37}$$

Using now the expression in (3.4.29) for the interaction and writing it in terms of the original g_i couplings, this is

$$\begin{aligned}\langle \mathcal{O}_n(x_1)\, \overline{\mathcal{O}}_n(x_2)\rangle &= G(x_1 - x_2)^n + \\ &2\,\frac{n^2}{2}\, g_1^2\, G(x_1 - x_2)^{n-2} \int d^6x \int d^6y\, G(x - x_1)\, G(x - x_2)\, G(x - y)\, G(y - x_1)\, G(y - x_2) + \cdots\,.\end{aligned} \tag{3.4.38}$$

In the second line in (3.4.38) we recognize precisely the diagram on the right panel in Fig. 3.7.[5]

[5]The factor of 2 in (3.4.38) deserves some discussion. Strictly speaking, the free correlator is $n!\,G(x_1-x_2)^n$. The NLO correction we are computing has an extra factor of 2 with respect to this $n!$, which is that in (3.4.38).

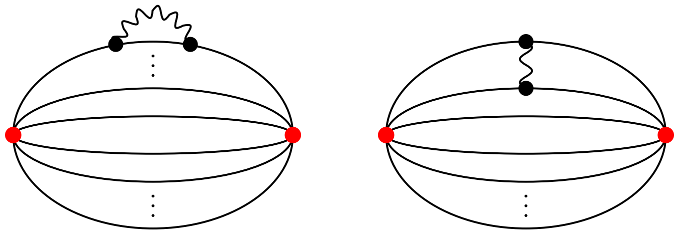


Figure 3.7: Leading order diagrams for the $\langle \mathcal{O}_n(x_1)\,\overline{\mathcal{O}}_n(x_2)\rangle$ correlator. Solid lines stand for ϕ propagators while wavy lines stand for ρ propagators.

Note in particular that the overall n^2 stands for the combinatorics: the ρ line connects each pair of ϕ lines, and there are $\frac{n\,(n-1)}{2} \sim \frac{n^2}{2}$ such lines, so that the diagram on the right panel in Fig. 3.7 is proportional to $n^2\,g_1^2$. Of course, at the same order in the g_i's there is the diagram on the left panel in Fig. 3.7, which would be proportional to $n\,g_2^2$. In our large n limit with fixed h_i, the diagram on the left panel is suppressed and only the diagram on the right panel contributes. Moreover, it is clear that the same logic goes through for the higher order terms in (3.4.38): the large n limit with fixed h_i limit will select the diagrams with the highest power of n. At weak coupling in the h_i's these are the natural "multi-ladder" generalization of the diagram on the right panel of figure 3.7 (see Fig. 3.9). These infinitely many diagrams exponentiate and give the formula (3.4.35), with the anomalous dimension given by (3.4.36). To be precise, in the leading correction in the second line of (3.4.38), all but the two lines connected by ρ are spectators. Thus we can factor out the n free propagators, so that, for all practical purposes, the relevant diagram is that of Fig. (3.8) –divided by $G(x_1 - x_2)^2$ to factor out the overall free correlator. Then, one can easily recognize that this is precisely $-S_{\text{int}}$ (*c.f.* (3.4.29)). It is thus this subdiagram what exponentiates in the large n, small h_i regime.

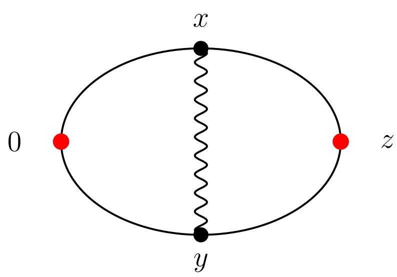


Figure 3.8: The subdiagram contributing to $-S_{\text{int}}$.

See appendix (5.E.2) for further discussion.

Exact resummation in the large n limit with couplings decreasing as $1/n$

It is possible to consider an alternative scaling where the leading order in the perturbation series in the h_i's couplings becomes exact. The correlation function (3.4.7) can be computed again by the saddle-point method, where we define rescaled fields as $\vec{\varphi} = g_1^{\frac{1}{2}}\vec{\phi}$. At the same time we define the couplings

$$\lambda_1 = g_1 n \ , \qquad \lambda_2 = g_2 n \ . \tag{3.4.39}$$

We now take the large n limit, this time with $\lambda_1,\ \lambda_2$ fixed. In this limit, the saddle-point equations become

$$\partial^2\varphi_I = 0 \ , \quad \partial^2\varphi_I^* = 0 \ , \quad I = 2, \cdots, N \ , \tag{3.4.40}$$

$$\partial^2\varphi_1 + \frac{\lambda_1}{\varphi_1^*}\delta(x - x_2) = 0 \ , \quad \partial^2\varphi_1^* + \frac{\lambda_1}{\varphi_1}\delta(x - x_1) = 0 \ , \tag{3.4.41}$$

$$\partial^2\eta = |\vec{\varphi}|^2 \ . \tag{3.4.42}$$

These equations reproduce the ones obtained for the leading order in the perturbation series in section 2.1, which, with the current scaling, become exact. This is the precise analog of the limit considered in [4, 5], with the difference that now there is an additional field η, which mediates the interaction. The saddle-point calculation in the double-scaling limit gives rise to the exponentiation of the second Feynman diagram of Fig. 3.7. The exponentiation corresponds to the full resummation of the Feynman diagrams of Fig. 3.9, which are the only surviving Feynman diagrams in this limit, all other diagrams being suppressed by powers of $1/n$. In the case of [4, 5], the relevant Feynman diagram corresponds to collapsing the η propagator to a point, i.e. setting $x = y$ in Fig. 3.8 (this gives rise to the "Kermit the frog"

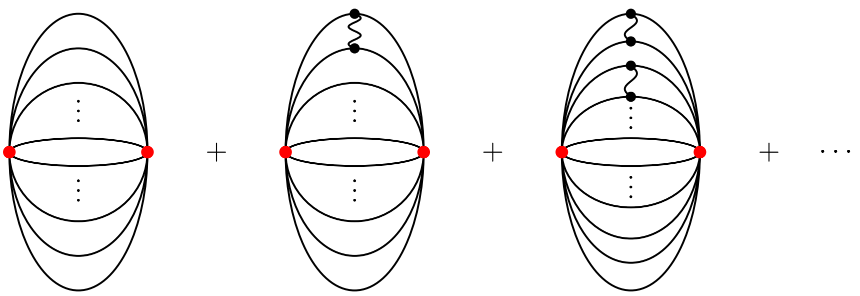


Figure 3.9: Resummation of the surviving Feynman diagrams in the large n limit with fixed $\lambda_1,\ \lambda_2$.

diagram of [4,5]).

In terms of the perturbation series in the $h_1,\ h_2$ couplings, organized as in section 2.1, this limit can be understood as follows. Recall that, on general grounds, the dimension of the operator is given by (3.4.10), where the Δ_k's are given by a perturbative series in $h_1,\ h_2$. At leading order in the $1/n$ expansion, we can keep the Δ_0 term alone. Δ_0 admits a perturbation series expansion in the $h_1,\ h_2$'s, which is schematically of the form (we collectively denote by h both $h_{1,2}$)

$$\begin{aligned}\Delta_{[n,0,\cdots,0]_{D_N}} &= n\,\Delta_0^{(0)} + \Delta_0^{(2)}\,n\,h^2 + \Delta_0^{(4)}\,n\,h^4 + \cdots \\ &= n\,\Delta_0^{(0)} + \Delta_0^{(2)}\,(\sqrt{n}\,h)^2 + \frac{1}{n}\,\Delta_0^{(4)}\,(\sqrt{n}\,h)^4 + \cdots\,. \end{aligned} \tag{3.4.43}$$

Here $\Delta_0^{(0)}$ is the classical dimension of a scalar in $d = 6-\epsilon$ dimensions, that is, $\Delta_0^{(0)} = 2 - \frac{\epsilon}{2}$.

In the limit where $h_i\sqrt{n} = \lambda_i$ is kept fixed, all but the leading correction are suppressed. Making use of the previous results, we obtain

$$\Delta_{[n,0,\cdots,0]_{D_N}} = 2\,n - \frac{\epsilon n}{2} + \Delta_0^{(2)}\,(\sqrt{n}\,h)^2 = 2\,n - \frac{\lambda_1^2}{64\,\pi^3} + \mathcal{O}(n^{-1})\,. \tag{3.4.44}$$

where we have used that $\epsilon \sim n^{-2}$ on the fixed point at fixed $\lambda_{1,2}$. Thus, the fixed $\lambda_1,\ \lambda_2$ limit reproduces the leading term of the perturbation series in $h_1,\ h_2$, where now $h_i \sim n^{-\frac{1}{2}}$, giving $g_i \sim \frac{1}{n}$ in terms of the original couplings.

On (the absence of) instanton contributions

Instanton contributions are typically of order $\exp[-\text{const.}/g_{1,2}^2]$ and are therefore exponentially suppressed at weak couplings. As the large n, double-scaling limit (3.4.8) requires $g_{1,2}^2 \sim 1/n$, instanton contributions will vanish exponentially as $\exp[-\text{const.}\,n]$. It is instructive to explicitly derive this result for the present model.

Let us first consider the equation (3.4.14) for ρ. Below we will show that the term $h_1|\vec{\Phi}|^2$ on the RHS can be neglected in a self-consistent approximation at weak coupling, as it carries higher powers of the couplings. Ignoring this term, the equation becomes

$$\partial^2\rho = \frac{h_2}{2}\,\rho\,. \tag{3.4.45}$$

It is well-known from long ago [210–212] –and recently described in this context in [206]– that this equation admits instanton solutions ρ_{inst} destabilizing the theory (and giving, in particular, imaginary parts to scaling dimensions). The solution is given

$$\rho_{\text{inst}} = -\frac{12}{h_2}\,\frac{4\,\lambda^2}{(1+\lambda^2\,(\vec{x}-\vec{a})^2)^2}\,. \tag{3.4.46}$$

Here $\vec{a}$ has the interpretation as the position of the instanton, while λ corresponds to its (inverse) size, both being moduli. Since the instanton solution is of order h_2^{-1}, this would justify neglecting the $|\vec{\Phi}|^2$ term in the (3.4.14) equation provided Φ is at most of order h_i^0. To check this, let us now turn to the Φ-equations. Of course, $\Phi_I = \Phi_I^* = 0$ for $I > 1$. The remaining equations are

$$\partial^2\Phi_1 + \frac{1}{\Phi_1^*}\,\delta(x-x_2) = -\frac{h_1}{h_2}\,\frac{48\,\lambda^2}{(1+\lambda^2\,(\vec{x}-\vec{a})^2)^2}\,\Phi_1\,, \tag{3.4.47}$$

$$\partial^2\Phi_1^* + \frac{1}{\Phi_1}\,\delta(x-x_1) = -\frac{h_1}{h_2}\,\frac{48\,\lambda^2}{(1+\lambda^2\,(\vec{x}-\vec{a})^2)^2}\,\Phi_1^*\,. \tag{3.4.48}$$

Since $\frac{h_1}{h_2} \sim \mathcal{O}(h_i^0)$, the solution to these equations is of order $\mathcal{O}(h_i^0)$, consistently with the assumption for the instanton solution to (3.4.14).

Evaluating the action on the solution, one obtains the instanton action given by

$$S_{\text{inst}} \sim n\,\frac{768\,\pi^3}{5\,h_2^2} + \mathcal{O}(h_i^0)\,. \tag{3.4.49}$$

Thus, in the large n limit (and actually for any value of h_i), the instanton action goes to ∞, and hence the instanton contribution is exponentially suppressed. The absence of instanton contributions in the large n, double scaling limit was first noticed in the supersymmetric context in [165]. Thus we conclude that, in the sector of large charge operators, instantons are completely suppressed even at finite N. This shows that large charge operators are free of instanton instabilities and therefore represent a stable sector with real scaling dimensions.

The quartic theory avatar

Let us consider the quartic $O(2N)$ theory with lagrangian

$$S = \int d^dx\left(\frac{1}{2}\left|\partial\vec{\varphi}\right|^2 + \frac{g}{4}\,(\vec{\varphi}^2)^2\right). \tag{3.4.50}$$

This theory has a UV fixed point at negative g in $4 < d < 6$ dimensions. It has been conjectured in [189] that the cubic theory (3.4.1) is a UV completion of this theory. Just as in the cubic theory, we will introduce the suitable complex combinations of fields to make explicit a $U(N)$ subgroup of the full $O(2N)$. The standard treatment [?, 187] that generates the $1/N$ expansion is by considering a Hubbard-Stratonovich transformation, which leads to the action

$$S = \int d^d x \left(|\partial \vec{\phi}|^2 + \sigma |\vec{\phi}|^2 - \frac{1}{4\,g}\,\sigma^2 \right) . \tag{3.4.51}$$

The original theory is recovered upon integrating out σ. At the (now UV) fixed point the last term can be dropped, and the dynamic is just described by the action

$$S = \int d^d x \left(|\partial \vec{\phi}|^2 + \sigma |\vec{\phi}|^2 \right) . \tag{3.4.52}$$

It is convenient to re-scale σ as in [189] so that the action becomes[6]

$$S = \int d^d x \left(|\partial \vec{\phi}|^2 + \frac{1}{\sqrt{N}}\,\sigma |\vec{\phi}|^2 \right) . \tag{3.4.53}$$

A 2-point function for σ is induced at one-loop [187]. In position space, it reads (see appendix (5.E.2), which includes a discussion of some relevant factors; see also [189])

$$\langle \sigma(x)\,\sigma(0)\rangle = \frac{C_d}{2\,(x^2)^2} \qquad C_d = \frac{2^{2+d}\,\Gamma(\frac{d-1}{2})\,\sin(\frac{\pi\,d}{2})}{\pi^{\frac{3}{2}}\,\Gamma(\frac{d}{2}-2)} . \tag{3.4.54}$$

Note that, if one substitutes $d = 6 - \epsilon$, one finds

$$\langle \sigma(x)\,\sigma(0)\rangle = \frac{1}{2}\,\tilde{C}_6\,G(x) ; \tag{3.4.55}$$

where $G(x)$ is the 6d (scalar) propagator, and

$$\tilde{C}_6 = (2\,N)\,g_1^{*\,2} , \tag{3.4.56}$$

being g_1^* the value of g_1 at the fixed point given by (3.4.2). Note that $g_1^{*\,2} \sim \epsilon$. Thus, to leading order in ϵ, $G(x)$ in (3.4.55) is just the 6d propagator.

