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An engineering interpretation of the complex modal mass in structural dynamics

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ABSTRACT

In this paper the modal mass is considered in the case of general damping where it is well-known that the modal mass becomes a complex valued quantity. The aim of this paper is to present a definition of the complex modal mass, which leads to an easier physical understanding of this modal parameter. It is demonstrated that based on the structural modification theory, the complex modal mass can be expressed as a linear combination of the un-damped modal masses, where a complex coefficient matrix can be defined that depends on the general damping matrix and on the mode shapes of the damped and the un-damped system. An important finding is that an apparent mass can be defined so that the modal mass is always equal to the product between the apparent mass and the length squared. The paper includes a 2-DOF example that illustrates the influence of the amount of general damping on modal quantities that are normally considered non-sensitive to damping.

1. Introduction

When the modal model is used to define the dynamic behavior of a structure, the modal parameters (natural frequencies, mode shapes and damping ratios) corresponding to each mode are needed [1,2,3,4]. In un-damped and proportional damped models, a mode shape (real components) is said to be scaled if it is mass normalized (normalized to the mass matrix) whereas it is defined as un-scaled when other kinds of normalization are used [1,2,3,4].

The most common normalization techniques used in un-damped and proportional damped models are mass normalization, normalization to the unit length of the mode shape (length scaling) and normalization to a component (usually to the largest component) equal to unity (DOF scaling) [1,2,3,4,5,6,7].

Another parameter, known as modal mass, has to be defined for each mode if un-scaled mode shapes are used. In [1] the modal mass is defined as a scaling parameter for the mode shapes, i.e., it is used to convert the original unscaled mode shape vector $\{\psi\}$ to the scaled (mass-normalized) mode shape vector $\{\phi\}$, i.e., the unscaled and the scaled vectors are related by the equation:

$$\{\phi\} = \frac{1}{\sqrt{m}}\{\psi\} \tag{1}$$

Where m is the modal mass of the unscaled mode shape $\{\psi\}$. The modal mass of the mode shape $\{\phi\}$ is dimensionless unity. On the

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other hand, the modal mass of mode shapes normalized to the unit length or to a component equal to unity have units of mass (kg in the international system) or inertia (kgm^2 in the international system).

The structural dynamic modification (SDM) theory [2,3,4,8], is a technique that predicts the dynamic behavior of a system (perturbed system), which is a modification of a known system (un-perturbed system). From the SDM theory it is derived that the mode shapes of the perturbed system can be expressed as a linear combination of the un-perturbed mode shapes [2,3,4,8].

In dynamic systems with general damping, the equation of motion has to be formulated in state space format and the mode shapes are extracted from the complex eigenvectors [2,3,4,9]. The modal masses are also complex because they contain information of both the mass and of the damping of the system, which complicates the interpretation [10,11,12]. An interpretation of the complex eigensolution that describes free and resonant vibrations of a generally damped linear structure is given in [10], where all elements of the complex solution are related to physical quantities.

Experimental mode shapes are, in general, complex, but normal modes are needed for comparison with undamped finite element models. In [11] a methodology to obtain normal modes from experimental complex modes is proposed. A simple method to normalize complex modes so that they are closest to their corresponding classical normal modes is proposed in [12].

In this paper we present a better definition of the complex modal mass, which leads to significant improvements in the physical understanding of this modal parameter. The SDM theory is used to relate the damped and the un-damped models, i.e., the damped model is considered a perturbation of the un-damped model where the modification is given by the damping matrix. It is demonstrated that the complex modal masses (and also other modal parameters) can be expressed as a linear combination of the un-damped modal masses, where the transformation matrix is complex and depends on the damping matrix and the mode shapes. The concepts and equations formulated in the paper are illustrated with an example.

The paper is organized as follows. After the introduction, the basic theory of general damped models and undamped models is presented in section 2. In section 3, the most common normalization techniques used to scale mode shapes and eigenvectors are presented. In section 4, the complex mode shapes are projected on the undamped normal mode shapes, from which an analytical expression that relates the modal masses of both the damped and the undamped models is derived. A similar procedure is followed in section 5, where the complex eigenvectors are projected on the undamped normal eigenvectors. In section 6, the concept of length of continuous and discrete mode shapes is extended to the complex modes case, and a new definition of the modal mass is formulated. In section 7, the theoretical formulations developed in this paper are illustrated performing simulations on a 2-DOF damped system. The paper finishes with the conclusions in section 8.

2. Basic theory

In this section, we briefly present the equations of motion of both damped and un-damped systems, their associated eigenvalue problems, as well as the modal parameters which are derived from both systems.

2.1. General damped model

The equation of motion of an N degree of freedom system is given by [1,2,3,4,5,6,7,9]:

$$[K]\{u\} + [C]\{\dot{u}\} + [M]\{\ddot{u}\} = \{p\} \tag{2}$$

Where $\{p\}$ is the force input vector, $\{u\}$ is the response vector and $[M]$, [C] and $[K]$ are the mass damping and stiffness matrices, respectively. In the case of general damping, the equation of motion has to be defined in a state space model format [1,2,3,4,7,9], i.e.:

$$[\boldsymbol{A}]\{\dot{\boldsymbol{y}}\} + [\boldsymbol{B}]\{\boldsymbol{y}\} = \{\boldsymbol{f}\} \tag{3}$$

where:

$$[\boldsymbol{A}] = \begin{bmatrix} [0] & [\boldsymbol{M}] \\ [\boldsymbol{M}] & [\boldsymbol{C}] \end{bmatrix} [\boldsymbol{B}] = \begin{bmatrix} -[\boldsymbol{M}] & [0] \\ [0] & [\boldsymbol{K}] \end{bmatrix} \{\boldsymbol{y}\} = \left\{ \begin{matrix} \{\dot{\boldsymbol{u}}\} \\ \{\boldsymbol{u}\} \end{matrix} \right\} \{\boldsymbol{f}\} = \left\{ \begin{matrix} \{0\} \\ \{\boldsymbol{p}\} \end{matrix} \right\}$$

