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# Correlation matrices of Gaussian Markov random fields over cycle graphs

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ABSTRACT

Gaussian Markov Random Fields over graphs have been widely used in many fields of application. Here, we address the matrix construction problem that arises in the study of Gaussian Markov Random Fields with uniform correlation, i.e., those in which all correlations between adjacent nodes in the graph are equal. We provide a characterization of the correlation matrix of a Gaussian Markov Random Field with uniform correlation over a cycle graph, which is circulant and has a sparse inverse matrix, and study the relationship with the stationary Gaussian Markov Process on the circle. Two methods for computing the correlation matrix are also provided. Ultimately, asymptotic results for cycle graphs of large order point out the relation between Gaussian Markov Random Fields with uniform correlation over cycle and path graphs.



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## 1. Introduction

Markov Random Fields (MRF) are random vectors for which the conditional statistical independence structure can be expressed as a graph in which the nodes represent the indices of the variables [20]. More precisely, the statistical dependence of two variables is linked to the paths that connect those variables in the graph. If the random vector is Multivariate Gaussian, the notion of statistical dependence becomes equivalent to that of linear dependence (see, e.g., [28]) and, thus, the conditional independence of the variables can be studied by using the inverse of the covariance matrix [35]. In this case, the random vector is called a Gaussian Markov Random Field (GMRF).

GMRFs have been used in several fields of application, for example, in the context of image processing tasks in astronomy (e.g., classification of Polarimetric SAR Images [19], and astronomical image restoration [29]) and of disease control [22,27]. Typically, the inverse of the covariance matrix of the model is sparse, therefore some research has also focused on different computational aspects of GMRFs, for instance, structure learning [25], sampling methods [5], simulation [31,36,37] and inference [17].

We are interested in GMRFs with uniform correlation, which are a particular type of GMRFs in which Pearson's correlation coefficient between adjacent variables in the graph is constant and equals a value in the open interval $(-1, 1)$. This can be a reasonable assumption in several contexts such as the understanding of an image as a grid of pixels or the study of statistical mechanics problems over a toroidal lattice [35]. The case of the cycle graph plays an important role in the definition of the Grenander model for identifying object contours on digital images (see [15]).

The main objective of this paper is to study the construction and properties of the correlation matrix of GMRFs over cycle graphs. We determine the link between the automorphism group of a graph and the symmetries of the correlation matrix of any GMRF with uniform correlation over the graph. Then, we prove that, when considering a cycle graph, the associated correlation matrix is circulant and, by using some properties of circulant matrices, a useful characterization of the correlation matrix in terms of its inverse is provided. From this characterization we can define a method to construct such correlation matrix and study the asymptotic behavior of GMRFs over cycle graphs of large order. In addition, we study the relationship of GMRFs over cycle graphs with the stationary Gaussian Markov Process (GMP) on the circle as defined in [32]. In this context, we design a method to compute the correlation matrix and provide some asymptotic results. This study is not only interesting in itself, but it also serves as a starting point for the study of more complex models such as a GMRF over the Cartesian product of cycle graphs. More precisely, the Cartesian product of two cycle graphs is a regular lattice over a toroidal topology, which is widely used in image analysis (see [8,33,35] for some examples). In addition, the presented results also establish a link between Graph Theory and Linear Algebra, a well-established relationship that has been developed over the years [1,24,30,44].

The remainder of the paper is organized as follows. In Section 2, we introduce the preliminary concepts and fix the notations that will be used throughout the paper. The link between the automorphism group of a graph and the symmetries of the correlation matrix is presented in Section 3. In Section 4, we provide the characterization of the correlation matrix of a GMRF with uniform correlation over a cycle graph, study the relationship with the stationary GMP on the circle and present two methods for computing the correlation matrix. Finally, in Section 5, we study the asymptotic behavior as the order of the cycle graph tends to infinity. Some conclusions and future research are presented in Section 6.

## 2. Preliminaries

In this section, preliminary concepts and notations are presented.

### *2.1. Simple undirected finite graphs*

In this subsection, we introduce some basic concepts of Graph Theory using as main reference [23]. A finite graph is a pair $G = (V, E)$, where $V$ is the set of nodes, which is required to be finite, and $E$ is the set of edges, which is a set of subsets of $V$ of cardinality equal to 2. In particular, and from now on, we will consider simple graphs, which are graphs containing no graph loops ($\{i, i\} \notin E$ for any $i \in V$) or multiple edges ($E$ is not a multiset). The adjacency matrix $A_G$ of a graph $G$ is the matrix such that $(A_G)_{i,j} = 1$ if $\{i, j\} \in E$ and $(A_G)_{i,j} = 0$ if $\{i, j\} \notin E$. The number of elements of $V$ is called the order of the graph. Unless stated otherwise, the set of nodes is assumed to be $V = \{1, \ldots, n\}$ since the elements of $V$ are typically used as indices. For simplicity, throughout this paper, we refer to simple undirected finite graphs simply as graphs.

If $\{i, j\} \in E$, then $i$ and $j$ are said to be adjacent. The set of nodes to which a node $i$ is adjacent is called the neighborhood of $N(i)$, and the number of elements of $N(i)$ is called the degree of $i$. A graph in which all the nodes are adjacent to each other is called a complete graph. A graph in which no node is adjacent to another is called an empty graph. A bipartite graph is a graph in which there exist two subsets $V_1$ and $V_2$ of $V$, called parts, such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ and $\{i, j\} \notin E$ if $i, j \in V_1$ or $i, j \in V_2$.

A sequence of nodes $(v_1, \ldots, v_k)$ is called a walk between $v_1$ and $v_k$ if $\{v_i, v_{i+1}\} \in E$ for any $i \in \{1, \ldots, k-1\}$. We say that $v_1$ and $v_k$ are connected if there exists a walk between $v_1$ and $v_k$ and we say that a graph is connected if any two nodes in the graph are connected. Given three pairwise disjoint subsets $A$, $B$ and $C$ of $V$, $C$ is said to separate $A$ and $B$ if any walk between a node in $A$ and a node in $B$ contains a node in $C$. If no edge is repeated in a walk, then the walk is called a path, and, if $v_1 = v_k$, then the path is called a cycle. A graph that is only formed by a cycle is called a cycle graph. A graph with no cycles and connected is called a tree graph. A graph in which two nodes have degree 1 and all other nodes have degree 2 is called a path graph.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a function $\phi : V_1 \to V_2$ is called a graph homomorphism if it holds that $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$. If $\phi$ is bijective, $\phi$ is called a graph isomorphism. A graph isomorphism in which a graph $G$ is mapped onto itself is called an automorphism of $G$. The set of automorphisms of a graph $G$ under the composition operation forms a group, see [18], denoted by $\mathrm{Aut}(G)$.