The quartic theory (3.4.50) exhibits the same $SU(N)$ global symmetry as the cubic theory (3.4.5). Thus, we can consider the same $\mathcal{O}_n = (\phi^1)^n$ operator which, by the same arguments as above, can only belong to the dimension $\Delta_{\rm cl}$ operator in the $[n, 0\cdots 0]_{D_N}$ representation of the original $O(2N)$. Thus, the correlator $\langle \mathcal{O}_n(x_1)\,\overline{\mathcal{O}}_n(x_2)\rangle$ determines its

[6] Recall that we are considering the $O(2N)$ model. Yet, as ϕ is complex, in the one-loop contribution to the σ propagator N ϕ's are running.

anomalous dimension. Since the σ propagator is itself induced at one-loop, in the present formulation we do not have an easy path integral representation for the correlator. Yet, we can compute it directly in perturbation theory. The leading correction to the free theory is given by the diagrams in Fig. 3.10.

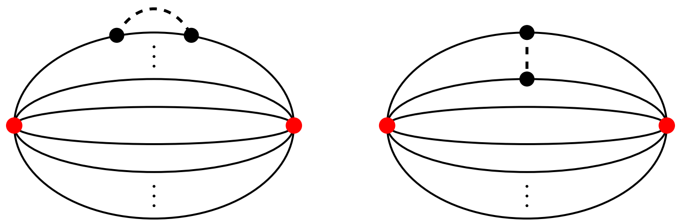


Figure 3.10: Diagrams contributing to the $\langle \mathcal{O}_n(x_1)\, \overline{\mathcal{O}}(x_2)\rangle$ correlator in the quartic theory to the leading-non-trivial order. Solid lines correspond to ϕ fields, while dashed lines are σ fields.

Just as for the cubic theory, the diagram on the left panel is suppressed with respect to the diagram on the right panel of Fig. 3.10 in the large n limit by a factor $1/n$. Hence, we only need to evaluate the diagram on the right. Moreover, the combinatorics of the diagram on the right panel of Fig. 3.10 are just as in the cubic case and thus, at large n, the diagram comes multiplied by $\frac{n^2}{2}$, giving (we include the aforementioned factor of 2 which cancels the $\frac{1}{2}$ in the σ propagator)

$$D = \frac{n^2}{2}\frac{\tilde{C}_6}{N}\, G(x_1-x_2)^{n-2}\int d^6x \int d^6y\, G(x-x_1)\, G(x-x_2)\, G(x-y)\, G(y-x_1)\, G(y-x_2)\,. \tag{3.4.57}$$

Using the explicit value of $\tilde{C}_6$ in (3.4.56), we find

$$D = n^2\, g_1^{*\,2}\, G(x_1-x_2)^{n-2}\int d^6x \int d^6y\, G(x-x_1)\, G(x-x_2)\, G(x-y)\, G(y-x_1)\, G(y-x_2)\,. \tag{3.4.58}$$

This precisely recovers the second line in (3.4.38) (evaluated at the fixed point), implying a striking match with the anomalous dimension computed from the cubic theory.

3.4.2 Extremal higher-point functions

The cubic theory

Let us now consider the correlation function of an arbitrary number of operators in representations $[n_i, 0, \cdots, 0]_{D_N}$, with $i = 1, \cdots, k$, and one operator in the conjugate representation $[\sum_{i=1}^{k} n_i, 0, \cdots, 0]_{D_N}$, with all n_i of order $n \gg 1$. Correlation functions of this form were dubbed *extremal* in [5]. The name is taken from superconformal field theories, where correlation functions of k chiral primary operators and one antichiral primary operator turn out to have a simpler structure because of supersymmetry. While here there is no supersymmetry, the extremal correlators are nevertheless far more simple than non-extremal correlators [5].

We shall now compute the correlation function

$$\langle \phi_1(x_1)^{n_1} \cdots \phi_1(x_k)^{n_k} \, \overline{\phi_1}(y)^{\sum_i n_i} \rangle \, . \tag{3.4.59}$$

The path integral representation of this correlator is given in terms of the action

$$\begin{aligned} S_{\text{eff}} &= n \int d^d x \Bigg(|\partial \vec{\Phi}|^2 + \frac{1}{2} \partial \rho^2 + h_1 \, \rho \, |\vec{\Phi}|^2 + \frac{h_2}{6} \, \rho^3 \\ &- \sum_{i=1}^{k} a_i \, \log(\Phi_1) \, \delta(x - x_i) - \sum_{i=1}^{k} a_i \, \log(\Phi_1^*) \, \delta(x - y) \Bigg) \, , \end{aligned} \tag{3.4.60}$$

where we have already extracted an overall factor of n and re-scaled fields and couplings just as in section 3.4.1. Moreover, we have written $n_i = a_i \, n$ and we shall consider the large n limit with all a_i fixed.

In the large n limit with fixed $h_{1,2}$ and a_i, the saddle point-approximation becomes exact. The corresponding saddle-point equations are

$$\partial^2 \Phi_I = h_1 \, \rho \, \Phi_I \, , \qquad \partial^2 \Phi_I^* = h_1 \, \rho \, \Phi_I^* \, , \qquad I = 2, \cdots, N \, , \tag{3.4.61}$$

$$\partial^2 \Phi_1 + \frac{\sum_{i=1}^{k} a_i}{\Phi_1^*} \, \delta(x - y) = h_1 \, \rho \, \Phi_1 \, , \qquad \partial^2 \Phi_1^* + \sum_{i=1}^{k} \frac{a_i}{\Phi_1} \, \delta(x - x_i) = h_1 \, \rho \, \Phi_1^* \, , \tag{3.4.62}$$

$$\partial^2 \rho = h_1 \, |\vec{\Phi}|^2 + \frac{h_2}{2} \, \rho^2 \, . \tag{3.4.63}$$

We follow the same procedure as in section 2.1, by solving these equations in the weak $h_{1,2}$ regime. For $I > 1$, one immediately has $\Phi_I = \Phi_I^* = 0$. In turn, for Φ_1, Φ_1^* and ρ one has (again we choose some constants judiciously)

$$\Phi_1 = G(x-y) + \mathcal{O}(h_i^2)\,, \qquad \Phi_1^* = \sum_{i=1}^{k} a_i \frac{G(x-x_i)}{G(x_i-y)} + \mathcal{O}(h_i^2)\,; \tag{3.4.64}$$

$$\rho = -h_1 \sum_{i=1}^{k} \frac{a_i}{G(x_i-y)} \int d^6z\, G(z-y)\, G(z-x_i)\, G(x-z) + \mathcal{O}(h_i^2)\,. \tag{3.4.65}$$

In order to compute the correlation function of interest we need to evaluate the action on this solution. Splitting S_{eff} in free and interaction pieces, with

$$S_{\text{free}} = n \int d^dx \left(|\partial\vec{\Phi}|^2 + \frac{1}{2}\partial\rho^2 - \sum_{i=1}^{k} a_i \,\log(\Phi_1)\,\delta(x-x_i) - \sum_{i=1}^{k} a_i \,\log(\Phi_1^*)\,\delta(x-y) \right), \tag{3.4.66}$$

and

$$S_{\text{int}} = n \int h_1\, \rho\, |\vec{\Phi}|^2 + \frac{h_2}{6}\,\rho^3\,, \tag{3.4.67}$$

up to a constant, one easily gets

$$e^{-S_{\text{free}}} = \prod_{i=1}^{k} G(x_i-y)^{n_i}\,. \tag{3.4.68}$$

On the other hand, one obtains the expression

$$-S_{\text{int}} = n\, h_1^2 \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{a_i\, a_j}{G(x_i-y)\, G(x_j-y)} \int d^6x \int d^6z\, G(x{-}y)\, G(x{-}x_i)\, G(z{-}y)\, G(z{-}x_j)\, G(x{-}z)\,. \tag{3.4.69}$$

For $x_i = x_j$, the integral is the same as the integral in (3.4.29), which leads to the result (3.4.32). More generally, we find

$$\int d^6x \int d^6z\, G(x-y)\, G(x-x_i)\, G(z-y)\, G(z-x_j)\, G(x-z) = \begin{cases} \frac{G(x_i-y)^2}{32\,\pi^3}\, \log|x_i-y|\,, & i=j\,; \\ \\ \frac{G(x_i-y)\,G(x_j-y)}{64\,\pi^3}\, \log\frac{|x_i-y|\,|x_j-y|}{|x_i-x_j|}\,, & i\neq j\,. \end{cases} \tag{3.4.70}$$

Thus, at the fixed point, we have

$$-S_{\mathrm{int}} = \sum_{i=1}^{k} \frac{6\,\epsilon\, n_i^2}{N}\, \log|x_i-y| + \sum_{i<j}^{k} \frac{6\,\epsilon\, n_i\, n_j}{N}\, \log\frac{|x_i-y|\,|x_j-y|}{|x_i-x_j|}\,. \tag{3.4.71}$$

Therefore, we finally find

$$\langle \phi_1(x_1)^{n_1} \cdots \phi_1(x_k)^{n_k}\, \overline{\phi_1}(y)^{\sum_i n_i}\rangle = \frac{\mathcal{N}}{\prod_{i=1}^{k} |x_i-y|^{4\,n_i - \frac{6\,\epsilon\, n_i\,(\sum n_j)}{N}}\, \prod_{i<j} |x_i-x_j|^{\frac{6\,\epsilon\, n_i\, n_j}{N}}}\,. \tag{3.4.72}$$

For $k=1$ this recovers the two-point correlation functions discussed above. For $k=2$, the formula (3.4.72) can be neatly encoded as

$$\langle \phi_1(x_1)^{n_1}\, \phi_1(x_2)^{n_2}\, \overline{\phi_1}(0)^{n_1+n_2}\rangle = \frac{\mathcal{N}}{|x_1|^{\Delta_1+\bar{\Delta}-\Delta_2}\, |x_2|^{\Delta_2+\bar{\Delta}-\Delta_1}\, |x_1-x_2|^{\Delta_1+\Delta_2-\bar{\Delta}}}\,, \tag{3.4.73}$$

which is the expected form for the three-point function in a CFT as dictated by conformal symmetry, with the precise expected dimensions for the operators

$$\Delta_i = 2\,n_i - \frac{3\,\epsilon\, n_i^2}{N}\,, \qquad \bar{\Delta} = 2\,(n_1+n_2) - \frac{3\,\epsilon\,(n_1+n_2)^2}{N}\,. \tag{3.4.74}$$

The quartic theory

In order to compute extremal correlators in these theories, we proceed as in section 3.4.1 by applying the saddle-point method. By expanding the interaction factor $e^{-S_{\mathrm{int}}}$ in powers of S_{int}, one can check that the diagrams contributing to the extremal correlators in the large n limit are those in figure 3.11. Just as in the 2-point case, one can easily show that these are indeed the Feynman diagrams that carry the highest power of n and are thus selected in our limit.

Just as for the 2-point functions, we can compute the relevant correlation function order

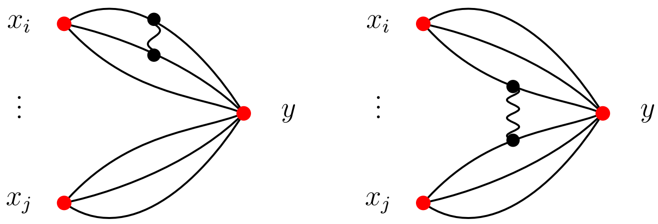


Figure 3.11: Types of diagrams that contribute to the extremal correlators in the cubic theory. These coincide with the relevant diagrams in the quartic theory upon replacing the ρ propagators by σ propagators.

by order in perturbation theory. In the end, since the combinatorics is just the same as in the cubic model, the relevant diagrams are formally identical. Thus, to check agreement of the two theories it is sufficient to check the first order. The corresponding diagrams are identical to the diagrams shown in figure 3.11, upon replacing the propagator lines of the elementary field ρ of the cubic model by the induced propagator of the HS field σ (denoted by a dashed line). However, (3.4.55), (3.4.56) show that, just as for the 2-point function, on the fixed point the diagrams with the exchanged scalar being σ will be identical to the diagrams in the cubic theory with the exchanged scalar ρ, thus ensuring the agreement of the two computations.