Eq. (3) leads to the following complex eigenvalue problem [1,2,3,4,7,9]:

$$([B] + [A]\lambda)\{E_\psi\} = \{0\} \tag{4}$$

Which generates 2 N complex eigenvalues (or poles) λ_r which appear in complex conjugate pairs:

$$[\Lambda] = \begin{bmatrix} \lambda_1 & 0 & . & 0 & 0 & . & 0 & 0 \\ 0 & \lambda_1^* & . & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & \lambda_r & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & \lambda_r^* & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & 0 & 0 & & \lambda_N & 0 \\ 0 & 0 & . & 0 & 0 & . & 0 & \lambda_N^* \end{bmatrix} \boldsymbol{r = 1:N} \tag{5}$$

Where superscript '*' indicates complex conjugate. The eigenvalues can be expressed as:

$$\lambda_r = -\zeta_r\omega_r + j\omega_{dr} \tag{6}$$

Where ω_r and ζ_r are the natural frequency and the damping ratio of the r-th mode, respectively, and ω_{dr} is the damped natural frequency given by:

$$\omega_{dr} = \omega_r\sqrt{1-\zeta_r^2} \tag{7}$$

The eigenvectors $\{E_{\psi_r}\} = \left\{ \begin{matrix} \lambda_r\{\psi_r\} \\ \{\psi_r\} \end{matrix} \right\}$ associated to the eigenvalues λ_r also appear in complex conjugate pairs. The $2N \times 2N$ matrix of eigenvectors $[E_\Psi]$ is given by [1,2,3,4,7,9]:

$$[E_\psi] = \begin{bmatrix} \lambda_1\{\psi_1\} & \lambda_1^*\{\psi_1\}^* & .\lambda_r\{\psi_r\} & \lambda_r^*\{\psi_r\}^* & . & \lambda_N\{\psi_N\} & \lambda_N^*\{\psi_N\}^* \\ \{\psi_1\} & \{\psi_1\}^* & .\{\psi_r\} & \{\psi_r\}^* & . & \{\psi_N\} & \{\psi_N\}^* \end{bmatrix} \tag{8}$$

Where $\{\psi_r\}$ is the r-th mode shape.

The eigenvalue problem given by Eq. (4) is equivalent to the eigenvalue problem derived from eq. (2) which is given by [1,2,3,4]:

$$\left([M]\lambda^2 + [C]\lambda + [K]\right)\{\psi\} = \{0\} \tag{9}$$

The equations to calculate the modal mass and the modal damping matrices are the same as those used in case of real modes [1,2,3,4,7,9,10], i.e:

$$[m] = [\psi]^T[M][\psi] \tag{10}$$

$$[c] = [\psi]^T[C][\psi] \tag{11}$$

but matrices $[m]$ and $[c]$ are not diagonal and the terms are complex [4,7]. However, the eigenvectors $\{E_{\psi_r}\}$ are orthogonal with respect to the matrices [A] and [B] [2,3,4,7,9,10,11], i.e:

$$[E_\Psi]^T[A][E_\Psi] = [a_\psi] \tag{12}$$

and

$$[E_\Psi]^T[B][E_\Psi] = [b_\psi] \tag{13}$$

Which means that the matrices $[a_\psi]$ and $[b_\psi]$ are diagonal [2,3,4,7,9,10,11].

From eq. (12) it is derived that:

$$a_{\psi r} = 2\lambda_r m_r + c_r = 2j\omega_{dr}m_r \tag{14}$$

Where a_{ψ_r} is the scaling parameter corresponding to the eigenvector $\{E_{\psi_r}\}$, and m_r (modal mass) and $c_r = 2\zeta_r\omega_r m_r$ (modal damping) are the r-th diagonal terms of matrices $[m]$ and $[c]$, respectively. The parameter a_{ψ_r} is complex and it has the units $\left(\frac{1}{s}kg\right)$ if the mode shapes are normalized to the unit length or to a component equal to unity [11,12,13,14].

Premultiplication of eq. (4) by $[E_\Psi]^T$ leads to [2,3,4,7,9]:

$$[\boldsymbol{b}_\psi] = -[\boldsymbol{a}_\psi][\Lambda] \tag{15}$$

2.2. Undamped model

The equation of motion of the corresponding un-damped model (same mass $[M]$ and stiffness $[K]$ matrices as the general damped model) in state space model format is expressed as [1,2,3,4,7,9]:

$$[\boldsymbol{A}_0]\{\dot{\boldsymbol{y}}\} + [\boldsymbol{B}_0]\{\boldsymbol{y}\} = \{\boldsymbol{f}\} \tag{16}$$

Where subindex '$0'$' denotes un-damped model and matrices $[A_0]$ and $[B_0]$ are given by:

$$[\boldsymbol{A}_0] = \begin{bmatrix} [0] & [\boldsymbol{M}] \\ [\boldsymbol{M}] & [0] \end{bmatrix} [\boldsymbol{B}_0] = \begin{bmatrix} -[\boldsymbol{M}] & [0] \\ [0] & [\boldsymbol{K}] \end{bmatrix} = [\boldsymbol{B}]$$

The corresponding eigenvalue problem is expressed as [1,2,3,4,7,9]:

$$([B_0] + [A_0]\lambda_{0r})\{E_{\psi_{0r}}\} = \{0\} \tag{17}$$

Where the poles $\lambda_{0r} = j\omega_{0r}$ are purely imaginary and ω_{0r} is the real un-damped natural frequency.

The eigenvectors $\{E_{\psi_{0r}}\} = \left\{ \begin{matrix} \lambda_{0r}\{\psi_{0r}\} \\ \{\psi_{0r}\} \end{matrix} \right\}$ are also purely imaginary, (but they can be normalized in such a way that they become real) [4,7,11,12].

The term $a_{\psi_{0r}}$ is also purely imaginary and given by:

$$a_{\psi 0r} = 2jm_{0r}\omega_{0r} \tag{18}$$

With m_{0r} being the real valued modal mass.

3. Normalization of mode shapes and eigenvectors.

In case of general damped models, both the eigenvectors and the mode shape vectors can be normalized with different techniques [12,13,14], which are briefly described in this section.