A cyclic permutation of maximal length is a graph automorphism $\phi : V \to V$ with $V = \{a_0, \ldots, a_{n-1}\}$ such that $\phi(a_i) = a_{i+1 \pmod n}$ for any $a_i \in V$, denoted by $\phi = (a_0, \ldots, a_{n-1})$. A graph is called circulant if the automorphism group of the graph contains a cyclic permutation of maximal length. In particular, cycle graphs are circulant.

### 2.2. Circulant matrices

In this subsection, we recall the definition and some basic properties of circulant matrices. As we shall see, this type of matrices will be central to our work.

**Definition 2.1.** [35] A matrix $C = (C_{i,j})_{1 \le i,j \le n}$ of dimension $n \times n$ is called a circulant matrix if there exists a generating vector $\vec{c} = (c_0, \ldots, c_{n-1})^T$ such that $C_{i,j} = c_{j-i \pmod n}$. A circulant matrix $C$ is then denoted by $C = \mathrm{circ}(\vec{c})$, where $\vec{c}$ is the generating vector.

The structure of a circulant matrix is the following:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ddots & & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_0 \end{pmatrix}.$$

In particular, we will deal with correlation matrices that are circulant. Since correlation matrices are always symmetric, we are interested in circulant matrices for which $c_j = c_{n-j \pmod n}$.

The following proposition includes some properties of circulant matrices. The proof can be achieved following a similar procedure as in [35]. Additionally, this result can also be found in [6].

**Proposition 2.1.** *[6][35] Let $C = circ(\vec{c})$ be a circulant matrix of dimension $n \times n$ with $\vec{c} = (c_0, \ldots, c_{n-1})^T$. It holds that:*

- *$\vec{v_j} = \left(1, w_j, \ldots, w_j^{n-1}\right)^T$ is an eigenvector of $C$ for any $j \in \{0, \ldots, n-1\}$ with associated eigenvalue $\lambda_j = \sum_{k=0}^{n-1} c_k w_j^k$,*

- $C^{-1} = circ(\vec{d})$ *with* $\vec{d} = (d_0, \ldots, d_{n-1})^T$ *and*

$$d_j = \frac{1}{n} \sum_{k=0}^{n-1} \frac{w_j^k}{\lambda_k}\,,$$

*where* $w_j = \exp(-2\pi j i/n)$ *and* $i$ *denotes the imaginary unit.*

*2.3. Multivariate Gaussian distribution*

A continuous random vector $\vec{X}$ has a Multivariate Gaussian distribution if any linear combination of its components has a Univariate Gaussian distribution (see [28]). The joint probability density function of a Multivariate Gaussian random vector has the following expression:

$$f(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} \exp\left(-\frac{(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}{2}\right), \qquad \forall \vec{x} \in \mathbb{R}^n\,,$$

where $\vec{\mu}$ is the mean vector and $\Sigma = (\Sigma_{i,j})_{1\leq i,j\leq n}$ is the covariance matrix. The correlation matrix $S$ is defined by $S_{i,j} = \frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}}$. If the components of a random vector are indexed by a set $I$, i.e. $\vec{X} = (X_i \mid i \in I)$, for any $A \subset I$, we consider the notations $\vec{X}_A = (X_i \mid i \in A)$ and $\vec{X}_{-A} = (X_i \mid i \in I\backslash A)$. Given $A, B \subseteq I$, we denote by $\Sigma_{A,B}$ the submatrix of $\Sigma$ whose rows are those indicated by $A$ and whose columns are those indicated by $B$.

We denote the set of positive-definite matrices by $\mathscr{P}$, i.e., the set of all symmetric matrices of dimension $n \times n$ (for a certain $n \in \mathbb{N}$) such that for any non-null vector $\vec{v}$ of dimension $n$ it holds that $\vec{v}^T\Sigma\vec{v} > 0$. Note that a matrix is positive definite if and only if its eigenvalues are strictly positive [4]. In this work, we require $\Sigma$ to be positive definite.

Two continuous random vectors $\vec{X}$ and $\vec{Y}$ of dimensions $n_X$ and $n_Y$ are said to be conditionally independent, see [34], given another continuous random vector $\vec{Z}$ of dimension $n_Z$ if there exist $h : \mathbb{R}^{n_X+n_Z} \to [0, \infty]$ and $g : \mathbb{R}^{n_Y+n_Z} \to [0, \infty]$ such that:

$$f(\vec{x}, \vec{y}, \vec{z}) = h(\vec{x}, \vec{z})g(\vec{y}, \vec{z})\,, \;\; \forall \vec{x} \in \mathbb{R}^{n_X}\,, \;\; \forall \vec{y} \in \mathbb{R}^{n_Y}\,, \;\; \forall \vec{z} \in \mathbb{R}^{n_Z}\,.$$

We denote by $\vec{X}_A \perp \vec{X}_B \mid \vec{X}_C$ the fact that $\vec{X}_A$ and $\vec{X}_B$ are conditionally independent given $\vec{X}_C$.

Interestingly, for a Multivariate Gaussian distribution with covariance matrix $\Sigma$, the conditional covariance matrix of $\vec{X}_A$ given a value of $\vec{X}_{-A}$, denoted by $\Sigma_{A|-A}$, satisfies that $\Sigma_{A|-A} = ((\Sigma^{-1})_A)^{-1} = \Sigma_A - \Sigma_{A,-A}(\Sigma_{-A})^{-1}\Sigma_{-A,A}$ (see [35]).

**Example 2.1.** Consider the following covariance matrix (indexed by $\{1,2,3,4\}$):

$$\Sigma = \begin{pmatrix} 1 & 0.5 & 0.4 & 0.3 \\ 1 & 2 & 0.3 & 1 \\ 0.4 & 0.3 & 0.5 & 0.2 \\ 0.3 & 1 & 0.2 & 3 \end{pmatrix} .$$

In order to determine $\Sigma_{\{2,3\}|\{1,4\}}$, we just need to compute the inverse of $\Sigma$,

$$\Sigma^{-1} = \begin{pmatrix} 1.55575 & -0.217943 & -1.11027 & -0.0089096 \\ -0.217943 & 0.673703 & -0.152834 & -0.192584 \\ -1.11027 & -0.152834 & 2.995 & -0.0376945 \\ -0.0089096 & -0.192584 & -0.0376945 & 0.400932 \end{pmatrix} ,$$

and compute the inverse of the block matrix associated with $\{2,3\}$:

$$\Sigma_{\{2,3\}|\{1,4\}} = \begin{pmatrix} 0.673703 & -0.152834 \\ -0.152834 & 2.995 \end{pmatrix}^{-1} = \begin{pmatrix} 1.50172 & 0.0766323 \\ 0.0766323 & 0.3378 \end{pmatrix} .$$

This property $\Sigma_{A|-A} = ((\Sigma^{-1})_A)^{-1} = \Sigma_A - \Sigma_{A,-A}(\Sigma_{-A})^{-1}\Sigma_{-A,A}$ implies that conditionally independent variables are characterized by the null elements of the inverse of the covariance matrix. The following theorem plays an important role on the characterization of GMRFs, defined in the upcoming Subsection 2.4.