3.4.3 Correlation functions for meson operators

The branching of the $[n,\,0,\cdots,\,0]_{D_N}$ of $O(2N)$ into $U(1)\times SU(N)$ in (3.4.4) has an interesting particularity for even n. For $[2\,n,\,0,\cdots,\,0]_{D_N}$, the RHS of (3.4.4) contains, for $k=n$, the representation $[n,\,0,\cdots,\,0,\,n]_{\mathbf{0}}$ of $U(1)\times SU(N)$. For $n=1$, from the point of view of the theory written as in (3.4.5), such operator corresponds to a “meson" operator $\mathcal{M}^i_j = \bar{\phi}_j\,\phi^i$ in the adjoint of $SU(N)$, which is neutral under $U(1)$. Then, higher n corresponds to symmetrized n-fold products of this meson operator. In particular, it follows that correlators of the $[2\,n,\,0,\cdots,\,0]_{D_N}$ can also be computed through correlators of n-fold symmetrized powers of meson operators.[7]

Consider now $\mathcal{M}^n = (\phi^1\,\bar{\phi}_2)^n = (\phi^1\,(\phi^2)^*)^n$. This operator clearly belongs to $\mathrm{Sym}^n([1,\,0\cdots 0,\,0]\otimes[0,\,0\cdots 0,\,1])$. Moreover, since that this operator does not contain any trace, it can only be an element of the $[n,\,0\cdots 0,\,n]$ $SU(N)$ representation. Thus, we may compute correlators of

[7] Anomalous dimensions for similar meson operators in the quartic $O(N)$ model in $4-\epsilon$ dimensions were recently computed in [?].

the $[n,\, 0\cdots 0,\, n]$ representation by means of the correlator $\langle \mathcal{M}^n(x_1)\, \overline{\mathcal{M}}^n(x_2)\rangle$. Once again, $\mathcal{M}^n$ can only sit in the $[n,\, 0\cdots 0,\, n]$ representation and thus there is no allowed mixing. While we could read-off the correlators and dimensions for the $[2\,n,\, 0, \cdots, 0]_{D_N}$ operators from the previous computation by simply replacing n by $2n$, in the following we will explicitly compute the correlators of $\mathcal{M}^n$, both as a consistency check of the previous results and also for a further application that will be discussed in section (3.4.3).

Anomalous dimensions for mesons

We can give a path integral representation for the two-point correlation function:

$$Z^{-1} \int D\phi\ \mathcal{M}^n(x_1)\, \overline{\mathcal{M}}^n(x_2)\, e^{-S} = Z^{-1} \int D\phi\ e^{-S_{\text{eff}}}\,, \tag{3.4.75}$$

where now

$$S_{\text{eff}} = \int d^dx \left(|\partial \vec{\phi}|^2 + \frac{1}{2}\partial\eta^2 + g_1\, \eta\, |\vec{\phi}|^2 + \frac{g_2}{6}\, \eta^3 - n\, \log(\phi_1 \phi_2^*)\, \delta(x-x_1) - n\, \log(\phi_1^* \phi_2)\, \delta(x-x_2) \right). \tag{3.4.76}$$

Upon performing the same re-scaling as in section 2.1, we find

$$S_{\text{eff}} = n \int d^dx \left(|\partial \vec{\Phi}|^2 + \frac{1}{2}\partial\rho^2 + h_1\, \rho\, |\vec{\Phi}|^2 + \frac{h_2}{6}\, \rho^3 - \log(\Phi_1 \Phi_2^*)\, \delta(x-x_1) - \log(\Phi_1^* \Phi_2)\, \delta(x-x_2) \right). \tag{3.4.77}$$

In the double-scaling limit $n \to \infty$ with fixed h_i, the saddle-point approximation becomes exact. The saddle-point equations are now given by

$$\partial^2 \Phi_I = h_1\, \rho\, \Phi_I\,, \qquad \partial^2 \Phi_I^* = h_1\, \rho\, \Phi_I^*\,, \qquad I = 3, \cdots, N\,, \tag{3.4.78}$$

$$\partial^2 \Phi_1 + \frac{1}{\Phi_1^*}\, \delta(x-x_2) = h_1\, \rho\, \Phi_1\,, \qquad \partial^2 \Phi_1^* + \frac{1}{\Phi_1}\, \delta(x-x_1) = h_1\, \rho\, \Phi_1^*\,, \tag{3.4.79}$$

$$\partial^2 \Phi_2 + \frac{1}{\Phi_2^*}\, \delta(x-x_1) = h_1\, \rho\, \Phi_2\,, \qquad \partial^2 \Phi_2^* + \frac{1}{\Phi_2}\, \delta(x-x_2) = h_1\, \rho\, \Phi_2^*\,, \tag{3.4.80}$$

$$\partial^2 \rho = h_1\, |\vec{\Phi}|^2 + \frac{h_2}{2}\, \rho^2\,. \tag{3.4.81}$$

Just as in section 3.4.1, we may solve these equations in perturbation theory. To order $\mathcal{O}(h_i^2)$, we find

$$\Phi_1 = \frac{G(x-x_2)}{\sqrt{G(x_1-x_2)}} + \mathcal{O}(h_i^2)\,, \qquad \Phi_1^* = \frac{G(x-x_1)}{\sqrt{G(x_1-x_2)}} + \mathcal{O}(h_i^2)\,, \tag{3.4.82}$$

$$\Phi_2 = \frac{G(x-x_1)}{\sqrt{G(x_1-x_2)}} + \mathcal{O}(h_i^2)\,, \qquad \Phi_2^* = \frac{G(x-x_2)}{\sqrt{G(x_1-x_2)}} + \mathcal{O}(h_i^2)\,, \tag{3.4.83}$$

$$\Phi_I = \Phi_I^* = 0\,, \qquad I = 3, \cdots, N\,, \tag{3.4.84}$$

$$\rho = -\frac{2\,h_1}{G(x_1-x_2)} \int d^6x_3\, G(x-x_3)\, G(x_3-x_1)\, G(x_3-x_2) + \mathcal{O}(h_i^2)\,. \tag{3.4.85}$$

The non-trivial part of the correlator is given by the interaction term of the action, which reads

$$-S_{\rm int} = \frac{4\,n\,h_1^2}{G(x_1-x_2)^2} \int d^6x \int d^6y\, G(x-x_1)\, G(x-x_2)\, G(x-y)\, G(y-x_1)\, G(y-x_2)\,. \tag{3.4.86}$$

This involves the same integral computed before in (3.4.29). We thus obtain

$$-S_{\rm int} = \frac{1}{16\,\pi^3}\, n\, h_1^2\, \log|x_1-x_2|^2\,. \tag{3.4.87}$$

Substituting the values of the couplings at the fixed point (3.4.2), we find that the anomalous dimension of the operator is

$$\gamma_{[n,\,0\cdots 0,\,n]} = -\frac{12\,\epsilon\, n^2}{N}\,. \tag{3.4.88}$$

As anticipated, this is precisely the result in (3.4.36) upon replacing n by $2\,n$. Moreover, just as in section 2, in the large n limit with fixed h_i instantons are exponentially suppressed. Thus, in the $n \to \infty$ limit these operators do not have imaginary parts in their scaling dimensions.

The quartic theory

Let us now study the correlation functions of the same $U(1)$-invariant operators $\mathcal{M}^n$, $\overline{\mathcal{M}}^n$ in the quartic model (3.4.53). The computation of the two-point correlation function turns out to be essentially identical to that of the $\mathcal{O}_n$, $\overline{\mathcal{O}}_n$ operators, with the only difference that the combinatorial factor $\mathcal{C}$ of the diagram is now slightly different. Note that our correlator is a two-point function for the meson $\phi_1^n\,(\phi_2^*)^n$. Therefore, the tree level diagram contains $2\,n$ lines: n of ϕ_1 and n of ϕ_2^*. In this case, the diagram on the right panel of Fig. 3.10 now consists on a σ joining any pair of such lines. There are $\frac{2n\,(2n-1)}{2} \sim 2\,n^2$ such diagrams, so that $\mathcal{C} = 2\,n^2$. Thus, borrowing the computation from section 2, the final result is

$$\gamma_{[n,\,0\cdots 0,\,n]} = -\frac{12\epsilon\, n^2}{N}\,, \tag{3.4.89}$$

which precisely agrees with (3.4.88). Note that, once again, we have only computed the first correction to the anomalous dimension for large n operators. However, using the scaling of section 3.4.1, one can prove that the combinatorics is such that higher order corrections exactly exponentiate [4] with the same anomalous dimension as in (3.4.88).

Projection to the $U(1)$-invariant sector through gauging

Theories with matter in vector representations of a group G play a relevant role in the AdS/CFT correspondence upon projection to the singlet sector –*i.e.* thinking of G as a color group. In particular, according to [178], in $d = 5$ the $O(N)$ model should have an AdS_6 gravity dual description in terms of the higher spin theories. One may imagine generalizations of this setup where one considers the color group to be the unitary group, and/or one adds more species of fields in vector representations. In this spirit, since $O(2N) \supset U(1) \times SU(N)$, and given that our operators are neutral under the $U(1)$, we may regard $SU(N)$ as a global symmetry with the $U(1)$ factor as the color group, and project to the singlet sector by gauging it.[8] To that matter, we start with the action (3.4.5) and gauge the $U(1)$ in $U(1) \times SU(N) \subset O(2N)$. Besides trading derivatives by gauge-covariant derivatives, one should add all marginal terms to the lagrangian. This gives rise to a higher-derivative theory, which was recently studied in [213]. Such higher-derivative theory was shown to admit three IR fixed points: in two of the fixed points the $g_{1,\,2}$ couplings attain the same value (3.4.2) as in [189] (they differ in the values of other couplings which, as it will be clear below, are not important for our purposes). The third fixed point, dubbed FP_2 in [213], corresponds to critical scalar QED and it will not be interesting for our purposes.[9]

Let us consider the computation of two-point correlation functions of the meson opera-

[8]It is not difficult to generalize our set-up to $U(M) \times SU(N)$. Yet we will stick to the abelian case, which can be nicely embedded in the $U(1)$ gauge theories discussed in [213].

[9]The one-loop β function for general six-dimensional renormalizable models containing the Yang-Mills part was computed in [214] for a general gauge group.

tors in the higher-derivative gauge theory. Including the operator insertions to compute the correlators of interest and dropping the gauge-fixing term, we need to consider[10]

$$
\begin{aligned}
S_{\text{eff}} &= \int d^dx \bigg(|D\vec{\phi}|^2 + \frac{1}{2}\left(\partial\eta\right)^2 + g_1\,\eta\,|\vec{\phi}|^2 + \frac{g_2}{6}\,\eta^3 + \frac{1}{4}(\partial F)^2 + \frac{g_3}{2}\,\eta\,F^2 \\
&\quad - n\,\log(\phi_1\phi_2^*)\,\delta(x-x_1) - n\,\log(\phi_1^*\phi_2)\,\delta(x-x_2)\bigg)\,, \qquad D_\mu = \partial_\mu - i\,e\,A_\mu\,.
\end{aligned} \tag{3.4.90}
$$

Upon performing the change of variables in (3.4.8) and defining

$$
A_\mu = \sqrt{n}\,a_\mu\,, \qquad g_3 = \frac{h_3}{\sqrt{n}}\,, \qquad e = \frac{q}{\sqrt{n}}\,, \tag{3.4.91}
$$

one gets

$$
\begin{aligned}
S_{\text{eff}} &= n\int d^dx \bigg(|D\vec{\Phi}|^2 + \frac{1}{2}\left(\partial\rho\right)^2 + h_1\,\rho\,|\vec{\Phi}|^2 + \frac{h_2}{6}\,\rho^3 + \frac{1}{4}(\partial f)^2 + \frac{h_3}{2}\,\rho\,f^2 \\
&\quad - \log(\Phi_1\Phi_2^*)\,\delta(x-x_1) - \log(\Phi_1^*\Phi_2)\,\delta(x-x_2)\bigg)\,;
\end{aligned} \tag{3.4.92}
$$

where $f = da$ and the covariant derivative is now $D_\mu = \partial_\mu - i\,q\,a_\mu$.

We can now take the corresponding double-scaling limit where $n \to \infty$ while $\{q,\, h_i\}$ are held fixed. Then, the computation of the correlators once again boils down to the evaluation of (3.4.92) on the solution to the saddle point equations. These equations are now, *a priori*, more complicated due to the presence of the gauge field. However, evaluating the gauge current $j^\mu \sim i\,(\vec{\Phi}^\dagger\,\partial^\mu\vec{\Phi} - \text{h.c.})$ on the leading perturbative solution above, it is straightforward to see that it exactly vanishes. Therefore the configuration does not source the gauge field and thus the computation becomes identical to the computation of section 2. This result could have been anticipated, since the operator insertions sourcing the saddle point equations carry no electric charge (they are gauge-invariant operators), so that, at the "classical level" (recall that the large n limit is a classical limit) the gauge field is not excited. In summary, the projection to the $U(1)$-invariant sector through gauging does not change the two-point correlation function of mesons.

[10] The notation is slightly changed with respect to [213]. We also tune all mass parameters to zero. These include, in particular, the standard kinetic term for the gauge field. As these parameters have a large classical β function, in searching for an IR fixed point, one is forced to set them to zero.

3.4.4 Conclusions

The $O(N)$ model with the familiar quartic potential in $4 < d < 6$ possesses a UV fixed point in perturbation theory. The proposed UV completion in terms of a cubic model [189] leads to a perturbative IR fixed point, although it faces the expected problems of non-perturbative instabilities due to the fact that the potential is not bounded from below. In this section we have investigated sectors of large charge operators where these instabilities are exponentially suppressed at finite N. In addition, this sector enjoys special simplifications. To begin with, these operators do not mix with other operators. Hence, their scaling dimension can be directly read off from their 2-point functions. Moreover, correlation functions can be computed by the saddle-point method, using the same techniques as in [4, 5]. In the large n limit the saddle point approximation becomes exact and it resums an infinite series of Feynman diagrams.