3.1. Normalization of eigenvectors

The best know technique to normalize eigenvectors is normalization to matrix $[A]$. The eigenvector $\{E_{\psi_r}\}$ is said to be normalized to matrix $[A]$ (or normalized to modal A), hereafter denoted as $\{E_{\phi_{Ar}}\}$, if the parameter $a_{\phi_{Ar}}$ is dimensionless unity, i.e:

$$\boldsymbol{a}_{\phi_{Ar}} = \{\boldsymbol{E}_{\phi_{Ar}}\}^T[\boldsymbol{A}]\{\boldsymbol{E}_{\phi_{Ar}}\} = 1 \tag{19}$$

The eigenvector $\{E_{\phi_{Ar}}\}$ and the unscaled eigenvector $\{E_{\psi_r}\}$ are related by means of the expression:

$$\{\boldsymbol{E}_{\phi_{Ar}}\} = \frac{1}{\sqrt{\boldsymbol{a}_{\psi r}}}\{\boldsymbol{E}_{\psi_r}\} \tag{20}$$

Where $a_{\psi r} = \{E_{\psi_r}\}^T[A]\{E_{\psi_r}\}$ is the scaling factor of the eigenvector $\{E_{\psi_r}\}$.

From eq. (20) it follows that the term a_{ψ_r} plays the same role as the modal mass with normal mode shapes. However, the term a_{ψ_r} (see eq. (14)) depends on the complex modal mass m_r and on the damped natural frequency ω_{dr}.

The eigenvectors $\{E_{\psi_r}\}$ cannot be normalized to the unit length of the vector because the units of $\{\psi_r\}$ and $\lambda_r\{\psi_r\}$ are different. For example, if $\{\psi_r\}$ is dimensionless then:

$$\{\psi_r\} \rightarrow dimensionless \quad \lambda_r\{\psi_r\} \rightarrow 1/s$$

However, it must be noticed that a change in the scaling of the mode shapes results in a change of parameters a_{ψ_r}. Similarly, normalization of the eigenvectors to matrix $[A]$ modifies the scaling of the mode shapes.

3.2. Normalization of mode shapes

The complex mode shapes can be normalized with the same techniques used in case of real modes [4,7,9,12,13,14].

3.2.1. Normalization to the unit length

The arbitrary normalized complex mode shape $\{\psi_r\}$ can be normalized to the unit length of the vector, hereafter denoted as $\{\psi_{Lr}\}$ by:

$$\{\boldsymbol{\psi}_{Lr}\} = \frac{\{\boldsymbol{\psi}_r\}}{\boldsymbol{L}_{\psi_r}} \tag{21}$$

Where L_{ψ_r} is the Euclidean length of the vector $\{\psi_r\}$ given by:

$$L_{\psi_r} = \sqrt{\{\psi_r\}^H \bullet \{\psi_r\}} \tag{22}$$

which is a real number. The superscript 'H' indicates complex conjugate. The length of the vector $\{\psi_{Lr}\}$ normalized to unit length $\{\psi_{Lr}\} = \frac{\{\psi_r\}}{L_{\psi r}}$ is dimensionless unity, i.e.,$L_{\psi_{Lr}} = 1$.

3.2.2. Normalization to a component equal to unity

The complex mode shapes can be normalized to a component equal to unity, hereafter denoted $\{\psi_{U_r}\}$, and it is commonly assumed to consider the largest component equal to a unity real number.

3.2.3. Normalization to the mass matrix

This normalization technique is used in un-damped models, but in case of general damping this normalization is commonly substituted by normalization to matrix $[A]$.

A complex mode shape $\{\psi_r\}$ is normalized to the mass matrix, hereafter denoted as $\{\phi_r\} = \{\psi_r\}/\sqrt{m_r}$ if the modal mass is dimensionless real unity, i.e. $m_r = \{\psi_r\}^T[M]\{\psi_r\} = 1$.

3.3. Relationship between normalizations

If we have mode shapes normalized to the unit length of the mode shape $\{\psi_{Lr}\}$, to a component equal to unity $\{\psi_{Ur}\}$, to the mass matrix $\{\phi_r\}$, or arbitrary normalized $\{\psi_r\}$, and we denote $a_{\psi_{Lr}}$, $a_{\psi_{Ur}}$, a_{ϕ_r} and a_{ψ_r} the corresponding complex scaling parameters, it is derived from eq. (14) that:

$$a_{\psi_{Lr}} = a_{\psi_{Ur}} \frac{1}{\{\psi_{Ur}\}^H \bullet \{\psi_{Ur}\}} = a_{\phi_r} \frac{1}{\{\phi_r\}^H \bullet \{\phi_r\}} = a_{\psi_r} \frac{1}{\{\psi_r\}^H \bullet \{\psi_r\}} \tag{23}$$

Or alternatively:

$$\frac{a_{\psi_{Lr}}}{L^2_{\psi_{Lr}} = 1} = \frac{a_{\psi_{Ur}}}{L^2_{\psi_{Ur}}} = \frac{a_{\phi_r}}{L^2_{\phi_r}} = \frac{a_{\psi_r}}{L^2_{\psi_r}} \tag{24}$$

From eq. (23) that the following relationship between the mode shapes normalized with different techniques is derived:

$$\frac{\{\psi_{L_r}\}}{\sqrt{a_{\psi_{Lr}}}} = \frac{\{\psi_{U_r}\}}{\sqrt{a_{\psi_{Ur}}}} = \frac{\{\phi_r\}}{\sqrt{a_{\phi_r}}} = \frac{\{\psi_r\}}{\sqrt{a_{\psi_r}}} \tag{25}$$

4. Projection of the complex mode shapes on the undamped normal modes

In this section the structural dynamic modification theory [8] is used to obtain the general damped system as a perturbation of the un-damped system, and an analytical expression that relates the modal masses of both the damped and the un-damped system is derived.