**Theorem 2.1.** *[35] Let $\vec{X}$ be a Multivariate Gaussian random vector with mean vector $\vec{\mu}$ and covariance matrix $\Sigma$. For any $i \neq j$, it holds that*

$$X_i \perp X_j \mid \vec{X}_{-\{i,j\}} \iff \left(\Sigma^{-1}\right)_{ij} = 0 .$$

### *2.4. Gaussian Markov random fields*

A Markov Random Field (MRF) associates the components of a random vector with the nodes of a graph, representing the conditional dependence of the components by means of the edges of the graph. Formally, given a graph $G = (V, E)$, a random vector $\vec{X} = (X_i \mid i \in V)$ is defined over the nodes of $G$. We are interested in three different properties of such random vector:

- The pairwise Markov property:

$$X_i \perp X_j \mid \vec{X}_{-\{i,j\}} \text{ for any } i, j \in V \text{ such that } \{i,j\} \notin E \text{ and } i \neq j .$$

- The local Markov property:

$$X_i \perp \vec{X}_{-\{i\}\cup N(i)} \mid \vec{X}_{N(i)} \text{ for any } i \in V .$$

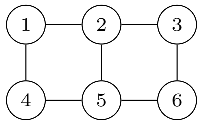


**Fig. 1.** Representation of a simple graph of order 6.

- The global Markov property:

$$\vec{X}_A \perp \vec{X}_B \mid \vec{X}_C \,,$$

for any pairwise disjoint $A, B, C \subset V$ with $A, B \neq \emptyset$ and where $C$ separates $A$ and $B$.

A random vector $\vec{X}$ that satisfies the global Markov property is called a Markov Random Field (MRF). If $\vec{X}$ is a Multivariate Gaussian random vector, then the three properties above are equivalent, [39], and we refer to $\vec{X}$ as a Gaussian Markov Random Field (GMRF). As a result of Theorem 2.1, given a GMRF, the Markov properties are characterized by the null elements of $\Sigma^{-1}$.

**Definition 2.2.** Let $G = (V, E)$ be a graph and $\vec{X}_V = \{X_i \mid i \in V\}$ be a Multivariate Gaussian random vector with mean vector $\vec{\mu}$ and covariance matrix $\Sigma$. The random vector $\vec{X}_V$ is called a GMRF over $G$ if $\{i, j\} \notin E$ implies that $(\Sigma^{-1})_{i,j} = 0$.

Note that the definition above may be found in some sources with the double implication, i.e. $\{i, j\} \notin E$ if and only if $(\Sigma^{-1})_{i,j} = 0$. However, for our purposes, it is more convenient to consider the definition with the single implication, which was already considered in [39]. In particular, this implies that we consider a Multivariate Gaussian distribution with diagonal covariance matrix to be a GMRF over any graph (not only over the graph with no edges).

**Example 2.2.** Consider a GMRF $\vec{X}_V$ with $V = \{1, \ldots, 6\}$ over the graph in Fig. 1. Since $\{1, 6\} \notin E$, it follows from the pairwise Markov property that $X_1$ and $X_6$ are conditionally independent given $X_{-\{1,6\}}$. Since $N(6) = \{3, 5\}$, it follows from the local Markov property that $X_1 \perp X_6 | X_{\{3,5\}}$. Finally, the subset of nodes $\{2, 5\}$ separates 1 and 6, therefore it follows from the global Markov property that $X_1 \perp X_6 | X_{\{2,5\}}$.

A GMRF in which all Pearson's correlation coefficients between adjacent variables are equal is called a GMRF with uniform correlation. This additional requirement allows us to find patterns in the correlation matrix of the distribution by studying the automorphism group of the graph (see upcoming Section 3.1).

**Definition 2.3.** Let $G = (V, E)$ be a graph. A Multivariate Gaussian random vector $\vec{X} = (X_i \mid i \in V)$ is called a GMRF with uniform correlation $\rho_0 \in (-1, 1)$ if the corresponding correlation matrix $S$ satisfies that:

- $S_{i,j} = \rho_0$ if $\{i, j\} \in E$;
- $(S^{-1})_{i,j} = 0$ if $\{i, j\} \notin E$ (and $i \neq j$).

Note that, since $S$ is a correlation matrix, its diagonal elements equal 1 (i.e., $S_{i,i} = 1$ for any $i \in \{1, \ldots, n\}$).

### 2.5. The GMRF with uniform correlation construction problem

Given a graph $G$ and a value $\rho_0$ for the uniform correlation, we aim at finding the correlation matrix of a GMRF with uniform correlation $\rho_0$ over $G$. This problem is a particular case of the GMRF construction problem defined in [39].

**Theorem 2.2.** *[39] Let $P$, $R \in \mathscr{P}$ and a graph $G = (V, E)$. There exists a unique $F \in \mathscr{P}$ that satisfies:*

- *$F_{i,j} = P_{ij}$ if $\{i, j\} \in E$ or if $i = j$,*
- *$(F^{-1})_{i,j} = R_{i,j}$ if $\{i, j\} \notin E$.*

We highlight the importance of finding the solution of this problem, typically in the case in which $R$ is the identity matrix, since it is related to several widely studied topics. Firstly, this problem is equivalent to finding the Maximum Likelihood Estimation (MLE) of the covariance matrix of a Multivariate Gaussian distribution given some linear constraints over the variables [11]. Secondly, it is also equivalent to finding the distribution that maximizes the differential entropy among all the random vectors with the variances and some of the covariances specified (since it maximizes the determinant of the associated covariance matrix) [16]. Finding the distribution that maximizes the differential entropy given some restrictions is a highly studied field, some examples can be found in [2,12,43]. Indirectly, in this paper we are finding the distribution that maximizes the differential entropy while setting to a fixed value the correlation between all pairs of variables whose nodes are adjacent on the cycle graph.

This problem was also introduced in [11] as a covariance selection model, which has been shown to be very useful for reducing the number of parameters in the estimation of the covariance matrix of a Multivariate Gaussian distribution (and, actually, of the exponential family, see [39]). Finally, there is a direct link between this problem and the positive definite completion of partial Hermitian matrices [16]. Please also note that finding the matrix $F$ in Theorem 2.2 is, in general, not easy. Numerical methods to solve the GMRF construction problem, not necessarily with uniform correlation, may be found in [3,11,39,41,42]. These algorithms are iterative approximations that converge to the solution. In this paper, we present two alternative methods for GMRFs with uniform correlation over cycle graphs (see Section 4). Our proposed methods are not based on the convergence to the solution but are based on finding a root of a non-linear function. The first one allows us to study in-depth the asymptotic properties of these distributions, whereas the second one is faster than the usual procedures in the literature, since the complexity in time grows linearly as the order of the graph increases. Unfortunately, they are restricted to GMRFs with uniform correlation over cycle graphs.