The large n limit selects the diagrams with the largest combinatorial factor. This is analogous to the Kermit-the-frog diagrams dominating the analogous limit in $d = 4 - \epsilon$ dimensions as described in [4]. The main difference with respect to the quartic $O(2)$ theory discussed in [4] is that, in the present cubic model, the dominant Feynman diagrams contain an additional ρ propagator. However, in the relevant integration region – which determines the logarithmic behavior as $x_1 \to x_2$ – the ρ propagator is constant, so the result of integration is essentially the same. On the other hand, one can also understand the matching with the UV fixed point of the quartic $O(2N)$ theory: in the latter, and upon performing a convenient Hubbard-Stratonovich transformation, the combinatorial factor shows that the same Feynman diagrams are the dominant ones in the double-scaling limit. Furthermore, (3.4.55) together with (3.4.56) ensure that the contribution of each diagram in the quartic theory is the same as in the cubic theory, hence explaining the agreement between the two calculations.

The fixed point in $d > 4$ (IR for the cubic theory, UV for the quartic theory) occurs only in perturbation theory. This can be seen by means of the exact renormalization group, where no such extremum of the effective potential exists. The instability manifests itself through instanton corrections which give small imaginary parts to scaling dimensions. However, as we have argued, in the large charge sector instanton instabilities are washed out: in the double scaling limit, the scaling dimensions of large charge operators become real, as the imaginary part goes exponentially to zero. We have also discussed a double scaling limit with couplings scaling as $1/n$ –the fixed λ limit – where our result using perturbation theory in h_i becomes exact. In terms of the standard expansion in Feynman diagrams, the fixed

λ limit selects an infinite series of diagrams which can be summed with infinite radius of convergence (in the end, it recovers the exponential of the classical action). The infinite radius of convergence is consistent with the absence of instantons in the strict $n \to \infty$ limit. As far as the $1/n$ expansion is concerned, the existence of instanton contributions of order $\exp(-\text{const.}\, n)$ indicates that the series is asymptotic. Since instantons provide an imaginary part, we expect that the Borel transform has poles on the real axes, which occurs when the asymptotic series is not of alternate type. It would also be interesting to study the convergence properties of the perturbative expansion in h_i of large charge correlators and its implications/relations with instanton instabilities.

In this work we have studied the large n limit to first order in perturbation theory in the h_i's. There are a number of motivations to study anomalous dimensions beyond this regime. To begin with, it would be important to test the agreement between the quartic and the cubic theory to higher orders, where the cubic interaction in the singlet scalar field may give rise to new diagrams contributing to the anomalous dimensions. To leading order in $1/N$, the cubic interaction has no counterpart in the quartic model, but we expect that the effect will be compensated by higher order corrections received by the HS propagator of the σ-field. Moreover, the strong h_i regime should be able to probe the $\epsilon \to 1$ region just as in the $d = 4 - \epsilon$ case discussed in [147, 173]. In this manner one may explore the large charge sector of a $d = 5$ "CFT". Such CFT is expected [178] to have an AdS_6 gravity dual description through the higher spin theories of [179–184]. In this context, the model projected to the $U(1)$ singlet is of special relevance [178]. This projection can be implemented through the gauging of the $U(1)$ symmetry, as done in section 4.2. This connects to the higher-derivative theory investigated in [213] (see also [215]). A straightforward yet very interesting extension is to consider a $U(M)$ model projected to the singlet sector with $SU(N)$ global symmetry (a model in the class of [216]). The $U(M)$ would play the role of the $O(N)/U(N)$ symmetry in [178], of which the singlet sector is kept, and the $SU(N)$ global symmetry would provide a global symmetry whose large charge sector, potentially free of instabilities, may be investigated.

Methods based on large charge expansions, as the one discussed in this work and in [4, 147, 173], or based on the effective action (see *e.g.* [156]), may be useful for a number of applications. An interesting one is to explore non-gaussianity effects in cosmological scenarios [217]. Another potential application is the study of non-conformal theories at large density (see [218, 219] for related investigations), in particular scalar QCD or brane constructions such as [220]. Lastly, the method used in this thesis for computing correlation functions of large charge operators should also be applicable to other higher-dimensional theories (see *e.g.* [221, 222] for an overview). It would be very interesting to see if a double scaling limit exists in these cases and what Feynman diagrams are resummed.

3.5 Concluding remarks

In this chapter, we have investigated a number of features of the large charge expansion of Quantum Field Theories, directly in terms of the microscopic fields in the UV lagrangian. The main takeaway is that there seem to be many situations where a double scaling limit exists that greatly simplifies the computation of observables in scalar theories.

Let us make a quick recap. In section 3.2 we explored the existence of a double scaling limit in the O(2) model, whereby we take the coupling to be very small and focus on operators of very large charge. In this limit, we saw that there is one family of Feynman diagrams that dominates the two point functions –the so called Kermit the Frog diagrams–, and moreover all orders in perturbation theory in the coupling can be easily resumed into an exponential. We also provided an alternative derivation of this double scaling limit from the path integral point of view. In these terms, it becomes clear how the inverse of the charge plays the same role as $\hbar$ in standard perturbation theory and the resumed Feynman diagrams conspire to give precisely the contribution from the saddle around this new classical limit.

This double scaling limit is not exclusive to the O(2) model. It also exists in other scalar theories and dimensions other than 4. The most obvious application is that one can use it to compute critical exponents such as the anomalous dimension of operators in the large charge sector. This is a result of the computation of the two point function, but we also saw in section 3.3 that higher-point correlation functions can be computed in the double scaling limit. A curious point in this analysis is that it seems that the computation of so called *extremal* correlators is significantly easier than the non-extremal ones, a fact reminiscent of the situation for supersymmetric theories, even though *a priori* the reasons for this simplification have nothing in common.

A different application that we found for our double scaling limit concerns the extension of the epsilon expansion towards dimensions higher than 4, in section 3.4. Here, the Wilson-Fisher fixed point is unstable rather than stable, and a natural question to ask is whether or not it admits a UV completion where it can be regarded as an IR fixed point of a new theory. This question is slippery *per se*, because the negative coupling in the cuartic theory at the fixed point, as well as the cubic potential of the putative completion, mean that the theories have instabilities. The first simplification is that we can look to the large charge sector of the theory, and here the instabilities are suppressed. Moreover, several observables can be easily computed in the double scaling limit, and we find that they agree between the two different models, which provides evidence that they are indeed different descriptions for the same fixed point.

The large charge expansion of Quantum Field Theories is a vast topic, and there are many directions for further progress. Among them, a particularly interesting one would be to find a similar double scaling limit to the one discussed here but in gauge theories, such that the coupling at play is the gauge coupling. This would greatly expand the landscape of theories where one can make use of the large charge expansion in the double scaling limit, possibly including models of direct relevance for the description of Nature. Another reason to hope that such limit might exist is that in the supersymmetric case, precisely this limit, involving as a global symmetry the R-symmetry, does exist.

Another exciting avenue is to continue exploring the applications of the large charge expansion to the behaviour of fixed points of different theories and their relation. For example, one could explore the phenomenon of bifurcations in RG flows, i.e. when two fixed points collide as we change the parameters of the theory. This can be seen to give rise to phase transitions, and moreover one can also determine the type of phase transition from the mathematical properties of the bifurcation.

Before concluding, let us make a final comment. In the introduction to Chapter 2, we discussed how supersymmetry can be invoked as a simplifying assumption in order to further our understanding of QFT. In this chapter, we studied the large charge limit of scalar non-supersymmetric theories. The remarkable thing is that the exploration of such limit in the first place was directly inspired by the investigation of the large R-charge limit of $\mathcal{N} = 2$ gauge theories mentioned above. Therefore, this work also serves as a nice example to bring home the fact that indeed one can learn interesting features of Quantum Field Theories without supersymmetry by studying their supersymmetric cousins.

Chapter 4

Conclusions and outlook

In this thesis, we pursued two avenues to study the behaviour of Quantum Field Theories beyond the classical perturbative regime. These two parts are fairly distinct: in the first one we considered essentially a new family of QFTs whose gauge group has non-trivial topology, and studied mostly their basic kinematical aspects; on the other hand, in the second part we focused on very simple theories where we were able to find a limit that allowed the computation of various observables. Each of these parts has their own set of remaining puzzles, which we have briefly discussed in sections 2.5 and 3.5 respectively. Here we will attempt to speculate on possible ways to relate the two approaches and where these considerations might lead to.

Quite possibly the most salient one has to do with the recent developments on generalised global symmetries in higher dimensions, of which the theories with disconnected gauge groups are examples. While in that particular context the non-invertible symmetry was a 1-form symmetry, there are many other cases where one can have a non-invertible 0-form symmetry, under which the charged objects are local operators. A natural question to ask is then how do the non-invertible 0-form symmetries act on the Hilbert space. A systematic understanding of this point doesn't exist yet, but it appears to be the case that in most examples, the construction of the non-invertible symmetry relies on a related invertible transformation which fails to be a symmetry (for example, due to anomalies) but that can be cured in a way that induces the non-invertibility. In these cases, typically there is a sector of the Hilbert space where the action of the categorical symmetry is the same as the naive invertible transformation, while on the rest of it it maps the state to zero. If this is true in general, then much in the same way that one restricts to a sector of fixed charge (under an ordinary 0-form global symmetry) in the Hilbert space, and write down a large charge EFT for it; it should be possible to focus on the sector where a non-invertible symmetry acts in the naive way, and then look at subsectors with fixed charge, and try to proceed in an analogous way. This would in principle allow to essentially extend results from the large charge expansion to operators that are not charged under a naive 0-form global symmetry.

Another interesting possibility would be to use generalized symmetries to study relation between different fixed points in order to establish various dualities. Related to this, it would be desirable to understand if and how they get mapped between different sides of established dualities. More generically, an important question is what is the fate of these topological operators along the RG flow; and whether phenomena known to exist with invertible symmetries, such as enhancements etc. also occur.

Conclusiones y direcciones futuras

En esta tesis, hemos perseguido dos avenidas para estudiar el comportamiento de Teorías Cuánticas de Campos más allá del régimen perturbativo clásico. Estas dos partes están sensiblemente diferenciadas: en la primera consideramos lo que esencialmente es una nueva familia de TCCs cuyo grupo gauge tiene topología no trivial, y estudiamos sus aspectos cinemáticos más básicos; por otro lado, en la segunda parte nos centramos en teorías muy simples en las que hemos sido capaces de encontrar un límite que permite el cálculo de ciertos observables. Cada una de estas partes tiene su propio conjunto de preguntas pendientes, que ya hemos discutido brevemente en las secciones 2.5 y 3.5 respectivamente. Aquí trataremos de especular sobre posibles formas de relacionar estas dos aproximaciones y a dónde estas consideraciones nos podrían llevar.

Quizás lo más relevante tenga que ver con los recientes desarrollos en simetrías globales generalizadas en dimensiones superiores, de las cuales las teorías con grupos de gauge disconexos son ejemplos. Mientras en ese contexto particular la simetría no invertible era una simetría de 1-forma, hay muchos otros casos donde uno puede encontrar una simetría de 0-forma no invertible, bajo la cual los objetos cargados son operadores locales. Entonces, una pregunta natural es cómo actúan dichas simetrías de 0-forma no invertibles en el espacio de Hilbert. Aún no existe una comprensión sistemática de este punto, pero parece que en la mayoría de los ejemplos, la construcción de la simetría no invertible depende de una transformación invertible relacionada que no llega a ser una simetría (por ejemplo, debido a anomalías) pero que puede ser corregida de una manera que induce la no invertibilidad. En estos casos, típicamente hay un sector del espacio de Hilbert donde la acción de la simetría generalizada es la misma que la transformación invertible ingenua, mientras que en el resto del espacio de Hilbert los estados mapean a cero. Si esto es cierto en general, entonces, de la misma manera que uno se restringe a un sector de carga fija (bajo una simetría global de 0-forma ordinaria) en el espacio de Hilbert, y escribe una EFT de carga grande para él, debería ser posible enfocarse en el sector donde una simetría no invertible actúa de manera ingenua, y luego mirar subsectores con carga fija, tratando de proceder de una manera análoga. Esto en principio permitiría esencialmente extender los resultados de la expansión de gran carga a operadores que no están cargados bajo una simetría global de 0-forma ingenua.

Otra posibilidad interesante sería usar las simetrías generalizadas para estudiar la relación entre diferentes puntos fijos a fin de establecer varias dualidades. Relacionado con esto, sería deseable entender si y cómo dichas simetrías se mapean entre diferentes lados de dualidades establecidas. Más en general, una pregunta importante es qué sucede con estos operadores topológicos a lo largo del flujo del grupo de renormalización, y si fenómenos conocidos en el contexto de simetrías invertibles, como su acrecentamiento, etc., también ocurren.

Chapter 5

Appendices

5.A Appendix to "Discrete gauge theories of charge conjugation"

5.A.1 Symmetric spaces and real forms

In this appendix we offer a lightning summary of some relevant facts on symmetric spaces. For a more thorough review, see Chapter 28 in [50].