The general damped system can be considered a perturbation of the un-damped system, where the modification is defined by the damping matrix $[C]$. The eigenvalue problem of the perturbed system is given by eq.(9), i.e.:

$$\left([M]\lambda_r^2 + [C]\lambda_r + [K]\right)\{\psi_r\} = \{0\} \tag{9}$$

Whereas that of the un-perturbed system (un-damped model) is given by:

$$\left([M]\lambda_{0r}^2 + [K]\right)\{\psi_{0r}\} = \{0\} \tag{26}$$

According to the structural dynamic modification theory [8], the perturbed mode shapes $\{\psi_r\}$ can be expressed as a linear combination of the un-damped (un-perturbed) mode shapes $\{\psi_{0r}\}$ by means of the expression [8]:

$$[\psi] = [\psi_0][P] \tag{27}$$

Where $[\psi]$ is the matrix of complex mode shapes of the damped system, $[\psi_0]$ is the matrix of mode shapes of the un-damped system, and $[P]$ is a transformation matrix. The columns of matrix $[P]$ can be obtained solving the eigenvalue problem [8]:

$$\left([m_0]\lambda_r^2 + [\psi_0]^T[C][\psi_0]\lambda_r + [m_0]\left[\omega_0^2\right]\right)\{p_r\} = \{0\} \tag{28}$$

Where $\{p_r\}$ indicates the r-th column of matrix $[P]$.

Matrix $[P]$ is diagonal if the inner product $[\psi_0]^T[C][\psi_0]$ is a diagonal matrix, i.e., if the normal mode shapes $[\psi_0]$ are orthogonal with respect to the damping matrix $[C]$. Thus, the complexity of the mode shapes depends on the level of orthogonality of the normal mode shapes with respect to damping matrix $[C]$.

The terms of matrix $[P]$ are, in general, complex. If eq. (27) is substituted in eq.(1 0), the complex modal mass m_r can be expressed as:

$$m_r = \{p_r\}^T[\psi_0]^T[M][\psi_0]\{p_r\} = \{p_r\}^T[m_0]\{p_r\} \tag{29}$$

From eq. (29) is concluded that since $[m_0]$ is diagonal, the complex modal mass can be obtained as a linear combination of the modal masses of the un-damped system, i.e.:

$$m_r = \sum_{k=1}^{N} p_{kr}^2 m_{0k} \tag{30}$$

The matrix $[P]$ is not diagonal and therefore all the modes of the un-damped system contribute to the complex modal mass. Although m_{0k} is real, the terms of matrix $[P]$ are complex and therefore m_r is complex. If both mode shapes are normalized with the same technique, the diagonal terms of matrix [P] will be close to ± 1 or to $\pm i$.

The modal mass matrix $[m] = [\psi]^T[M][\psi]$ and the modal stiffness matrix given by:

$$[\boldsymbol{k}] = [\boldsymbol{\psi}]^T[\boldsymbol{K}][\boldsymbol{\psi}] \tag{31}$$

are not diagonal matrices [4,7] and the components are complex. The diagonal terms of eqs. (10) and (31) are related by:

$$\boldsymbol{\omega}_r^2 = \boldsymbol{k}_r/\boldsymbol{m}_r \tag{31}$$

Where k_r and m_r indicate the r-th diagonal terms of matrices $[k]$ and $[m]$.

5. Projection of the complex eigenvectors on the undamped REAL eigenvectors

In this section the structural dynamic modification theory [8] is used to obtain the modal parameters (poles and parameters a_ψ) of the general damped system, as a perturbation of the un-damped system in a state space model format. Relationships between the poles and parameters a_ψ of both the damped and the un-damped system are derived.

The general damped system can be obtained perturbing the un-damped system (eq. (16)) with the matrix $[\Delta A]$ where:

$$[\Delta A] = \begin{bmatrix} [0] & [0] \\ [0] & [C] \end{bmatrix} \tag{32}$$

The eigenvalue problem of the perturbed system is given by [8]:

$$([B_0] + ([A_0] + [\Delta A])\lambda_r)\{E_{\psi_r}\} = \{0\} \tag{33}$$

According to the structural dynamic modification theory, the perturbed eigenvectors $\{E_{\psi_r}\}$ can be expressed as a linear combination of the un-damped (unperturbed) eigenvectors $\{E_{\psi_{0r}}\}$ by means of [8]:

$$[E_\psi] = [E_{\psi_0}][T] \tag{34}$$

where $[T]$ is a transformation matrix. The columns of matrix $[T]$ can be obtained solving the eigenvalue problem [8]:

$$\left([b_{\psi_0}] + \left([a_{\psi_0}] + [E_{\psi_0}][\Delta A][E_{\psi_0}]\right)\lambda_r\right)\{T_r\} = 0 \tag{35}$$

Or:

$$\left([b_{\psi_0}] + \left([a_{\psi_0}] + \{\psi_r\}^T[C]\{\psi_r\}\right)\lambda_r\right)\{T_r\} = 0 \tag{36}$$

Premultiplication of eq. (4) by $[E_\Psi]^T$ and substitution of eq. (34) into the first term of such equation, leads to:

$$[T]^T[E_{\psi 0}]^T[B][E_{\psi 0}][T] = -[a_\psi][\Lambda] \tag{37}$$

Due to the fact $[B] = [B_0]$, eq. (37) can also be expressed as:

$$-[T]^T[a_{\psi 0}][\Lambda_0][T] = -[a_\psi][\Lambda] \tag{38}$$

from which it follows that the term $a_{\psi r}$ of the complex system is related to the terms $[a_{\psi 0}]$ of the undamped model by means of the expression:

$$a_{\psi r} = \frac{\{T_r\}^T[a_{\psi 0}][\Lambda_0]\{T_r\}}{\lambda_r} \tag{39}$$

i.e., the parameters a_{ψ_r} can be obtained as a linear combination of the parameters $a_{\psi 0r}$.