Setting $R = I_n$, it is concluded that there exists a (unique) correlation matrix of a GMRF with uniform correlation $\rho_0$ over $G$ as long as there exists a positive definite matrix $P$ satisfying that $P_{i,i} = 1$ for any $i \in V$ and $P_{i,j} = \rho_0$ if $\{i, j\} \in E$. In particular, for $\rho_0 \geq 0$, such matrix $P$ always exists. For instance, we may consider $P$ such that $P_{i,i} = 1$ with $i \in V$ and $P_{i,j} = \rho_0$ for any $i, j \in V$ with $i \neq j$. If $G$ is bipartite (with parts $V_1$ and $V_2$) and $\rho_0 < 0$, such matrix also exists. For instance, we may consider the matrix $P$ obtained as the solution for the case in which the uniform correlation equals $-\rho_0$ (which is assured to exist since $-\rho_0 > 0$) and consider the (positive definite) matrix $\hat{P}$ such that $\hat{P}_{i,j} = -P_{i,j}$ if $i \in V_1$ and $j \in V_2$ or $i \in V_2$ and $j \in V_1$ and $\hat{P}_{i,j} = P_{i,j}$ otherwise. Unfortunately, if $G$ is not bipartite, there exists a value $\rho_a \in (-1, 0)$ for which said positive definite matrix $P$ does not exist if $\rho_0 \in (-1, \rho_a)$. For instance, there does not exist a GMRF with uniform correlation $\rho_0 = -0.6$ over the complete graph with three nodes. Note that the uniqueness of the solution implies that any GMRF with uniform correlation $\rho_0$ over a graph $G$ has the same correlation matrix.

## 3. Graph automorphisms and the correlation matrix

Given a GMRF with uniform correlation over a graph, the study of the automorphism group of the graph allows us to identify elements of its correlation matrix $S$ that must be equal to each other. We devote this section to prove this statement and to study the particular case in which the graph is circulant (concluding that the correlation matrix must also be circulant).

### *3.1. General symmetries of a GMRF with uniform correlation*

The easiest graphs for which we can solve the GMRF with uniform correlation construction problem are the complete graph and the empty graph. For the complete graph, given a value of $\rho_0$ for which $S$ exists, $S$ is such that $S_{i,i} = 1$ for any $i \in V$ and $S_{i,j} = \rho_0$ for any $i, j \in V$ such that $i \neq j$. For the empty graph, $S$ is the identity matrix, regardless of the value of $\rho_0$. In both cases all the nodes are interchangeable, so any bijective function $\phi : V \to V$ is an automorphism of the graph.

The next result links directly the automorphisms of a graph to the structure of the correlation matrix of a GMRF with uniform correlation over the graph.

**Proposition 3.1.** *Let $G = (V, E)$ be a graph with $V = \{1, \dots, n\}$ and consider $\vec{X} = (X_1, \dots, X_n)^T$ a GMRF with uniform correlation over $G$. If $\phi$ is an automorphism of $G$, then the correlation matrix of $\vec{Y} = \left(X_{\phi(1)}, \dots, X_{\phi(n)}\right)^T$ is the same as the correlation matrix of $\vec{X}$.*

**Proof.** The correlation matrix $S$ of $\vec{X}$ is such that:

- $S_{i,i} = 1$, for any $i \in V$,
- $S_{i,j} = \rho_0$, for any $\{i,j\} \in E$,
- $\left(S^{-1}\right)_{i,j} = 0$, for any $\{i,j\} \notin E$ with $i \neq j$.

It holds that $\vec{Y}$ is a GMRF with uniform correlation $\rho_0$ over $\hat{G} = (\phi(V), \hat{E})$ with $\hat{E} = \{(\phi(i), \phi(j)) | \{i,j\} \in E\}$. The correlation matrix $\hat{S}$ of $\vec{Y}$ is another positive definite matrix such that:

- $\hat{S}_{\phi(i),\phi(i)} = 1$, for any $\phi(i) \in \phi(V)$,
- $\hat{S}_{\phi(i),\phi(j)} = \rho_0$, for any $\{\phi(i), \phi(j)\} \in \hat{E}$,
- $\left(\hat{S}^{-1}\right)_{\phi(i),\phi(j)} = 0$, for any $(\phi(i), \phi(j)) \notin \hat{E}$ with $\phi(i) \neq \phi(j)$.

Since $\phi$ is bijective, so is surjective, it holds that $\phi(V) = V = \{1, \ldots, n\}$ and, since $\phi$ is a graph homomorphism, it holds that $\{i,j\} \in E$ if and only if $\{\phi(i), \phi(j)\} \in \hat{E}$. Since $\vec{Y} = \left(X_{\phi(1)}, \ldots, X_{\phi(n)}\right)^T$, $S$ and $\hat{S}$ are both a solution of the same GMRF with uniform correlation construction problem. Due to the uniqueness of the solution (see Theorem 2.2), it is concluded that $S = \hat{S}$. $\square$

This result allows us to identify equal elements of the correlation matrix just by studying the automorphism group of the graph (Aut($G$)).

**Remark 3.1.** Although the correlation matrices of $\vec{X}$ and $\vec{Y}$ coincide, for the covariance matrices to coincide it is necessary that $\sigma_i = \sigma_{\phi(i)}$ for any $i \in V$ (since the covariance matrix is unequivocally determined by the correlation matrix and the variances of the variables). Furthermore, for both $\vec{X}$ and $\vec{Y}$ to be identically distributed, it is also necessary that $\mu_i = \mu_{\phi(i)}$ for any $i \in V$ (since a Multivariate Gaussian distribution is unequivocally determined by the mean vector and the covariance matrix).

### *3.2. GMRF with uniform correlation over circulant graphs*

From Proposition 3.1, it is possible to see that any GMRF with uniform correlation over a circulant graph has a circulant correlation matrix.

**Proposition 3.2.** *Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and consider $\vec{X} = (X_1, \ldots, X_n)^T$ a GMRF with uniform correlation over $G$. If $\phi = (a_0, \ldots, a_{n-1}) \in$ Aut($G$) is a cyclic permutation of maximal length of $V$, then the correlation matrix of $\vec{Y} = (X_{a_0}, \ldots, X_{a_{n-1}})^T$ is circulant.*

**Proof.** Let $S$ be the correlation matrix of $\vec{Y} = (X_{a_0}, \ldots, X_{a_{n-1}})^T$ and denote its first row by $S_1 = (s_0, \ldots, s_{n-1})$. Let $\phi^d$ denote the graph automorphism resulting of applying