Let G be a Lie group and H a closed subgroup. In general, the quotient G/H is not a group, but it is a well-behaved topological space, called a *homogeneous space*. For instance, $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ is the sphere S^{2N-1} seen as the unit sphere of $\mathbb{C}^N$.[1]

Consider now the following situation: suppose G is a connected Lie group, with an involution (i.e. an automorphism of order 2) Θ such that the subgroup $K = \{g \in G|\Theta(g) = g\}$ is compact. Then the homogeneous space $X = G/K$ is a *symmetric space*, i.e. a Riemannian manifold in which around every point there is an isometry reversing the direction of every geodesic. The involution Θ, and the corresponding involution on the Lie algebra $\mathfrak{g}$ of G, which we denote θ, is called a *Cartan involution*. Let $\mathfrak{k}$ be the Lie algebra of K, or equivalently the $+1$ eigenspace of θ in $\mathfrak{g}$. It is natural to also introduce the -1 eigenspace, that we call $\mathfrak{p}$. We have clearly

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,. \tag{5.A.1}$$

[1]Similarly, S^{N-1} seen as the unit sphere of $\mathbb{R}^N$ is $\mathrm{SO}(N)/\mathrm{SO}(N-1)$, and S^{4N-1} seen as the unit sphere of $\mathbb{H}^N$ is $\mathrm{Sp}(N)/\mathrm{Sp}(N-1)$.

$$
\begin{array}{lccc}
 & \text{non-compact space} & & \text{compact space} \\
\text{Groups } \Theta\text{:} & K \subset G,\ X = G/K & \xleftrightarrow{\text{duality}} & K \subset G_c,\ X_c = G_c/K \\
 & \updownarrow & & \updownarrow \\
\text{Lie algebra:} & \theta \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} & \xleftrightarrow{\text{same complexification}} & \theta' \quad \mathfrak{g}_c = \mathfrak{k} \oplus i\mathfrak{p}
\end{array}
$$

Figure 5.1: Summary of the duality relations between homogeneous space X and X_c.

Now let's introduce another Lie algebra

$$\mathfrak{g}_c = \mathfrak{k} \oplus i\mathfrak{p}\,. \tag{5.A.2}$$

Both $\mathfrak{g}$ and $\mathfrak{g}_c$ have the same complexification $\mathfrak{g}_\mathbb{C}$. The involution θ induces an involution on $\mathfrak{g}_c$ defined by

$$x + iy \to x - iy \tag{5.A.3}$$

where $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$.

Now we go back to the level of the groups. Under good assumptions, $\mathfrak{g}_c$ is the Lie algebra of a *compact* and *connected* Lie group G_c, and both G and G_c can be embedded in the complexification $G_\mathbb{C}$. Moreover (5.A.3) can be lifted to G_c, which means $X_c = G_c/K$ is also a symmetric space.

In summary, we have two symmetric spaces X and X_c, one non-compact and one compact, which are said to be in *duality* (see figure 5.1). For instance, the sphere S^2 can be realized as the compact symmetric space $\mathrm{SU}(2)/\mathrm{SO}(2)$, the hyperbolic plane $\mathcal{H}$ as the non-compact symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and they are in duality. The duality between symmetric spaces is a generalization of this elementary example.

The pairs of (irreducible, simply connected) symmetric spaces have been classified by Cartan. There are three types of pairs:

- The Euclidean spaces;
- The pair with $G_c = (K \times K)/K$ and $G = (K_\mathbb{C})_\mathbb{R}$ where K is a compact simple Lie group (a member of the Killing-Cartan ABCDEFG classification);
- A pair in Table 28.1 of [50], which corresponds to the classification of noncompact real forms of the simple Lie algebras.

Here we are interested in those symmetric spaces where $G_c = \mathrm{SU}(N)$ for some N. Looking at the classification, we find that the candidates come from the third type, and are reported in the first three columns of Table 2.2.

5.A.2 Results for the unrefined Hilbert series

In this appendix we collect the results obtained for the unrefined Hilbert series $\mathrm{HS}^{\mathrm{I,II}}_{(N,F)}(t)$ with N colors and F flavors. Note that when N is even both the type I-action and the type II-action are possible.

$N=3,\ F=8,10$ **with action** Θ_{I}

$$\mathrm{HS}^{\mathrm{I}}_{(3,8)}(t) = \frac{1}{(1-t)^{32}(1+t)^{24}(1+t^2)^{12}(1+t+t^2)^{16}}\Big(1+8t+60t^2+352t^3+2180t^4+12240t^5 + 63615t^6+297072t^7+1271655t^8+5001104t^9+18251874t^{10}+62027176t^{11}+197358994t^{12}+ 589894792t^{13}+1662662266t^{14}+4431761456t^{15}+11202560833t^{16}+26916075192t^{17}+61602528492t^{18}+ 134547288976t^{19}+280922141406t^{20}+561538929032t^{21}+1076105342885t^{22}+1979421972312t^{23}+ 3498766636248t^{24}+5948607168296t^{25}+9737172113226t^{26}+15357420491872t^{27}+23355546914320t^{28}+ 34271353352936t^{29}+48550884100169t^{30}+66437452982600t^{31}+87857610599498t^{32}+ 112323553804264t^{33}+138879963090049t^{34}+166117154759136t^{35}+192266666483228t^{36}+ 215374877940064t^{37}+233536846417860t^{38}+245150314372704t^{39}+249146681474602t^{40}\ \ldots + \text{palindrome } + \ldots\ t^{80}\Big)\ ,$$

$$\mathrm{HS}^{\mathrm{I}}_{(3,10)}(t) = \frac{1}{(1-t)^{44}(1+t)^{32}(1+t^2)^{16}(1+t+t^2)^{22}}\Big(1+10t+94t^2+708t^3+5594t^4+40304t^5+ 267596t^6+1604770t^7+8823246t^8+44685068t^9+210162976t^{10}+922138360t^{11}+3793387031t^{12}+ 14685693384t^{13}+53699356234t^{14}+186024512912t^{15}+612212660929t^{16}+1918747129356t^{17}+ 5739475779538t^{18}+16417980228736t^{19}+44992209839201t^{20}+118311677930184t^{21}+ 298973084347420t^{22}+727001567961864t^{23}+1703229868953967t^{24}+3848902875668712t^{25}+ 8398044127896305t^{26}+17709753210444906t^{27}+36126291437128415t^{28}+71345154443802538t^{29}+ 136509440283280531t^{30}+253232898276985664t^{31}+455739121278331778t^{32}+796177311646870288t^{33}+ 1350951695000313907t^{34}+2227550842846449570t^{35}+3570900466255197137t^{36}+ 5567741522682300884t^{37}+8447064933353162776t^{38}+12474366711895916734t^{39}+ 17937609369569411305t^{40}+25123443718887660186t^{41}+34283553514238981759t^{42}+ 45592869670297954474t^{43}+59103639171661870052t^{44}+74701493375989226532t^{45}+ 92071217634978085051t^{46}+110680303143430918394t^{47}+129787178088343520066t^{48}+$$

$1484781225759038789 90t^{49} + 16573258710515209345 3t^{50} + 18051160744393661031 6t^{51}+$
$19185926860574930315 0t^{52} + 19900374240360908702 0t^{53} + 20144324563752255022 4t^{54}+$
$+ \ ... \ + \text{palindrome} + \ ... + t^{108}\Big) \ .$

$N = 4,\ F = 8, 10$ **with action** Θ_{II}

$$\text{HS}^{\text{II}}_{(4,8)}(t) = \frac{1}{(1-t^2)^{34}(1+t^2)^{17}}\Big(1 + 11t^2 + 749t^4 + 8520t^6 + 123173t^8 + 975504t^{10}+$$
$$7079801t^{12} + 37130520t^{14} + 168290287t^{16} + 606231681t^{18} + 1880386783t^{20} + 4837617956t^{22}+$$
$$10783278743t^{24} + 20384258878t^{26} + 33595129641t^{28} + 47516178744t^{30} + 58828027690t^{32}+$$
$$62834962052t^{34} + \ ... \ + \ \text{palindrome} \ + ... + t^{68}\Big) \ ,$$

$$\text{HS}^{\text{II}}_{(4,10)}(t) = \frac{1}{(1-t^2)^{50}(1+t^2)^{25}}\Big(1 + 20t^2 + 1880t^4 + 40559t^6 + 932570t^8 + 13749498t^{10}+$$
$$172341355t^{12} + 1684998864t^{14} + 13851616125t^{16} + 94630953820t^{18} + 552972551103t^{20}+$$
$$2770203725095t^{22} + 12073883443120t^{24} + 45987359734926t^{26} + 154444878746850t^{28}+$$
$$459222671967535t^{30} + 1216126216507310t^{32} + 2877699662424911t^{34} + 6109680294283385t^{36}+$$
$$11666292937742595t^{38} + 20092424985476080t^{40} + 31261869088087670t^{42} + 44025712808863775t^{44}+$$
$$56169284503495746t^{46} + 64994327796765700t^{48} + 68224551337259378t^{50} + \ ... \ + \text{palindrome} \ + ... \ t^{100}\Big)$$

$N = 4,\ F = 8, 10$ **with action** Θ_{I}

$$\text{HS}^{\text{I}}_{(4,8)}(t) = \frac{1}{(1-t^2)^{34}(1+t^2)^{17}}\Big(1 + 19t^2 + 621t^4 + 9672t^6 + 115781t^8 + 1012392t^{10} + 6929353t^{12}+$$
$$37647616t^{14} + 166763191t^{16} + 610159441t^{18} + 1871499527t^{20} + 4855440684t^{22} + 10751422823t^{24}+$$
$$20435224870t^{26} + 33521903017t^{28} + 47610887368t^{30} + 58717583354t^{32} + 62951199956t^{34}+$$
$$... \ + \ \text{palindrome} \ + ... + t^{68}\Big) \ ,$$

$$\text{HS}^{\text{I}}_{(4,10)}(t) = \frac{1}{(1-t^2)^{50}(1+t^2)^{25}}\Big(1 + 30t^2 + 1640t^4 + 43719t^6 + 903050t^8 + 13965248t^{10}+$$
$$171040855t^{12} + 1691679084t^{14} + 13821738043t^{16} + 94749067680t^{18} + 552555331397t^{20}+$$
$$2771531440035t^{22} + 12070052718828t^{24} + 45997431130604t^{26} + 154420650803330t^{28}+$$
$$459276181907479t^{30} + 1216017405986190t^{32} + 2877903862084869t^{34} + 6109325929218841t^{36}+$$
$$11666862552680995t^{38} + 20091575715527008t^{40} + 31263044887405650t^{42} + 44024199831283511t^{44}+$$
$$56171095173235402t^{46} + 64992311468943920t^{48} + 68226641217885546t^{50}+$$

$$+ \dots + \text{palindrome} + \dots t^{100} \Big)$$

$N = 5, F = 10$ **with action** Θ_{I}

$$\mathrm{HS}^{\mathrm{I}}_{(5,10)}(t) = \frac{1}{(1-t)^{52}(1+t)^{48}(1+t^2)^{24}(1+t^2+t^3+t^4)^{26}} \Big(1 + 22t + 284t^2 + 2706t^3 + 21955t^4 +$$
$$160914t^5 + 1095989t^6 + 6979246t^7 + 41658165t^8 + 233574566t^9 + 1234569365t^{10} + 6174964900t^{11} +$$
$$29339025390t^{12} + 132880692724t^{13} + 575483327555t^{14} + 2389678052368t^{15} + 9537108858707t^{16} +$$
$$36658340475690t^{17} + 135959694126589t^{18} + 487352408392372t^{19} + 1690878189035940t^{20} +$$
$$5685865819978940t^{21} + 18553353915421956t^{22} + 58812746144565240t^{23} + 181295374749401949t^{24} +$$
$$543973294401568114t^{25} + 1590097569959523153t^{26} + 4531884343550335332t^{27} + 12602966622005009583t^{28} +$$
$$34222732445449084068t^{29} + 90801798406026318027t^{30} + 235550865278275435154t^{31} +$$
$$597781158693692309598t^{32} + 1484941206577385534578t^{33} + 3612556855586202953706t^{34} +$$
$$8611425331844868499654t^{35} + 20123123002882735041990t^{36} + 46117967367942045961984t^{37} +$$
$$103701230641717242512770t^{38} + 228882161202628401398674t^{39} + 496044838564012314603553t^{40} +$$
$$1056013484156029947574972t^{41} + 2209065184079799283904974t^{42} + 4542364802182471087464116t^{43} +$$
$$9183898349160013048150427t^{44} + 18263102474622174109283076t^{45} + 35731344980304035652518168t^{46} +$$
$$68797198502279183054832396t^{47} + 130392515255665999661450280t^{48} + 243334669278371355251281076t^{49} +$$
$$447227865448283414970636444t^{50} + 809705050788821331767991526t^{51} + 1444419199557525884710374569t^{52} +$$
$$2539330673373178708242199168t^{53} + 4400400061161378562047041542t^{54} +$$
$$7517882728502831968413954866t^{55} + 12665124362834156846827184294t^{56} +$$
$$21043140994302550778160376372t^{57} + 34488341247592291002683019204t^{58} +$$
$$55765427955534322478937405226t^{59} + 88972575754699507596936788405t^{60} +$$
$$140091005097562491907119219110t^{61} + 217715405474146926177559832432t^{62} +$$
$$334004367894475080619545506914t^{63} + 505889843814484679526388720852t^{64} +$$
$$756580164670751794484968322138t^{65} + 1117380660642869503684842943144t^{66} +$$
$$1629837707627349788223179891574t^{67} + 2348184609132241268753454614722t^{68} +$$
$$3342030098734627900076782048544t^{69} + 4699182325256166921736227036528t^{70} +$$
$$6528444132112829102068477963998t^{71} + 8962152087005380332380602472677t^{72} +$$
$$12158166797853174056382018906264t^{73} + 16300962310475160862617427342532t^{74} +$$
$$21601416397311748248950447669348t^{75} + 28294881210629359687423170146093t^{76} +$$
$$36637125430881682085950853304052t^{77} + 46897794251366524433817202742132t^{78} +$$
$$59351139575785230185385458458708t^{79} + 74263933013520921469754722695984t^{80} +$$