The matrix $[a_\psi]$ can also be expressed as:

$$[a_\psi] = [E_\psi]^T[A][E_\psi] = [E_\psi]^T[A_0][E_\psi] + [E_\psi]^T[\Delta A][E_\psi] \tag{40}$$

If eq. (34) is substituted in the first term of eq. (40), it results in:

$$[a_\psi] = [T]^T[a_{\psi 0}][T] + [E_\psi]^T[\Delta A][E_\psi] = [T]^T[a_{\psi 0}][T] + [\psi]^T[C]\{\psi\} \tag{41}$$

And the terms a_{ψ_r} can also be formulated as:

$$a_{\psi r} = \{T_r\}^T [a_{\psi 0}] \{T_r\} + c_r \quad (42)$$

Where a_{ψ_r} and c_r are the diagonal terms of matrices $[a_\psi]$ and $[c]$, respectively.

If the eigenvectors $[E_\Psi]$ and $[E_{\Psi_0}]$ are normalized to matrices $[A] and [A_0]$, respectively, eq. (38) becomes:

$$[T]^T [\Lambda_0][T] = [\Lambda] \quad (43)$$

Which means that the poles of the damped system can be obtained as a linear combination of the poles of the un-damped system, i.e.:

$$\lambda_r = \{T\}_r^T [\Lambda_0] \{T\}_r \quad (44)$$

Whereas the parameter a_{ψ_r} results in

$$a_{\psi_r} = \{T_r\}^T \{T_r\} + c_r \quad (45)$$

6. Engineering interpretation of the complex modal mass

6.1. Modal mass in un-damped systems

In this section, the concept of length of continuous and discrete mode shapes defined for un-damped models in [13,14], as well as the relationship between modal mass and length of mode shapes, is briefly outlined.

It was demonstrated in [13,14] that the modal mass of a discrete un-damped dynamic system with constant mass-density can be expressed as:

$$\boldsymbol{m}_{0r} = \{\boldsymbol{\psi}_{0r}\}^T [\boldsymbol{M}] \{\boldsymbol{\psi}_{0r}\} = \boldsymbol{M}_T \boldsymbol{L}^2_{\psi_{0r}} \quad (46)$$

Where M_T is the total mass of the system and $L^2_{\psi_{0r}}$ is the square length of the mode shape $\{\psi_{0r}\}$ defined by the expression:

$$\boldsymbol{L}^2_{\psi_{0r}} = \frac{\{\boldsymbol{\psi}_{0r}\}^T [\boldsymbol{V}] \{\boldsymbol{\psi}_{0r}\}}{\boldsymbol{V}_T} \quad (47)$$

With V_T being the total volume of the system and $[V]$ the volume matrix. This length definition has the same unit as the mode shape, it is a pure geometrical quantity and it does not depend on the number of DOF's considered to discretize the model [13,14].

In the general case of varying mass density, eq. (46) can be expressed as:

$$m_{0r} = M_{Tar} L^2_{\psi_{0r}} \quad (48)$$

Where M_{Tar} is the apparent total mass of the r-th mode, which is different for each mode and depends on how the mass is distributed in the structure (see [13,14] for information on how to obtain M_{Tar}).

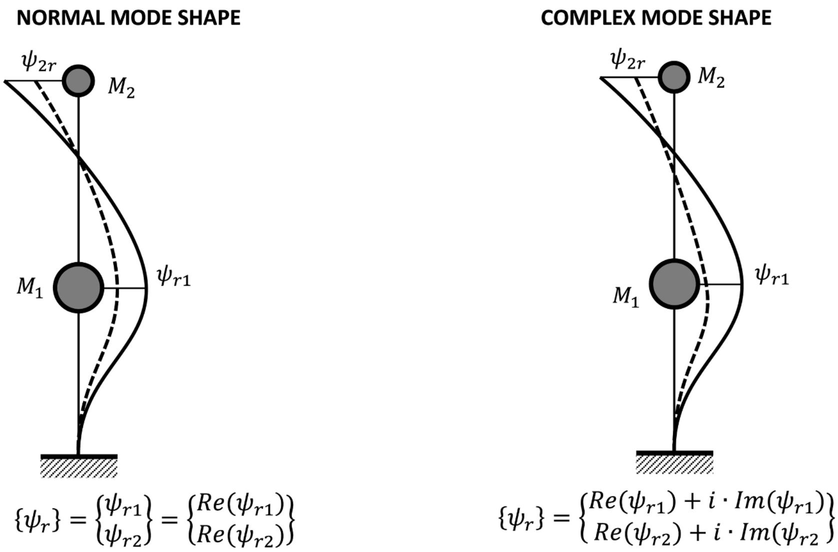

Fig. 1. 2nd normal and complex mode shapes of a cantilever beam model.

6.2. Modal mass in damped systems

The concept of length of continuous and discrete mode shapes defined for un-damped models in [13,14] is here extended to the complex modes case, which provides a better interpretation of the complex modal mass of damped models.

In algebra, the length (also denoted Euclidean length) of a complex vector $\{\psi_r\}$ is calculated with eq. (22), which is a real scalar. This might suggest calculating the modal mass using the expression:

$$m_r = \{\psi_r\}^H[M]\{\psi_r\} \tag{49}$$

Which is also a real scalar. Although eq. (49) might be appealing, it cannot be used because the products $[E_\Psi]^H[A][E_\Psi]$ and $[E_\Psi]^H[B][E_\Psi]$ are not diagonal, i.e., there is no orthogonality.

The eigenvectors are determined by the eigenvalue problem and what comes out of an eigenvalue algorithm is a complex eigenvector $\{E_{\psi_r}\}$ and a complex mode shape $\{\psi_r\}$) scaled by some complex constant a_{ψ_r}. For a complex mode shape, there is not always a point on the structure at which the modal displacement is zero at all times within a periodic cycle [2,3,4,7,9,10]. Let us consider a single mode $\{\psi_r\}$ of a discrete system of a cantilever beam (Fig. 1). For normal modes, the relative displacement location of each mass M_k is given by:

$$u_1(t) = Re(\psi_{r1})\cos(\omega_r t)$$

$$u_2(t) = Re(\psi_{r2})\cos(\omega_r t) \tag{50}$$

And the ratio $\frac{u_1(t)}{u_2(t)}$ does not change with time t.