$d$ times the graph automorphism $\phi = (a_0, \ldots, a_{n-1})$. The group structure of Aut$(G)$ assures that $\phi^d \in$ Aut$(G)$ and it obviously holds that $\phi^d(a_i) = (a_{i+d\ (mod\ n)})$ for any $i \in \{0, \ldots, n-1\}$. From Proposition 3.1, it follows that the correlation matrix of $\vec{Z} = \left(X_{\phi^d(a_0)}, \ldots, X_{\phi^d(a_n)}\right)^T = \left(X_{a_{d\ (mod\ n)}}, X_{a_{1+d\ (mod\ n)}}, \ldots, X_{a_{n-1+d\ (mod\ n)}}\right)^T$ is $S$. Thus, the elements of the $a_d$ $(mod\ n)$-th row of $S$ satisfy that $S_{a_d\ (mod\ n),j} = S_{1,j-a_d\ (\mathrm{mod}\ n)+1}$ $= s_{j-a_d\ (\mathrm{mod}\ n)}$ for any $j \in \{1, \ldots, n\}$. Since this holds for any $d \in \{1, \ldots, n\}$ the latter property is verified for all the rows $a_1, \ldots, a_{n-1}$ and is concluded that $S$ is a circulant matrix. $\square$

We conclude that, for a circulant graph, it suffices to find an appropriate reindexing of the variables of the GMRF with uniform correlation in order to guarantee that the correlation matrix is circulant. In particular, this result holds for cycle graphs.

## 4. Correlation matrix of a GMRF with uniform correlation over a cycle graph

In this section, we first provide a characterization of (the inverse of) the correlation matrix of a GMRF with uniform correlation over cycle graphs. This characterization serves as a source of inspiration for a method that allows to compute such correlation matrix. In Subsection 4.2, we provide an alternative method for the computation of the correlation matrix based on the relationship between the distribution and the stationary GMP over the circle.

### *4.1. General results*

We focus on the case in which the uniform correlation is not equal to zero, since the solution in that case is immediately given by the identity matrix. Firstly, we prove that the inverse of the correlation matrix of a GMRF with uniform correlation over a cycle graph is circulant with only three non-zero elements in each row.

**Lemma 4.1.** *Let $\vec{X} = (X_1, \ldots, X_n)^T$ be a GMRF with uniform correlation $\rho_0 \neq 0$ over the cycle graph of order $n$, $C_n$, with correlation matrix $S$. It holds that $S^{-1} = \alpha I_n + \beta A_{C_n}$, for some $\alpha, \beta \in \mathbb{R}$.*

**Proof.** Since a cycle graph is circulant, it follows from Proposition 3.2 that it is possible to reindex the variables in such a way that the correlation matrix $\hat{S}$ of $\vec{Y} = (X_{a_1}, \ldots, X_{a_n})^T$ is circulant. From Proposition 2.1 the inverse of a symmetric circulant matrix also is a symmetric circulant matrix, it follows that $\hat{S}^{-1} = \mathrm{circ}(\vec{d})$ for some appropriate $\vec{d} = (d_0, \ldots, d_{n-1})^T$. From Theorem 2.1 it follows that $d_i \neq 0$ only if $i \in \{n-1, 0, 1\}$. Since $\hat{S}^{-1}$ is symmetric, it follows that $\vec{d} = (\alpha, \beta, 0, \ldots, 0, \beta)^T$ for some $\alpha, \beta \in \mathbb{R}$. Returning to the original indexing of the variables, it is concluded that $S^{-1} = \alpha I_n + \beta A_{C_n}$. $\square$

To complete the characterization of the correlation matrix of a GMRF with uniform correlation over a cycle graph, the best possible scenario would be to express $\alpha$ and $\beta$ as a simple function of $\rho_0$ and $n$. Although formulas to find the inverse of circulant matrices with only three non-zeros elements in each row have been developed (see [38]), it is not evident how to express $\alpha$ and $\beta$ in terms of $\rho_0$ and $n$, since we should solve a non-linear equation in which the solution does not have a nice expression. However, it is possible to express $\alpha$ as a function of $\beta$ and $\rho_0$ and, ultimately, provide a characterization of the correlation matrix in terms of $\beta$ as a root of a polynomial with coefficients that only depend of $n$ and $\rho_0$. Since there might be more than one root of the aforementioned polynomial, we also provide some restrictions associated with the positive definiteness of the matrix that will allow us to select the correct root of the polynomial.

**Theorem 4.1.** *Let $S$ be a matrix of dimension $n \times n$. It holds that $S$ is the correlation matrix of a GMRF with uniform correlation $\rho_0 \neq 0$ over $C_n = (V, E)$ with $V = \{1, \dots, n\}$ if and only if $S^{-1} = (1 - 2\beta\rho_0)I_n + \beta A_{C_n}$, with $A_{C_n}$ being the adjacency matrix of $C_n$ and $\beta \in \mathbb{R}$, verifying:*

*(1) $\beta$ is root of the polynomial $P(t)$:*

$$(2t)^{n-1}\prod_{j=0}^{n-1} K_j + \sum_{j=1}^{n-1}(2t)^{j-1}\left(\sum_{\vec{v}\in\mathcal{P}(V)_j}\prod_{i=1}^{j} K_{v_i} - \frac{1}{n}\sum_{k=1}^{n}\sum_{\vec{v}\in\mathcal{P}(V_{-k})_j}\prod_{i=1}^{j} K_{v_i}\right),$$

*with $K_j = \cos(2\pi j/n) - \rho_0$, $\mathcal{P}(I)$ denoting the powerset of $I$ and $\mathcal{P}(I)_k = \{A \in \mathcal{P}(I) \mid \#A = k\}$.*

*(2) $\beta > \frac{1}{2(\rho_0 - 1)}$.*

*(3) $\beta < \frac{1}{2(\rho_0+1)}$ if $n$ is even and $\beta < \frac{1}{2\left(\rho_0 - \cos\left(\frac{\pi(n-1)}{n}\right)\right)}$ if $n$ is odd.*

**Proof.** ($\Rightarrow$) Without loss of generality, suppose that $S$ is a circulant matrix. If this is not the case, consider the appropriate reindexing of the variables, as in Proposition 3.2. Since $S\ S^{-1} = I_n$, it holds that $\vec{c}\ ^T\vec{d} = 1$, where $\vec{c}$ and $\vec{d}$ respectively denote the generating vectors of the circulant matrices $S$ and $S^{-1}$. Since $S$ is a correlation matrix of a GMRF with uniform correlation $\rho_0$, it holds that $\vec{c} = (1, \rho_0, c_2, \dots, c_{n-2}, \rho_0)^T$. From Lemma 4.1, it follows that $\vec{d} = (\alpha, \beta, 0, \dots, 0, \beta)^T$. Therefore, it holds that $\vec{c}\ ^T\vec{d} = \alpha + \beta\rho_0 + 0 + \dots + 0 + \beta\rho_0 = \alpha + 2\beta\rho_0$. It is concluded that $\alpha + 2\beta\rho_0 = 1$, and taking into account (by Lemma 4.1) that $S^{-1} = \alpha I_n + \beta A_{C_n}$, we can conclude that $S^{-1} = (1 - 2\beta\rho_0)I_n + \beta A_{C_n}$.