$$918806865082836057212757426304 80t^{81} + 1124065609076466213116562071387 40t^{82}+$$

$$1359886253608144130940299427964 82t^{83} + 1626964169738456864290645387808 35t^{84}+$$

$$1925030117630131951882818402339 04t^{85} + 2252680223475788301931919859991 01t^{86}+$$

$$2607240526623854079116265350903 60t^{87} + 2984681368175134663818090766227 98t^{88}+$$

$$3379595477906856419930883403691 46t^{89} + 3785250736218532509770774761721 04t^{90}+$$

$$4193724308376181634544541205906 22t^{91} + 4596119399624982810148891141866 30t^{92}+$$

$$4982859649980094724026417435389 14t^{93} + 5344049696789726575317268662568 18t^{94}+$$

$$5669884286916456773762571387927 06t^{95} + 5951083146852647297715548575851 74t^{96}+$$

$$6179325197877664166284010963123 04t^{97} + 6347654093950598233866876830657 51t^{98}+$$

$$6450827739983193362098456818505 52t^{99} + 6485587470111656814576017566178 02t^{100} \ldots +$$

$$\text{palindrome } + \ldots + t^{200}\Big)$$

$N = 6$, $F = 12, 14$ **with action** Θ_{I} We report the results only for the disconnected component $\mathrm{HS}^{\mathrm{I},-}_{(N,F)}(t)$

$$\mathrm{HS}^{\mathrm{I},-}_{(6,12)}(t) = \frac{1}{(1-t^2)^{42}(1+t^2)^{37}}\Big(1 + 7t^2 + 69t^4 + 358t^6 + 2038t^8 + 8419t^{10} + 35209t^{12}$$
$$+ 118646t^{14} + 392133t^{16} + 1091925t^{18} + 2941220t^{20} + 6833264t^{22} + 15255425t^{24}$$
$$+ 29803863t^{26} + 55760142t^{28} + 92180215t^{30} + 145662506t^{32} + 204720814t^{34} + 274750067t^{36}$$
$$+ 329305773t^{38} + 376711462t^{40} + 385626520t^{42} + \ \ldots \text{ palindrome } \ldots \ + \ t^{84}\Big) \ ,$$

$$\mathrm{HS}^{\mathrm{I},-}_{(6,14)}(t) = \frac{1}{(1-t^2)^{54}(1+t^2)^{49}}\Big(1 + 9t^2 + 101t^4 + 654t^6 + 4357t^8 + 22320t^{10} + 111704t^{12}$$
$$+ 469641t^{14} + 1895000t^{16} + 6669349t^{18} + 22380498t^{20} + 66872433t^{22} + 190076679t^{24}$$
$$+ 487466405t^{26} + 1188492526t^{28} + 2638404185t^{30} + 5568826504t^{32} + 10772076177t^{34}$$
$$+ 19818706650t^{36} + 33573603786t^{38} + 54119513030t^{40} + 80595879849t^{42} + 114256971885t^{44}$$
$$+ 149990270920t^{46} + 187496330812t^{48} + 217354673235t^{50} + 239983501133t^{52}+$$
$$245894331898t^{54} + \ \ldots\text{palindrome}\ldots \ + t^{108}\Big) \ .$$

5.B Appendix to "Discrete gauging and Hasse diagrams"

5.B.1 Definition of $\widetilde{\mathrm{SU}}(N)$

The groups. We are interested in semidirect products $\mathrm{SU}(N) \rtimes_\Theta \mathbb{Z}_2$, defined by a group morphism $\Theta : \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{SU}(N))$. There are essentially two inequivalent choices for Θ, see Table 2 in [1]. For $g \in \mathrm{SU}(N)$, we define $\Theta^I_{+1}(g) = \Theta^{II}_{+1}(g) = g$ and

$$\Theta^I_{-1}(g) = (g^{-1})^T = \overline{g}\,, \qquad \Theta^{II}_{-1}(g) = -J_N(g^{-1})^T J_N = -J_N\overline{g}J_N \ , \tag{5.B.1}$$

where the bar denotes complex conjugation and the matrix J_{2N} reads

$$J_{2N} := \begin{pmatrix} 0 & -\mathbb{I}_{N\times N} \\ \mathbb{I}_{N\times N} & 0 \end{pmatrix} \ . \tag{5.B.2}$$

Moreover we note that Θ^{II}_{-1} is defined only for N even. When we discuss both cases together, we simply use the letter Θ. Spelling out the definition of the semidirect product, the group $\widetilde{\mathrm{SU}}(N)_{I,II}$ is the Cartesian product $\mathrm{SU}(N) \times \mathbb{Z}_2$ with group law defined by

$$(g,\epsilon)\cdot(g',\epsilon') = (g\Theta_\epsilon(g'), \epsilon\epsilon')\,. \tag{5.B.3}$$

Explicitly, we can write $\widetilde{\mathrm{SU}}(N)_{I,II}$ as a union of two connected components

$$\widetilde{\mathrm{SU}}(N)_{I,II} = \{(g,1) \mid g \in \mathrm{SU}(N)\} \cup \{(g,-1) \mid g \in \mathrm{SU}(N)\} \tag{5.B.4}$$

with the product rules

$$(g,1)\cdot(g',1) = (gg',1) \tag{5.B.5}$$
$$(g,1)\cdot(g',-1) = (gg',-1) \tag{5.B.6}$$
$$(g,-1)\cdot(g',1) = (g\Theta(g'),-1) \tag{5.B.7}$$
$$(g,-1)\cdot(g',-1) = (g\Theta(g'),1)\,. \tag{5.B.8}$$

From this we also have

$$(g,\epsilon)^{-1} = (\Theta_\epsilon(g^{-1}),\epsilon)\,. \tag{5.B.9}$$

and

$$(g',\epsilon')\cdot(g,\epsilon)\cdot(g',\epsilon')^{-1} = (g'\Theta_{\epsilon'}(g)\Theta_\epsilon(g')^{-1},\epsilon)\,. \tag{5.B.10}$$

The Lie Algebra. The Lie algebra of $\widetilde{\mathrm{SU}}(N)_{I,II}$ is

$$\mathfrak{g} = \{X \in \mathfrak{gl}(N, \mathbb{C}) \mid \mathrm{Tr}(X) = 0 \text{ and } X + X^\dagger = 0\}\,. \tag{5.B.11}$$

The involutions $\Theta_{-1}^{I,II}$ on $\widetilde{\mathrm{SU}}(N)_{I,II}$ descend to involutions on the Lie algebra defined by

$$\theta_{-1}^{I}(X) = -X^T\,, \qquad \theta_{-1}^{II}(X) = J_N X^T J_N\,. \tag{5.B.12}$$

This is also valid on the complexified Lie algebra, where the condition that $X + X^\dagger = 0$ is dropped. We can rewrite equation (5.B.10) for $(g, \epsilon) = (1 + X, 1) \equiv 1 + X$ with $X \in \mathfrak{g}$, and get the adjoint representation of $\widetilde{\mathrm{SU}}(N)_{I,II}$:

$$(g, \epsilon) \cdot X \cdot (g, \epsilon)^{-1} = g\theta_\epsilon(X)g^{-1}\,. \tag{5.B.13}$$

It is useful to compute the trace of $\theta^{I,II}$, and this can be done by expressing it on any basis of $\mathfrak{g}$. We use as a basis $\{(A_{ij})_{1\leq i<j\leq N}, (B_{ij})_{1\leq i<j\leq N}, (C_i)_{1\leq i<N}\}$ with $(A_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}$, $(B_{ij})_{kl} = i\,(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$ and $(C_i)_{kl} = i\,(\delta_{ik}\delta_{il} - \delta_{i+1,k}\delta_{i+1,l})$. The matrices A are eigenvectors of θ^I with eigenvalue $+1$ and the matrices B and C are eigenvectors with eigenvalue -1, so

$$\mathrm{Tr}\left(\theta^I\right) = 1 - N\,. \tag{5.B.14}$$

For θ^{II} with $N = 2n$ even, we note that the matrices A and B are permuted (with signs) and the eigenvectors are $A_{i,i+n}$ and $B_{i,i+n}$ with eigenvalue $+1$. Finally there is a contribution $+1$ from $\theta^{II}(C_n) = \sum_{1\leq i<N} C_i$, so the trace of θ^{II} is $2n + 1$:

$$\mathrm{Tr}\left(\theta^{II}\right) = 1 + N\,. \tag{5.B.15}$$

5.B.2 Maximal tori and Cartan Subgroups

Before writing characters for representation of a Lie group G, it is necessary to pick a subgroup which is parametrized by a collection of variables z_i (called fugacities, which can assume continuous or discrete range). For connected compact Lie groups, there is an obvious choice, which is a maximal torus $\mathrm{U}(1)^r$ where r is the rank of the group. The situation is much less clear when one considers disconnected groups. For general considerations, we refer the reader to [223, Chapter VII] and [224, Chapter I] for a discussion of the various Cartan subgroups, and to the series of papers by Lusztig starting with [225] for characters of disconnected groups. The case of $\widetilde{\mathrm{SU}}(N)_{I,II}$ is discussed more specifically in [226].

Here we simply give a brief and explicit exposition of the situation in the simplest non trivial case of $\widetilde{\mathrm{SU}}(3)_I$, the generalization to $\widetilde{\mathrm{SU}}(N)_{I,II}$ being straightforward.

Let us define the diagonal and anti-diagonal matrices

$$D(z_1, z_2, z_3) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} \qquad A(z_1, z_2, z_3) = -\begin{pmatrix} 0 & 0 & z_1 \\ 0 & z_2 & 0 \\ z_3 & 0 & 0 \end{pmatrix} . \tag{5.B.16}$$

The minus sign is there to ensure that $\det D(z_1, z_2, z_3) = \det A(z_1, z_2, z_3) = z_1 z_2 z_3$. We have

$$D(z_1, z_2, z_3) \in \mathrm{SU}(3) \Longleftrightarrow A(z_1, z_2, z_3) \in \mathrm{SU}(3) \Longleftrightarrow |z_1| = |z_2| = |z_3| = z_1 z_2 z_3 = 1 \tag{5.B.17}$$

Obviously we have a group morphism $T = \mathrm{U}(1)^2 \to \mathrm{SU}(3)$ given by $(z_1, z_2) \mapsto D\left(z_1, \frac{z_2}{z_1}, \frac{1}{z_2}\right)$. T has three interesting properties:

A. It is a maximal torus[2] of $\mathrm{SU}(3)$.

B. It is a large Cartan subgroup [224] of $\mathrm{SU}(3)$, i.e. it is equal to the set of elements that normalize a certain maximal torus (namely itself) and fixes the fundamental Weyl chamber.

C. Any element in $\mathrm{SU}(3)$ is conjugate to at least one element of T.

We want to see how this can be extended to $\widetilde{\mathrm{SU}}(3)$. The crucial point is that the three properties **A**, **B** and **C** are not equivalent in the context of disconnected groups.

In $\widetilde{\mathrm{SU}}(3)$, T is still a maximal torus. The corresponding large Cartan subgroup is the set of elements $g \in \widetilde{\mathrm{SU}}(3)$ such that $g^{-1}Tg = T$ and $g^{-1}Bg = B$ where B is the set of elements of the form $(M, 1)$ with M upper triangular. We find that the large Cartan subgroup is given by

$$T^+ = \{\varphi(z_1, z_2, \epsilon) \mid z_1, z_2 \in \mathrm{U}(1) , \quad \epsilon = \pm 1\} , \tag{5.B.18}$$

where we have defined

$$\varphi(z_1, z_2, \epsilon) = \begin{cases} \left(D\left(z_1, \frac{z_2}{z_1}, \frac{1}{z_2}\right), 1\right) & \text{if } \epsilon = 1 \\ \left(A\left(z_1, \frac{z_2}{z_1}, \frac{1}{z_2}\right), -1\right) & \text{if } \epsilon = -1 . \end{cases} \tag{5.B.19}$$

[2] A maximal torus is a compact, connected, abelian subgroup.

The product rules give

$$\varphi(z_1, z_2, \epsilon) \cdot \varphi(y_1, y_2, \eta) = \begin{cases} \varphi(z_1 y_1, z_2 y_2, \epsilon\eta) & \text{if } \epsilon = 1 \\ \varphi(z_1 y_2, z_2 y_1, \epsilon\eta) & \text{if } \epsilon = -1 \ . \end{cases} \tag{5.B.20}$$

This means that φ is an injective group morphism $\mathrm{U}(1)^2 \rtimes \mathbb{Z}_2 \to \widetilde{\mathrm{SU}}(3)$ where the semidirect product $\mathrm{U}(1)^2 \rtimes \mathbb{Z}_2$ is defined by

$$(z_1, z_2, \epsilon) \cdot (y_1, y_2, \eta) = \begin{cases} (z_1 y_1, z_2 y_2, \epsilon\eta) & \text{if } \epsilon = 1 \\ (z_1 y_2, z_2 y_1, \epsilon\eta) & \text{if } \epsilon = -1 \ , \end{cases} \tag{5.B.21}$$

so that the semidirect product can be identified with the wreath product $\mathrm{U}(1) \wr S_2$. Clearly, this group is not Abelian, and as a consequence its image T^+ by φ is not Abelian either.