In case of complex modes, the relative displacement location of each mass M_k at any given instant in time (Fig. 1) is a function of the corresponding real and imaginary components of the mode shape, i.e.:

$$u_1(t) = \psi_{r1}e^{\lambda_r t} + \psi_{r1}^* e^{\lambda_r^* t}$$

$$u_2(t) = \psi_{r2}e^{\lambda_r t} + \psi_{r2}^* e^{\lambda_r^* t} \tag{51}$$

And the ratio $\frac{u_1(t)}{u_2(t)}$ changes with time t. This means that the length of the mode shape is also changing with time, and it must be complex.

For systems with constant mass density, $[M] = \rho[V]$ and Eq. (10) can be expressed as:

$$m_r = \{\psi_r\}^T[M]\{\psi_r\} = M_T\frac{\{\psi_r\}^T[V]\{\psi_r\}}{V_T} \tag{52}$$

In order to extend eq. (46) to complex modes, the length of a complex mode shapes must be defined as:

$$L_{\psi r} = \sqrt{\frac{\{\psi_r\}^T[V]\{\psi_r\}}{V_T}} \tag{53}$$

The definition given by eq. (53) uses the transpose of the mode shapes (not the complex conjugate) and the length is, in general, complex [4,7]. This is a difference with the Euclidean length used in algebra, which leads to a real scalar because it uses the conjugate transpose.

Substitution of Eq. (53) in Eq. (52) leads to:

$$m_r = M_T L_{\psi_r}^2 \tag{54}$$

Eq. (54) demonstrates that the expression which relates the modal mass and the length of the mode shapes in constant mass-density systems is the same for real and complex modes [13,14]. Due to the fact that M_T is a scalar, the modal mass m_r must also be complex if L_{ψ_r} is complex.

As the general damped system can be consider a perturbation of the un-damped system, where the modification is defined by the damping matrix $[C]$, both models have the same mass $[M]$, and volume $[V]$, matrices. If eq. (27) is substituted into eq. (52), this can be expressed as:

$$L_{\psi_r}^2 = \{p_r\}^T[L_0]^2\{p_r\} \tag{55}$$

Where $[L_0]$ is a matrix containing the length of the un-damped normal modes, which is diagonal in constant mass-density systems [.

From eq. (55) is inferred that the length of the complex mode shapes can be expressed as a linear combination of the length of the mode shapes of the un-damped model. Substitution of eq. (55) in eq. (52) gives:

$$m_r = M_T\{p_r\}^T[L_0]^2\{p_r\} \tag{56}$$

Where the modal mass depends on the total mass M_T, the length of the un-damped mode shapes, and the terms of matrix $[P]$.

In the general case of varying mass density, the modal mass can be expressed as [13,14]:

$$m_r = M_{Tar} L^2_{\psi_r} \tag{58}$$

Where M_{Tar} is the apparent total mass given by [13,14]:

$$M_{Tar} = V_T \frac{\sum_n M_n L^2_{\psi_{nr}}}{\sum_n V_n L^2_{\psi n_r}} \tag{59}$$

With n being the number of different regions of the structure with different mass density and $L^2_{\psi_{nr}}$ is the complex square length of the mode shapes in the region n. This apparent total mass M_{Tar} is different from the total mass and changes value from mode shape to mode shape. Thus, the deviation of the apparent total mass M_{Tar} from the total mass M_T, is a measure of how much the mass distribution deviate from the uniform distribution [13,14].

7. Example

In order to illustrate the equations developed in this paper, the modal parameters of the 2-DOF system shown in Table 1 were determined. Three different types of damping situations were considered: un-damped, proportional damping and general damping.

For the damped systems, the same damping ratios were considered. It can be observed in Table 1 that although the damping ratios are the same for both damped systems, the damping matrices are significantly different. With respect to the natural frequencies, they are close but not equal, which means that the natural frequencies ω_r of the general damped model are different from both the proportionally damped model and the undamped model.

In order to show the effect of the damping, several models with the same mass and stiffness matrices and different damping ratios were simulated. Two types of damping were considered: proportional and general damping. In case of general damping, the damping matrix considered in the simulations was:

$$[C] = \beta \begin{bmatrix} 8 & -2 \\ -2 & 3 \end{bmatrix} \tag{57}$$

Where $0 \leq \beta \leq 20$, whereas the damping matrix corresponding to proportional damping was:

$$[C] = \beta \begin{bmatrix} 6.5565 & -3.0973 \\ -3.0973 & 5.1887 \end{bmatrix} \tag{58}$$

The damping ratios of both systems are equal for $\beta = 1$. In case of proportional damping, the damping ratios are proportional to β, whereas the damping ratios of the general damping case are approximately proportional to β (Fig. 2).

Fig. 3 shows the influence of the damping on the natural frequencies. The figure confirms that the natural frequency ω_r does not coincide with the natural frequency of the un-damped model. It can also be observed that the natural frequency can increase or decrease with damping. The first natural frequency increase with increasing damping whereas that of the second mode decrease with damping. The same can be said for the damped natural frequencies (Fig. 4).

The complex modal masses m_r corresponding to mode shapes normalized with the maximum component equal to unity are presented in Figs. 5 and 6. The modal mass for the proportional damped model is real. With respect to the damped model, the real part of the modal masses decreases with damping whereas the imaginary part increases with increasing damping reaching a maximum for $\beta \approx 8$ for the first mode and for $\beta \approx 10$ for the second mode. The.

8. Conclusions

The dynamic behavior of a structure can be defined using the modal model for which the natural frequencies, mass normalized mode shapes and damping ratios are needed. If the mode shapes are not normalized to the mass matrix, a new modal parameter for each mode has to be defined, known as modal mass.

Table 1
Matrices and modal parameters of a 2 DOF system.