Let us first prove (1). From the formula for computing the inverse of a circulant matrix in Proposition 2.1, it follows that $c_0 = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\lambda_k}$, where $\lambda_0, \dots, \lambda_{k-1}$ of $S^{-1}$ are the eigenvalues of $S^{-1}$. We recall that the element $c_0$ equals 1, since $S$ is a correlation matrix. Therefore, the expression can be rewritten as $\sum_{k=0}^{n-1}\frac{1}{\lambda_k} = n$. Since the determinant of $S^{-1}$ equals the product of the eigenvalues of $S^{-1}$, we may multiply each side of the equality by the proper expression to obtain: $\sum_{k=0}^{n-1}\prod_{j=0,\, j\neq k}^{n-1}\lambda_j = n|S^{-1}|$. We substitute the eigenvalues of $S^{-1}$ by their expanded expression detailed in Proposition 2.1 bearing in mind that $w_k$ is defined as $w_k = \exp(2\pi ik/n)$ and $\vec{d} = (\alpha, \beta, 0, \dots, 0, \beta)$, thus obtaining

$$\lambda_k = \sum_{p=0}^{n-1} d_p w_k^p = \alpha + \beta(\exp(2\pi ik/n) + \exp(-2\pi ik/n)) = \alpha + 2\beta\cos(2\pi k/n)\,.$$

Since $\alpha + 2\beta\rho_0 = 1$, it is obtained that $\lambda_k = 1 + 2\beta(\cos(2\pi k/n) - \rho_0)$. Thus, we can express the determinant of $S^{-1}$ as follows

$$|S^{-1}| = \prod_{k=0}^{n-1}(1 + 2\beta(\cos(2\pi k/n) - \rho_0))\,.$$

We want to express the latter equality as a polynomial of $\beta$. We consider the notation $L = \{0, \dots, n-1\}$. Please note that any addend of the expression of $|S^{-1}|$ that is multiplied by a factor $\beta^j$ is also multiplied by a factor $2^j$ and a term $\prod_{k=l_1,\dots,l_j}(\cos(2\pi k/n) - \rho_0)$ for different $l_i, \dots, l_j \in L$. Moreover, the coefficient of the polynomial associated with $\beta^j$ is $2^j$ multiplied by the sum of the terms $\prod_{k=l_1,\dots,l_j}(\cos(2\pi k/n) - \rho_0)$ for all possible combinations of different $l_i, \dots, l_j \in L$. We denote the subsets of cardinality $j$ of $L$ as $\mathcal{P}(L)_j$. For any $l \in \mathcal{P}(L)_j$, we denote the $j$ elements of $l$ by $l_1, \dots, l_j$, i.e., $l = \{l_1, \dots, l_j\}$. By considering this notation, the determinant $|S^{-1}|$ may be expressed as follows:

$$|S^{-1}| = \sum_{j=0}^{n}(2\beta)^j \sum_{l\in\mathcal{P}(L)_j}\prod_{i=1}^{j}(\cos(2\pi l_i/n) - \rho_0)\,.$$

We denote, for any $k \in L$, the subset of $L$ defined as $\{l \in L \mid l \neq k\}$ by $L_{-k}$. Similarly to the determinant, which is the product of all the eigenvalues, the product of all but one of the eigenvalues, $\prod_{j\in L_{-k}}\lambda_j$ with $k \in L$ has the following expression:

$$\prod_{j\in L_{-k}}\lambda_j = \sum_{j=0}^{n-1}(2\beta)^j \sum_{l\in\mathcal{P}(L_{-k})}\prod_{i=1}^{j}(\cos(2\pi l_i/n - \rho_0)\,.$$

The right part of the equation $\sum_{k=0}^{n-1}\prod_{j=0,\,j\neq k}^{n-1}\lambda_j = n|S^{-1}|$ may be expressed as follows:

$$\sum_{k=0}^{n-1}\prod_{j=0,\,j\neq k}^{n-1}\lambda_j = \sum_{j=0}^{n-1}(2\beta)^j\sum_{k=0}^{n-1}\sum_{l\in\mathcal{P}(L_{-k})_j}\prod_{i=1}^{j}(\cos(2\pi l_i/n) - \rho_0)\,.$$

We consider the notation $K_j = \cos(2\pi j/n) - \rho_0$ and divide by $n$ in the equation $\sum_{k=0}^{n-1}\prod_{j=0,\,j\neq k}^{n-1}\lambda_j = n|S^{-1}|$, thus reaching the following expression:

$$(2\beta)^n\prod_{j=0}^{n-1}K_j + \sum_{j=0}^{n-1}(2\beta)^j\left(\sum_{\vec{v}\in\mathcal{P}(L)_j}\prod_{i=1}^{j}K_{l_i} - \frac{1}{n}\sum_{k=0}^{n-1}\sum_{l\in\mathcal{P}(L_{-k})_j}\prod_{i=1}^{j}K_{v_i}\right) = 0\,.$$

Note that the constant term of the polynomial equals 0. Finally, since the constant term is 0, after dividing the previous expression by $\beta$ (which is possible since $\beta = 0$ holds if and only if $\rho_0 = 0$), the resultant expression is the one provided in (1).

We now prove Conditions (2) and (3). The matrix $S^{-1}$ has to be positive definite. A necessary and sufficient condition for a matrix to be positive definite is that all its eigenvalues are strictly positive. If $\beta < 0$, the smallest eigenvalue is $\lambda_0 = \alpha + 2\beta = 1 - 2\beta\rho_0 + 2\beta$. If $\beta > 0$ and $n$ is even, the smallest eigenvalue is $\lambda_{n/2} = \alpha - 2\beta = 1 - 2\beta\rho_0 - 2\beta$, whereas if $\beta > 0$ and $n$ is odd, the smallest eigenvalues are $\lambda_{(n+1)/2} = \lambda_{(n-1)/2} = \alpha + 2\beta cos\left(\frac{\pi(n-1)}{n}\right) = 2\beta\rho_0 + 2\beta cos\left(\frac{\pi(n-1)}{n}\right)$.

($\Leftarrow$) Let $S^{-1} = \alpha I_n + \beta A_{C_n}$ with $\alpha = (1 - 2\beta\rho_0)$ and $\beta$ satisfying (1), (2) and (3). It holds that $S^{-1}$ is symmetric, circulant and, since (2) and (3) hold, the smallest eigenvalue is greater than 0 and, thus, $S^{-1}$ is positive definite. Therefore, $S$ is positive definite and, in particular, $S$ is the covariance matrix of a GMRF over $C_n$ since the elements associated with non-adjacent nodes of $S^{-1}$ are equal to zero.