A natural Abelian subgroup of $\mathrm{U}(1) \wr S_2$ is $T = \mathrm{U}(1)^2$ considered above. This is in fact the *small Cartan subgroup* [224] associated to T, defined as the centralizer of T, which in the present case is equal to T. Clearly this is not relevant for our study of the disconnected component of $\widetilde{\mathrm{SU}}(3)$.

Another natural Abelian subgroup is $\mathrm{U}(1) \times \mathbb{Z}_2$ where the first factor is the diagonal subgroup of T. Its image in $\widetilde{\mathrm{SU}}(3)$ is

$$T^0 = \{\varphi(z, z, \epsilon) \mid z \in \mathrm{U}(1) \, , \quad \epsilon = \pm 1\} \ . \tag{5.B.22}$$

Property **C** fails here: clearly not every element of $\widetilde{\mathrm{SU}}(3)$ is conjugate to an element of T^0. Note however that every element of the disconnected part of $\widetilde{\mathrm{SU}}(3)$ is conjugate to an element of the disconnected part of T^0. This property is crucial in establishing a Weyl integration formula over $\widetilde{\mathrm{SU}}(3)$ [40].

Finally, consider the subgroup

$$\mathcal{T} = \{\psi(z_1, z_2, \epsilon) \mid z_1, z_2 \in \mathrm{U}(1) \, , \quad \epsilon = \pm 1\} \ , \tag{5.B.23}$$

where we have defined

$$\psi(z_1, z_2, \epsilon) = \left(D\left(z_1, \frac{z_2}{z_1}, \frac{1}{z_2} \right), \epsilon \right) \ . \tag{5.B.24}$$

Group	$F \oplus \overline{F}$ character
$T = T^-$	$z_1 + \frac{z_2}{z_1} + \frac{1}{z_2} + z_2 + \frac{z_1}{z_2} + \frac{1}{z_1}$
T^+	$\left(\frac{1+\epsilon}{2}\right)\left(z_1 + \frac{z_2}{z_1} + \frac{1}{z_2} + z_2 + \frac{z_1}{z_2} + \frac{1}{z_1}\right)$
T^0	$(1+\epsilon)\left(z + 1 + \frac{1}{z}\right)$
$\mathcal{T}$	$\left(\frac{1+\epsilon}{2}\right)\left(z_1 + \frac{z_2}{z_1} + \frac{1}{z_2} + z_2 + \frac{z_1}{z_2} + \frac{1}{z_1}\right)$

Table 5.1: Character for the $F \oplus \overline{F}$ representation of $\widetilde{\mathrm{SU}}(3)$ for various fugacity subgroups.

Group	Adjoint Character
T	$2 + \frac{z_1^2}{z_2} + z_1 z_2 + \frac{z_2^2}{z_1} + \frac{z_1}{z_2^2} + \frac{1}{z_1 z_2} + \frac{z_2}{z_1^2}$
T^+	$\left(\frac{1+\epsilon}{2}\right)\left(2 + \frac{z_1^2}{z_2} + \frac{z_2^2}{z_1} + \frac{z_1}{z_2^2} + \frac{z_2}{z_1^2}\right) + \epsilon\left(z_1 z_2 + \frac{1}{z_1 z_2}\right)$
T^0	$(1+\epsilon)\left(1 + z + z^{-1}\right) + \epsilon\left(z^2 + z^{-2}\right)$
$\mathcal{T}$	$\left(\frac{1+\epsilon}{2}\right)\left(\frac{z_1^2}{z_2} + z_1 z_2 + \frac{z_2^2}{z_1} + \frac{z_1}{z_2^2} + \frac{1}{z_1 z_2} + \frac{z_2}{z_1^2}\right) + 2\epsilon$

Table 5.2: Character for the adjoint representation of $\widetilde{\mathrm{SU}}(3)$ for various fugacity subgroups.

The product rules give

$$\psi(z_1, z_2, \epsilon) \cdot \psi(y_1, y_2, \eta) = \begin{cases} \psi(z_1 y_1, z_2 y_2, \epsilon\eta) & \text{if } \epsilon = 1 \\ \psi(z_1 y_1^{-1}, z_2 y_2^{-1}, \epsilon\eta) & \text{if } \epsilon = -1 \end{cases} . \tag{5.B.25}$$

This means that ψ is an injective group morphism $\mathrm{U}(1)^2 \rtimes \mathbb{Z}_2 \to \widetilde{\mathrm{SU}}(3)$ where the semidirect product $\mathrm{U}(1)^2 \rtimes \mathbb{Z}_2$ is defined by

$$(z_1, z_2, \epsilon) \cdot (y_1, y_2, \eta) = \begin{cases} (z_1 y_1, z_2 y_2, \epsilon\eta) & \text{if } \epsilon = 1 \\ (z_1 y_1^{-1}, z_2 y_2^{-1}, \epsilon\eta) & \text{if } \epsilon = -1 \end{cases} \tag{5.B.26}$$

This is a different from (5.B.21). In (5.B.21) the $\mathbb{Z}_2$ acts on $\mathrm{U}(1)^2$ by permuting the two factors, while here is inverts elements in both factors and preserves the order. The subgroup $\mathcal{T}$ is not a Cartan subgroup, as it does not preserve the fundamental Weyl chamber. However its matrices are all diagonal, and therefore is well suited for deriving branching rules.

5.B.3 Characters

A representation of $\widetilde{\mathrm{SU}}(N)$ is a vector space V with a group morphism $\rho : \widetilde{\mathrm{SU}}(N) \to GL(V)$. Picking a basis for V, a finite dimensional representation is given by matrices $\rho(g, 1)$ and $\rho(g, -1)$ for each $g \in \mathrm{SU}(N)$, satisfying the product rule $\rho(g, \epsilon)\rho(g', \epsilon') = \rho(g\Theta_\epsilon(g'), \epsilon\epsilon')$. The character χ of this representation is the trace of these matrices: $\chi(g,\epsilon) = \mathrm{Tr}(\rho(g,\epsilon))$. Note that using (5.B.10) we have $\chi(g,\epsilon) = \chi(g'\Theta_{\epsilon'}(g)\Theta_{\epsilon}(g')^{-1},\epsilon)$ for any g, g', ϵ and ϵ'.

In particular

$$\chi(g,1) = \chi(hgh^{-1},1) \tag{5.B.27}$$

$$\chi(g,-1) = \chi(hg\Theta_{-1}(h)^{-1},-1) \tag{5.B.28}$$

$$\chi(g,1) = \chi(h\Theta_{-1}(g)h^{-1},1) \tag{5.B.29}$$

$$\chi(g,-1) = \chi(h\Theta_{-1}(g)\Theta_{-1}(h)^{-1},-1) \tag{5.B.30}$$

In order to express these characters we pick a diagonal form $g = \mathrm{Diag}(z_1,\ldots,z_N)$, as explained in the previous subsection. Let's see what the constraints above tell us about the function $\chi(z_i,\epsilon)$, taking the case of type I to illustrate. The third lines says that the character for $\epsilon = 1$ is invariant under $z \to z^{-1}$. The second line says that $\chi(g,-1) = \chi(hgh^T,-1)$ for any $h \in \mathrm{SU}(N)$. In particular for $h = \mathrm{Diag}(h_i)$ with $|h_i| = 1$ this gives $\chi(z_i,-1) = \chi(h_i^2 z_i,-1)$. In other words, $\chi(z_i,-1)$ can not depend on the z_i at all! Therefore it is a pure number that can be evaluated for $z_i = 1$.

5.C Appendix to "Non-invertible symmetries from discrete gauging and completeness of the spectrum"

5.C.1 Gukov-Witten operators and principal extensions

In the main text, we have studied the 1-form symmetries of pure gauge theories with disconnected gauge groups, using the known result that the topological Gukov-Witten operators should correspond to the conjugacy classes of elements in the centralizer of the identity component of the group. In this appendix, we find the same topological GW operators by direct computation of their linking with Wilson lines in the adjoint. This computation has two steps: first we identify all possible GW operators, and then we use (2.4.5) to find which of them link trivially with the Wilson line; these will be the topological ones.

Gukov-Witten operators where introduced in [129,130] as codimension two operators that preserve a certain amount of supersymmetry. This was done by finding solutions to Hitchin's equations for the gauge fields, with a singularity at the locus of the operator and prescribed boundary conditions. Said boundary conditions are specified by the monodromy when going around the singular locus, which is an element of the gauge group and whose conjugacy class is a gauge invariant that labels the different Gukov-Witten operators.

If the gauge group is connected, finding all posible GW operators is easy. A basic theorem of Lie theory tells us that in this case, any element of the group is conjugate to at least one element of the maximal torus, i.e. if we call G the group and T its maximal torus, the map

$$\begin{aligned} \mathcal{C} : G \times T \rightarrow & \quad G \\ (g,t) \quad \mapsto & \quad g^{-1} t g \end{aligned} \tag{5.C.1}$$

is surjective. More precisely, elements of T that are related by the action of the Weyl group $\mathcal{W}_{\mathfrak{g}}$ will give rise to the same conjugacy classes. For our purposes, this implies that the possible GW operators (topological or not) are labelled by elements of $T/\mathcal{W}_{\mathfrak{g}}$.

This statement is no longer true if the group is not connected. Still, if we restrict ourselves to the case of principal extensions (namely the group is a semidirect product of its connected component times its group of outer automorphisms), we have lemma 2.1 of [40], which is enough for our purposes. The statement in the case where the outer automorphism group is isomorphic to $\mathbb{Z}_2$ is that, if we call G^0 the identity component of the group and Θ the map such that $G = G^0 \rtimes_\Theta \mathbb{Z}_2$, then the map

$$\begin{aligned} \varphi : G \times T^\Theta \rightarrow & \quad G^{\text{disc}} \\ (g,t) \quad \mapsto & \quad g^{-1} \Theta(t) g \end{aligned} \tag{5.C.2}$$

is surjective onto the disconnected component of the group. Here $G^{\text{disc}} = \Theta \cdot G^0$ denotes said disconnected component, and T^Θ is the subgroup of the maximal torus of G^0 which is left invariant by the action of Θ. Therefore, GW operators specified by a monodromy transformation in the disconnected component of the gauge group can be labelled by elements in T^Θ.

Once all the GW operators have been identified, we look for the ones that link trivially with an adjoint Wilson line: these are the topological ones that generate the 1-form symmetry. From the linking coefficient (2.4.5) it follows that we need to solve

$$\chi_{\text{Adj}}(a) = \dim \text{Adj}\,, \tag{5.C.3}$$

where $a \in T$ if we are considering a GW operator in the connected component and $a = \Theta(t)$, $t \in T^\Theta$ if we are considering one in the disconnected component.

Example: $\widetilde{SU}(3)_I$

As an example, we can consider the principal extension of $SU(3)$. Its Lie algebra has three positive roots, α_1, α_2 and $\alpha_1+\alpha_2$, and the outer automorphism exchanges α_1 and α_2: thus, the invariant subgroup of the torus, T^Θ, precisely corresponds to the root $\alpha_1+\alpha_2$. In order to write down the characters, it's more convenient to use a modified basis for the fugacities, instead of the usual one, such that the one parameter subgroup corresponding to $\alpha_1+\alpha_2$ is parametrized by z_1^2. This can be achieved by selecting fugacities $z_1 z_2^3$ and z_1/z_2^3 for the α_1 and α_2 directions respectively [1,85]; note that in these terms the action of the outer automorphism is $z_2 \mapsto 1/z_2$. With this, the character of the adjoint evaluated in a conjugacy class in the connected and disconnected components gives

$$\chi_{\mathrm{Adj}}(t) = 2 + z_1^2 + \frac{1}{z_1^2} + z_1 z_2^3 + \frac{1}{z_1 z_2^3} + \frac{z_1}{z_2^3} + \frac{z_2^3}{z_1}\,, \quad t \in T \tag{5.C.4}$$

$$\chi_{\mathrm{Adj}}(\Theta(t')) = -z_1^2 - \frac{1}{z_1^2}\,, \quad t' \in T^\Theta \tag{5.C.5}$$

The dimension of the adjoint of $\widetilde{SU}(3)_I$ is equal to 8. Since the fugacities are complex numbers of modulus 1, we find that the topological GW operators correspond to elements in the connected component such that $z_1 = 1$ and $z_2^3 = 1$, or $z_1 = z_2 = -1$. The solutions with $z_1 = z_2 = 1$ and $z_1 = z_2 = -1$ are in fact one and the same, which can be seen from the fact that with this fugacity parametrization, the character of the fundamental is $\chi_F = z_1 z_2 + 1/z_2^2 + z_2/z_1$. The corresponding Gukov-Witten operator is the identity of the 1-form symmetry. The other two solutions $z_1 = 1$, $z_2 = e^{i\pi/3}$ and $z_1 = 1$, $z_2 = e^{2i\pi/3}$ correspond to different elements of the gauge group, but ones that get identified via conjugation with the generator of the $\mathbb{Z}_2$. Therefore, there is one non-trivial GW operator, with quantum dimension two, that generates the 1-form symmetry. This is the same result obtained from the centralizer computation in the main text.