	UN-DAMPED	PROPORTIONAL DAMPING	GENERAL DAMPING
Stiffness and mass matrices	$[K] = \begin{bmatrix} 6000 & -2000 \\ -2000 & 6000 \end{bmatrix} [M] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$		
Damping matrix		$[C] = \begin{bmatrix} 6.5565 & -3.0973 \\ -3.0973 & 5.1887 \end{bmatrix}$	$[C] = \begin{bmatrix} 8 & -2 \\ -2 & 3 \end{bmatrix}$
Poles λ_r	0 + 39.2756i 0 + 58.7999i	−0.5106 + 39.2723i −1.9933 + 58.7661i	−0.5099 + 39.2797i −1.9901 + 58.7550i
Natural frequencies ω_r (rd/s)	39.2756 58.7999	39.2756 58.7999	39.2830 58.7887
Damping ratios ζ_r	0 0	0.0130 0.0339	0.0130 0.0339
Natural frequencies ω_{dr} (rd/s)	39.2756 58.7999	39.2723 58.7661	39.2797 58.7550

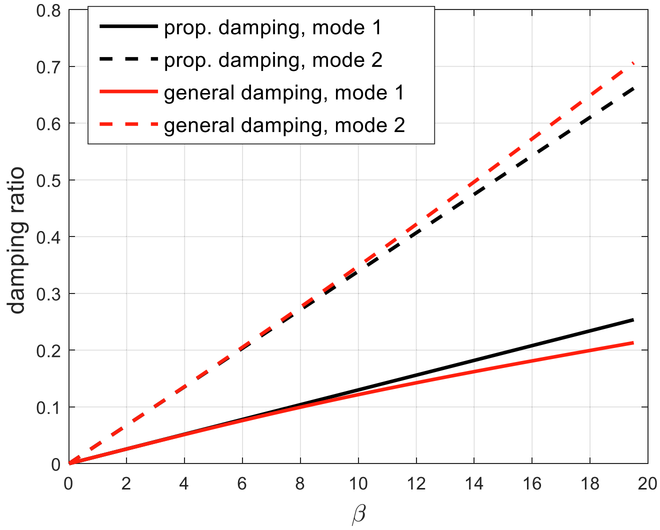

Fig. 2. Damping ratios of the proportional and general damped models.

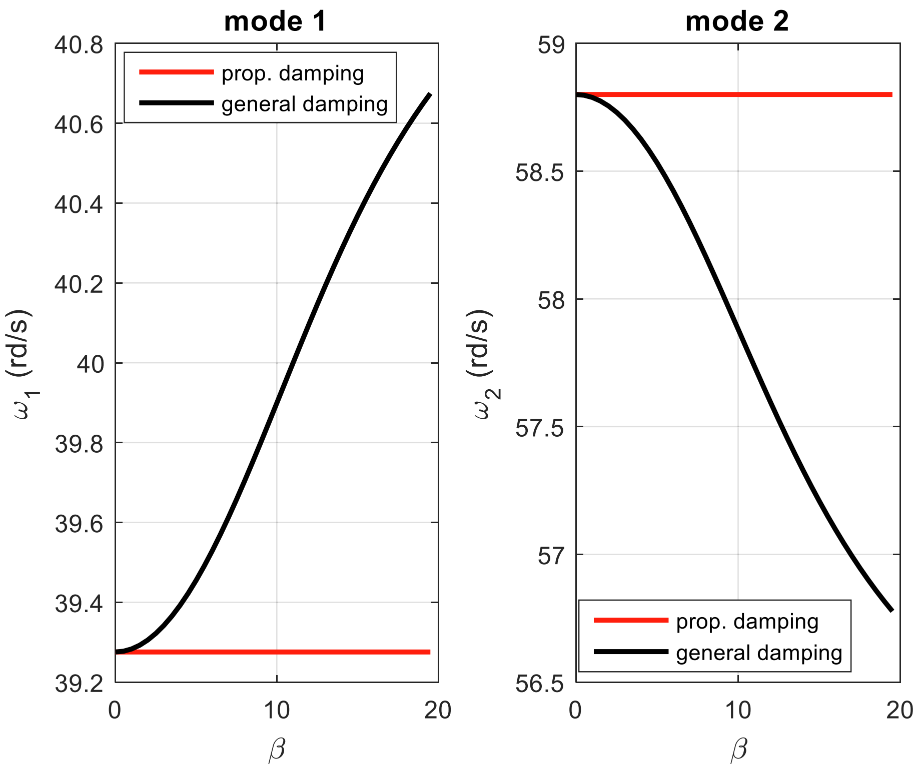

Fig. 3. Natural frequencies of the system for proportional and general damped models.

In general damped models, the modal mass is complex, and its interpretation is difficult. In this paper, the structural dynamic modification theory (SDM) has been used to give an engineering interpretation of this modal parameter. The general damped model has been considered a perturbation of the corresponding un-damped model. It has been demonstrated that the complex modal mass can be considered a linear combination of the modal masses of the un-damped model. Moreover, the natural frequencies, the parameter a_r and the poles of the general damped systems can also be expressed as linear combination of the corresponding un-damped modal parameters.

In [13,14] the concept of length of continuous and discrete mode shapes was defined for un-damped models. In this paper, this concept has been extended to the complex modes case.