Proceeding inversely to the steps in the proof of (1) for the converse implication, we conclude that all elements of the diagonal of $S$ are equal to one. Therefore, $S$ is a correlation matrix. Since $S$ is circulant and symmetric, all Pearson's correlation coefficients between $X_i$ and $X_{i+1 \pmod n}$ are the same for any $i \in \{1, \ldots, n\}$. It is concluded that $S$ is the correlation matrix of a GMRF with uniform correlation over $C_n$. Denote the value of this uniform correlation by $\hat{\rho_0}$. From the relation between $\alpha$ and $\beta$, it holds that $\beta = \frac{1-\alpha}{2\hat{\rho_0}}$, whereas from the hypothesis it holds that $\beta = \frac{1-\alpha}{2\rho_0}$. Thus, it necessarily holds that $\hat{\rho_0} = \rho_0$. □

From the results presented in Theorem 4.1, it is possible to design a method for computing the correlation matrix of any GMRF with uniform correlation over a cycle graph (if such GMRF with uniform correlation exists) and, in particular, to obtain $\beta$.

**Step 1.** Compute the coefficients of the polynomial $P(t)$ in Theorem 4.1.
**Step 2.** Compute the roots of $P(t)$.
**Step 3.** Find the root that satisfies Conditions (2) and (3) in Theorem 4.1.
**Step 4.** Find the inverse of $S^{-1} = (1 - 2\beta\rho_0)I_n + \beta A_{C_n}$.

As has already been mentioned, there exist some combinations of $n$ and $\rho_0$ for which the solution of the GMRF with uniform correlation construction problem does not exist. If such solution exists, the above procedure allows us to obtain the value of $\beta$ (and $\alpha$). Admittedly, a brute force implementation of the method above has exponential complexity, since the computation of the coefficients of the polynomial involves the power set of the set of nodes of the graph. Another issues are the computation of the roots of the polynomial and, once $\beta$ is already obtained, the computation of the inverse of the matrix $S^{-1}$. In this direction, some work has been done in recent years concerning the computation of the inverse of circulant matrices, see [6,7,14]. Nevertheless, the development of an efficient algorithm for the computation of $\beta$ and the correlation matrix is here left as a future study subject.

In the following, we provide an illustrative example for the computation of $\beta$ and the correlation matrix for a GMRF with uniform correlation over a cycle graph.

**Example 4.1.** Consider the cycle graph of order 6 and $\rho_0 = 0.7$. The values $K_i$ (with $i \in \{0, \dots, 5\}$) are the following ones:

$$K_0 = 0.3\,,\;\; K_1 = -0.2\,,\;\; K_2 = -1.2\,,\;\; K_3 = -1.7\,,\;\; K_4 = -1.2\,,\;\; K_5 = -0.2\,.$$

We compute the coefficients of $P(t)$, resulting in the following polynomial:

$$P(t) = -1.880064t^5 + 6.9888t^4 - 2.624t^3 - 10.64t^2 + 7.8t - 1.4\,.$$

The five roots of $P(t)$ are the following ones:

$$r_1 = 2.5,\;\; r_2 \approx -1.247,\;\; r_3 \approx 1.713,\;\; r_4 \approx 0.4167,\;\; r_5 \approx 0.3345\,.$$

Note that root $r_2$ is the only one contained in the interval $\left[\frac{1}{2(\rho_0-1)}, \frac{1}{(2(\rho_0+1)}\right] = [-1.666, 0.294]$. Thus, we identify $\beta = r_2 \approx -1.247$ (and obtain $\alpha = 1 - 2\beta\rho_0 \approx 2.746$). Finally, we construct the inverse of the correlation matrix:

$$S^{-1} \approx \begin{pmatrix} 2.746 & -1.247 & 0 & 0 & 0 & -1.247 \\ -1.247 & 2.746 & -1.247 & 0 & 0 & 0 \\ 0 & -1.247 & 2.746 & -1.247 & 0 & 0 \\ 0 & 0 & -1.247 & 2.746 & -1.247 & 0 \\ 0 & 0 & 0 & -1.247 & 2.746 & -1.247 \\ -1.247 & 0 & 0 & 0 & -1.247 & 2.746 \end{pmatrix}.$$

By inverting the matrix above, we verify that the diagonal elements equal 1 and the elements associated with adjacent nodes equal the uniform correlation $\rho_0 = 0.7$:

$$S \approx \begin{pmatrix} 1 & 0.7 & 0.541 & 0.492 & 0.541 & 0.7 \\ 0.7 & 1 & 0.7 & 0.541 & 0.492 & 0.541 \\ 0.541 & 0.7 & 1 & 0.7 & 0.541 & 0.492 \\ 0.492 & 0.541 & 0.7 & 1 & 0.7 & 0.541 \\ 0.541 & 0.492 & 0.541 & 0.7 & 1 & 0.7 \\ 0.7 & 0.541 & 0.492 & 0.541 & 0.7 & 1 \end{pmatrix}.$$

*4.2. Relationship with the stationary Gaussian Markov process on the circle*

In this subsection, we consider a stationary Gaussian Markov Process on the circle, as defined in Section 1 of [32]. Let $X_C = \{X_i \mid i \in [0, 2\pi)\}$ be a stochastic process over the unit circle, that is, a collection of random variables defined in a common probability space and indexed by the points of the unit circle. It is said that $X_C$ is a stationary GMP over the circle if:

- it is a Gaussian process: any finite subset of random variables has Multivariate Gaussian distribution.
- it is stationary: the process $Y_C = \{X_{i+k \ (mod \ 2\pi)} \mid i \in [0, 2\pi)\}$ has the same distribution as $X_C$ for any $k \in \mathbb{R}$.
- it is simply Markovian: for any interval $[a, b] \in [0, 2\pi)$ it holds that:

$$X_{(a,b)} \perp X_{[0,a)\cup(b,2\pi)} \mid X_{\{a,b\}} \, .$$

The explicit expression of the Pearson's correlation coefficient between two variables in a stationary GMP over the circle, $\rho_{a,b}$, is specified in the proof of Proposition 1.1 in [32] (in particular, see Equation (1.7)). For any $a, b \in [0, 2\pi)$ such that $|a - b| = t$, it holds that:

$$\rho_{a,b} = \frac{\cosh\left[(t-\pi)\sqrt{-\lambda/\upsilon}\right]}{\cosh\left[\pi\sqrt{-\lambda/\upsilon}\right]} \, ,$$

with $\lambda\upsilon < 0$ and cosh denoting the hyperbolic cosine function. For simplicity, we will denote $\sqrt{-\lambda/\upsilon}$ by $\varphi$ and refer to it as the parameter of the distribution.

This process can be seen as a continuous version of the GMRF with positive uniform correlation over cycle graphs. In fact, the distribution of the variables associated with a finite set of equispaced points on the circle is a GMRF with uniform correlation over the cycle graph.