An important remark is that $\chi_{\mathrm{Adj}}(\Theta(t')) = \dim(\mathrm{Adj})$ has no solutions for GW operators corresponding to the disconnected component. This is completely generic and due to the fact that, since T^Θ has always a smaller dimension than T, there will be fewer monomials in the corresponding character than it's needed to have solutions to the equation. Therefore, GW operators labelled by classes in the disconnected component can never be topological.

5.D Appendix to "The large charge limit of scalar field theories and the Wilson-Fisher fixed point at $\epsilon = 0$"

5.D.1 Real space renormalization

A relevant integral in our discussion is

$$\mathcal{K} = \frac{1}{G(0,\, x)^2} \int d^4z\, G(0,\, z)^2\, G(z,\, x)^2\,, \tag{5.D.1}$$

where the propagator is

$$G(x,\, y) = \frac{1}{4\,\pi^2}\,\frac{1}{(x-y)^2}\,. \tag{5.D.2}$$

After rotation to euclidean signature, the relevant integral to compute is

$$G(0,\, x)^2\,\mathcal{K} = -\frac{i}{(4\,\pi^2)^4}\int d^4z\,\frac{1}{z^4\,(x-z)^4}\,. \tag{5.D.3}$$

The integral can be easily computed following the regularization method of [175], i.e. using that, in $d = 4$,

$$\frac{1}{z^4} = -\frac{1}{4}\partial^2\left(\frac{\log z^2\,\Lambda^2}{z^2}\right)\,. \tag{5.D.4}$$

Note that there will be an identical contribution from the divergence at $z = x$, to be regulated just in the same way, and hence the value of the integral will be twice of the contribution at, say $z = 0$. We now substitute (5.D.4) into the integrand of (5.D.3) and integrate by parts. The resulting integral is convergent upon giving a small imaginary part to z, which does not affect the coefficient of the logarithmic term. The integral is then easily computed by going to polar coordinates. One arrives at

$$\mathcal{K} = -\frac{i}{8\,\pi^2}\,\log(\Lambda^2\,x^2)\,. \tag{5.D.5}$$

One may alternatively use the method of [227], which leads to the same result.

Next, consider the $d = 3$ case. The relevant integral is now

$$\int d^3z\,\frac{1}{z^3\,(x-z)^3}\,. \tag{5.D.6}$$

In this case, it can be regularized using the formula

$$\frac{1}{z^3} = -\frac{1}{2}\partial^2 \left(\frac{\log z^2\Lambda^2}{z^2}\right) , \tag{5.D.7}$$

and following just the same steps as in the $d = 4$ case.

5.E Appendix to "On the UV completion of the $O(N)$ model in $6 - \epsilon$ dimensions: a stable large charge sector"

5.E.1 Fourier transforms formulæ

In euclidean signature, the Fourier transform of $1/(x^2)^\alpha$ is given by

$$\frac{1}{(x^2)^\alpha} = \frac{(4\pi)^{\frac{d}{2}}\,\Gamma(\frac{d}{2}-\alpha)}{4^\alpha\,\Gamma(\alpha)} \int \frac{d^d p}{(2\pi)^d}\,\frac{e^{-ipx}}{(p^2)^{\frac{d}{2}-\alpha}} \,. \tag{5.E.1}$$

As an application, it follows that the Green's function in position space is

$$G(x) = \frac{4^{\frac{d-2}{2}}\,\Gamma(\frac{d-2}{2})}{(4\pi)^{\frac{d}{2}}\,(x^2)^{\frac{d-2}{2}}} \,. \tag{5.E.2}$$

5.E.2 Combinatorics and propagators in the (complex) scalar theory with cubic interaction

Let us consider the cubic theory with interaction Hamiltonian

$$H_{\text{int}} = \int d^d x\, g\,\rho\,\bar{\phi}\,\phi \,. \tag{5.E.3}$$

In the following we shall use the shorthand notation $G^{(\rho)}_{xy}$, $G^{(\phi)}_{xy}$ to denote ρ, ϕ propagators from x to y ($G^{(\rho,\,\phi)}_{xy} = G^{(\rho,\,\phi)}_{yx}$).

Using Dyson's formula, the expectation value of any quantity can be written as $\langle O(x)\rangle \equiv \langle O(x)\,e^{-H_{\text{int}}}\rangle$ where the LHS is to be evaluated in the interacting picture (that is, for all practical purposes, with free fields).

The 1PI diagram for the real scalar self-energy

The (leading) 1PI diagram for ρ in the cubic theory is

$$\frac{1}{2}\, g^2 \int d^d x \int d^d y\, \langle \rho(0)\, \rho(x)\, \bar{\phi}(x)\, \phi(x)\, \rho(y)\, \bar{\phi}(y)\, \phi(y)\, \rho(z) \rangle\,. \tag{5.E.4}$$

Note that the $\frac{1}{2}$ comes from the expansion to second order of the exponential of the interacting Hamiltonian. This gives

$$\frac{1}{2}\, g^2 \int d^d x \int d^d y\, G^{(\phi)}_{xy}\, G^{(\phi)}_{xy}\, \langle \rho(0)\, \rho(x)\, \rho(y)\, \rho(z) \rangle\,. \tag{5.E.5}$$

Performing the ρ contractions we obtain (we omit the disconnected terms)

$$\frac{1}{2}\, g^2 \int d^d x \int d^d y\, G^{(\phi)}_{xy}\, G^{(\phi)}_{xy}\, G^{(\rho)}_{0x}\, G^{(\rho)}_{yz} + \frac{1}{2} \int d^d x \int d^d y\, G^{(\phi)}_{xy}\, G^{(\phi)}_{xy}\, G^{(\rho)}_{0y}\, G^{(\rho)}_{xz}\,. \tag{5.E.6}$$

The two integrals are just the same and we finally find

$$g^2 \int d^d x \int d^d y\, G^{(\phi)}_{xy}\, G^{(\phi)}_{xy}\, G^{(\rho)}_{0x}\, G^{(\rho)}_{yz}\,. \tag{5.E.7}$$

The σ propagator in the quartic theory

The above discussion is of direct application to the quartic theory, where loop effects induce a 2-point function for σ (akin to the ρ in the previous discussion). The tree-level σ propagator from the action in (3.4.51) is just $-2\,g$. Denoting the 1PI diagram by $-\Gamma$, it then follows that the σ propagator is [187]

$$G^{(\sigma)} = -\frac{2\,g}{1-2\,g\,\Gamma}\,. \tag{5.E.8}$$

At the UV fixed point, when $g \to \infty$, $G^{(\sigma)}$ it is just the inverse of the 1PI diagram for σ. In momentum space, (5.E.7) is given by[3]

$$\Gamma(p) = \int \frac{d^d q}{(2\pi)^d}\, \frac{1}{(p-q)^2\, q^2}\,. \tag{5.E.9}$$

[3]The signs can be checked by going back to Lorentzian signature, where propagators have an extra i. In turn, the cubic vertex is $-\frac{i}{\sqrt{N}}$. Thus the 1PI diagram is given by Γ_L, where Γ_L is the (Lorentzian) loop integral alone ($(-\frac{i}{N})^2$ from the vertices, i^2 from the ϕ propagators, and an overall N from the N ϕ's running in the loop). Hence $G^{(\sigma)} = -\frac{2\,i\,g}{1+2\,i\,g\,\Gamma_L} \sim \frac{1}{\Gamma_L}$. Wick-rotating to the Euclidean $\Gamma_L = i\,\Gamma$, and so $G^{(\sigma)} = i\,\frac{1}{\Gamma}$. Stripping off the i to go back to Euclidean signature gives $G^{(\sigma)} = \frac{1}{\Gamma}$, with Γ given by (5.E.9).

Introducing Feynman parameters

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2\, q^2} = \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx\, \frac{1}{(q^2+\Delta)^2}\,. \qquad \Delta = x(1-x)\, p^2\,. \tag{5.E.10}$$

Computing the integral, we find

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2\, q^2} = -\frac{(4\pi)^{\frac{3-d}{2}}}{2^d \sin\left(\frac{d\pi}{2}\right) \Gamma\left(\frac{d-1}{2}\right)} (p^2)^{\frac{d}{2}-2}\,. \tag{5.E.11}$$

Hence

$$\Gamma(p) = -2\, \tilde{C}_d^{-1}\, (p^2)^{2-\frac{d}{2}}\,, \qquad \tilde{C}_d = 2^{d+1}\, (4\pi)^{\frac{d-3}{2}} \sin\left(\frac{d\,\pi}{2}\right) \Gamma\left(\frac{d-1}{2}\right). \tag{5.E.12}$$

Thus, in momentum space, the σ propagator is

$$\langle \sigma(p)\, \sigma(-p) \rangle = \frac{\tilde{C}_d}{2\, (p^2)^{\frac{d}{2}-2}}\,. \tag{5.E.13}$$

In position space this is

$$\langle \sigma(x)\, \sigma(0) \rangle = \frac{C_d}{2\, (x^2)^2}\,, \qquad C_d = \frac{16}{(4\pi)^{\frac{d}{2}}\, \Gamma\left(\frac{d}{2}-2\right)} \tilde{C}_d\,. \tag{5.E.14}$$

The $\langle \phi(0)^2\, \bar{\phi}^2(z) \rangle$ correlator to NLO

Let us consider the calculation of the correlation function $\langle \phi(0)^2\, \bar{\phi}^2(z) \rangle$ to NLO order. To LO this is just the free correlator. Evaluating it through Wick contractions one easily finds

$$\langle \phi(0)\, \phi(0)\, \bar{\phi}(z)\, \bar{\phi}(z) \rangle = 2\, G_{0z}^{(\phi)}\,. \tag{5.E.15}$$

The NLO correction comes from evaluating

$$\frac{1}{2}\, g^2 \int d^d x\, d^d y\, \langle \phi(0)\, \phi(0)\, \rho(x)\, \bar{\phi}(x)\, \phi(x)\, \rho(y)\, \bar{\phi}(y)\, \phi(y)\, \bar{\phi}(z)\, \bar{\phi}(z) \rangle\,. \tag{5.E.16}$$

This gives

$$\frac{1}{2}\, g^2 \int d^d x\, d^d y\, G_{xy}^{(\rho)}\, \langle \phi(0)\, \phi(0)\, \bar{\phi}(x)\, \phi(x)\, \bar{\phi}(y)\, \phi(y)\, \bar{\phi}(z)\, \bar{\phi}(z) \rangle\,. \tag{5.E.17}$$

In order to have connected diagrams, the $\phi(0)$'s must be contracted with either the $\bar{\phi}(x)$ or the $\bar{\phi}(y)$, for which each case has two possible contractions (one for each of the two $\phi(0)$'s) thus giving

$$\begin{aligned}&\frac{1}{2}\,g^2\int d^dx\,d^dy\,G^{(\rho)}_{xy}\,2\,G^{(\phi)}_{0x}\,\langle\phi(0)\,\phi(x)\,\bar{\phi}(y)\,\phi(y)\,\bar{\phi}(z)\,\bar{\phi}(z)\rangle \\ &+\frac{1}{2}\,g^2\int d^dx\,d^dy\,G^{(\rho)}_{xy}\,2\,G^{(\phi)}_{0y}\,\langle\phi(0)\,\bar{\phi}(x)\,\phi(x)\,\phi(y)\,\bar{\phi}(z)\,\bar{\phi}(z)\rangle\,.\end{aligned} \tag{5.E.18}$$

By the same token, the remaining $\phi(0)$ can only be contracted either with the $\bar{\phi}(x)$ or with the remaining $\bar{\phi}(y)$, giving

$$2\,g^2\int d^dx\,d^dy\,G^{(\rho)}_{xy}\,G^{(\phi)}_{0x}\,G^{(\phi)}_{0y}\,\langle\phi(x)\,\phi(y)\,\bar{\phi}(z)\,\bar{\phi}(z)\rangle\,. \tag{5.E.19}$$

Making the final contractions, we find

$$4\,g^2\int d^dx\,d^dy\,G^{(\rho)}_{xy}\,G^{(\phi)}_{0x}\,G^{(\phi)}_{0y}\,G^{(\phi)}_{xz}\,G^{(\phi)}_{yz}\,. \tag{5.E.20}$$

Thus, all in all, to NLO, the correlator is given by

$$\langle\phi(0)\,\phi(0)\,\bar{\phi}(z)\,\bar{\phi}(z)\rangle = 2\,G^{(\phi)}_{0z} + 4\,g^2\int d^dx\,d^dy\,G^{(\rho)}_{xy}\,G^{(\phi)}_{0x}\,G^{(\phi)}_{0y}\,G^{(\phi)}_{xz}\,G^{(\phi)}_{yz}\,. \tag{5.E.21}$$

We may now write this as

$$\langle\phi(0)\,\phi(0)\,\bar{\phi}(z)\,\bar{\phi}(z)\rangle = 2\left(G^{(\phi)}_{0z} + 2\,g^2\int d^dx\,d^dy\,G^{(\rho)}_{xy}\,G^{(\phi)}_{0x}\,G^{(\phi)}_{0y}\,G^{(\phi)}_{xz}\,G^{(\phi)}_{yz}\right)\,, \tag{5.E.22}$$

where we have extracted the factor of 2 of the free correlator, leaving behind the relative factor of 2 pointed out in the main text. Note that the factor of 2 stands for the 4 possible relative ways to paste the two halves of (3.8) multiplied by the factor $\frac{1}{2}$ that arises from expanding the Dyson series. This extends in a straightforward way to the more general case of correlators of $\mathcal{O}_n$, $\overline{\mathcal{O}}_n$ operators, with the only difference that the overall 2 becomes $n!$.

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