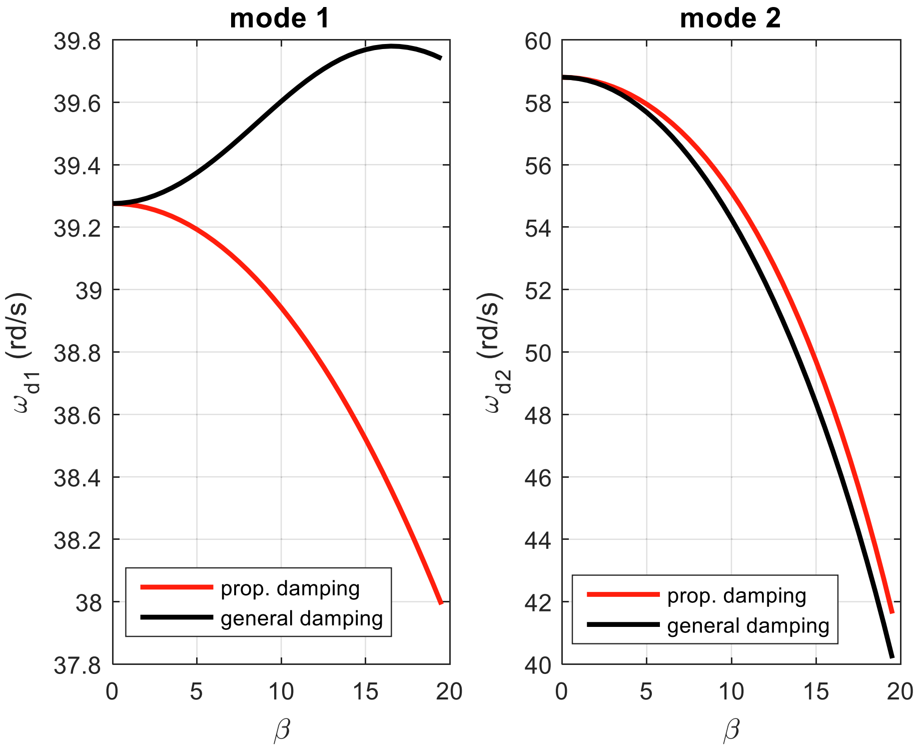

Fig. 4. Damped natural frequencies of the system for proportional and general damped models.

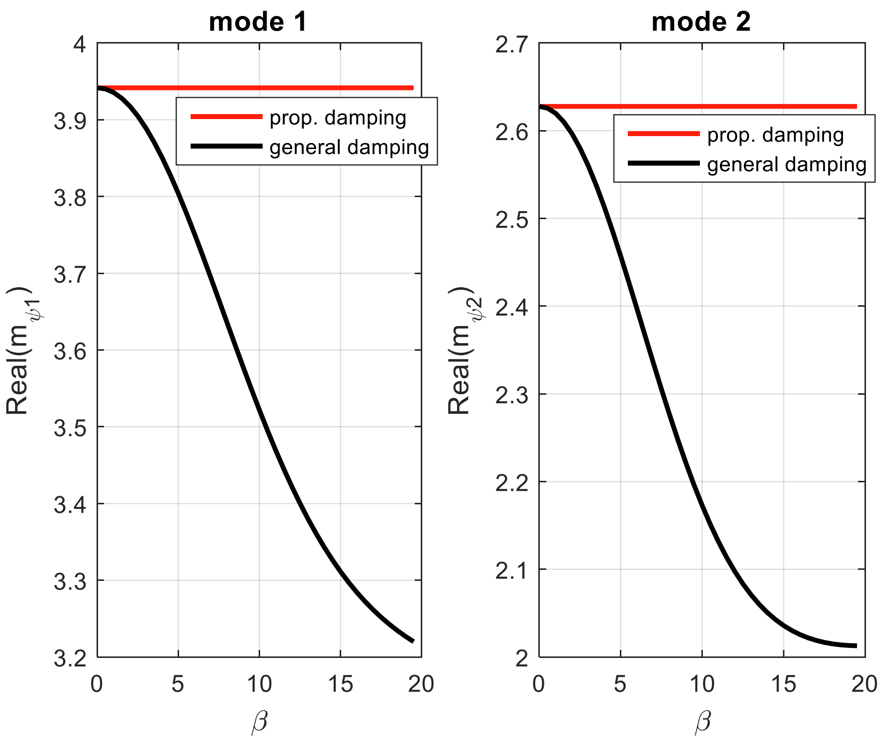

Fig. 5. Real part of the complex modal masses..m_r

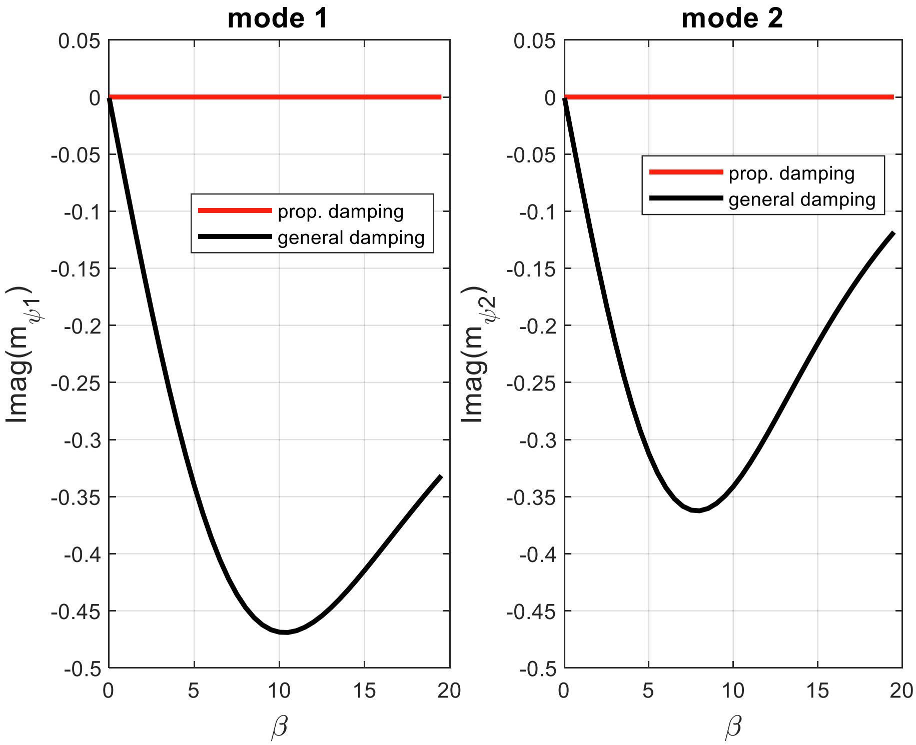

Fig. 6. Imaginary part of the complex modal masses..m_r

A new and better definition of the modal mass in general damped models, which is physically meaningful and does not depend on the number of DOF's consider to discretize the model, has been formulated. It is demonstrated that if the mass density of the system is constant, the modal mass is always equal to the product between the total mass of the structure and the magnitude of the length squared, but with a phase to introduce a complex number.

If the mass density is not constant, the concept of apparent mass is proposed, and the modal mass is always equal to the product between the apparent mass and the magnitude of the complex length squared.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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