**Proposition 4.1.** *Let $X_C$ be a stationary GMP on the circle with parameter $\varphi$. Let $V = \{0, 2\pi/n, 4\pi/n, \ldots, 2\pi(n-1)/n\}$ be a set of $n$ equispaced points on the circle. It holds that $\vec{X}_V$ is a GMRF with uniform correlation $\rho_0 = \frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]}{\cosh\left[\pi\varphi\right]}$ over $C_n = (V, E)$, where $E$ is defined by*

$$\{i, j\} \in E \textit{ if and only if } j \in \{i - \frac{2\pi}{n} \ (mod \ 2\pi), i + \frac{2\pi}{n} \ (mod \ 2\pi)\} \, .$$

**Proof.** We will prove that $\vec{X}_V$ satisfies the local Markov property over $C_n$. For any $i \in V$, the neighborhood of $i$ is defined as $N(i) = \{i^-, i^+\}$, where $i^+ = i + \frac{2\pi}{n} \ (mod \ 2\pi)$ and $i^- = i - \frac{2\pi}{n} \ (mod \ 2\pi)$. Without loss of generality, suppose that $i^- < i^+$ (otherwise, we may work with $Y_C = \{X_{j-i^- \ (mod \ 2\pi)} \mid j \in [0, 2\pi)\}$). Since $X_C$ is simply Markovian, it holds that:

$$X_{(i^-,i^+)} \perp X_{[0,i^-)\cup(i^+,2\pi))} \mid X_{i^-,i^+} \, .$$

Considering that $i \in (i^-, i^+)$, $V \backslash \{i^-, i, i^+\} \subset [0, i^-) \cup (i^+, 2\pi))$ and $N(i) = \{i^-, i^+\}$ it holds that:

$$X_i \perp \vec{X}_{-\{i\}\cup N(i)} \mid \vec{X}_{N(i)} \text{ for any } i \in V \, ,$$

i.e., the local Markov property holds. Since $X_C$ is a Gaussian process, $\vec{X}_V$ has Multivariate Gaussian distribution. From the equivalence between all three Markov properties when the distribution is Multivariate Gaussian [35], it is concluded that $\vec{X}_V$ is a GMRF over $C_n$.

The uniform correlation property is obtained from the fact that $\{i,j\} \in E \iff j \in \{i - \frac{2\pi}{n} \pmod n, i + \frac{2\pi}{n} \pmod n\} \iff |i-j| \in \left\{\frac{2\pi}{n}, \frac{2\pi(n-1)}{n}\right\} \iff \rho_{i,j} = \frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]}{\cosh[\pi\varphi]} = \rho_0$. For the last step, we recall that the hyperbolic cosine is an even function. □

Given a value of $\rho_0 \in (0,1)$ and $n \in \mathbb{N}$, we are now interested in finding the specific value of $\varphi$ that allows us to construct the GMRF with uniform correlation $\rho_0$ over the cycle graph of order $n$. The case in which $n = 1$ is trivial. Fortunately, for any $n > 1$, the value of $\varphi$ exists and is unique.

**Proposition 4.2.** *Consider $\rho_0 \in (0,1)$ and $n \in \mathbb{N}$ such that $n > 1$. There exists a unique $\varphi \in (0,\infty)$ such that $\frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]}{\cosh[\pi\varphi]} = \rho_0$.*

**Proof.** Let $f : [0,\infty) \to (0,1]$ be the function defined as follows:

$$f(\varphi) = \frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]}{\cosh[\pi\varphi]}\,.$$

This function is a continuous function such that $f(0) = 1$ and, since $\left|\left(\frac{2\pi}{n}-\pi\right)\varphi\right| < |\pi\varphi|$, it follows that:

$$\lim_{\varphi\to\infty} f(\varphi) = \lim_{\varphi\to\infty} \frac{e^{\left(\frac{2\pi}{n}-\pi\right)\varphi} + e^{-\left(\frac{2\pi}{n}-\pi\right)\varphi}}{e^{\pi\varphi}+e^{-\pi\varphi}} = 0\,.$$

The derivative $\frac{\partial f(\varphi)}{\partial\varphi}$ has the following expression:

$$\frac{\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]\sinh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]\cosh[\pi\varphi] - \cosh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right][\pi\varphi]\sinh[\pi\varphi]}{\left(\cosh[\pi\varphi]\right)^2}\,.$$

By applying some basic properties of hyperbolic functions (e.g., $\sinh(a+b) = \cosh(a)\sinh(b) + \cosh(b)\sinh(a)$), the previous derivative may be expressed as:

$$\frac{\partial f(\varphi)}{\partial\varphi} = \frac{\frac{2\pi}{n}\sinh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]\cosh[\pi\varphi] - [\pi\varphi]\sinh\left[\frac{2\pi}{n}\varphi\right]}{\left(\cosh[\pi\varphi]\right)^2}\,.$$

Since $\sinh(a) < 0$ if $a < 0$, $\sinh(a) > 0$ if $a > 0$ and $\cosh(a) > 0$ for any $a \in \mathbb{R}$, it holds that $\frac{\partial f(\varphi)}{\partial\varphi} < 0$ for any $\varphi \in (0,\infty)$.

Since $f(0) = 1$, $\lim_{\varphi\to\infty} f(\varphi) = 0$ and $\frac{\partial f(\varphi)}{\partial\varphi} < 0$ for any $\varphi \in (0,\infty)$, it is concluded that $f : [0,\infty) \to (0,1]$ is a bijective function, and, therefore, the result holds. □

The previous result allows us to define another alternative method to compute the correlation matrix of a GMRF with positive uniform correlation $\rho_0 \geq 0$ over the cycle graph of order $n$.

**Step 1.** Find $\varphi$ such that $\frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\varphi\right]}{\cosh[\pi\varphi]} = \rho_0$.
**Step 2.** Compute $s_j = \frac{\cosh\left[\left(\frac{2\pi j}{n}-\pi\right)\varphi\right]}{\cosh[\pi\varphi]}$ for $j \in \{0, \ldots, n-1\}$.
**Step 3.** Construct the correlation matrix $S = \text{circ}(\vec{s})$ with $\vec{s} = (s_0, \ldots, s_{n-1})^T$.

As it happened with $\beta$, it is not easy to provide an explicit function of $\varphi$ in terms of $n$ and $\rho_0$. However, this method results to be more efficient than the one proposed in the previous subsection and the ones used in the literature for general GMRFs, whose time complexity is of the order of $O(n^{4+\epsilon})$, with $\epsilon > 0$ (see, for instance [3,42]). In this case, the complexity of the first step is not dependent on the value of $n$ and the second and third steps have a complexity of $O(n)$. This contrasts with the case presented in the previous section, for which the computation of $\beta$ required the construction of a polynomial that led to an exponential complexity, the search for the correct root and the inversion of a matrix of dimension $n \times n$.

Unfortunately, there does not exist an analogous notion for the stationary GMP on the circle for negative (uniform) correlation. For a negative value of $\rho_0$, if $n$ is even $G$ is bipartite, thus we can use the existence of GRMFs with negative uniform correlation over bipartite graphs and the construction of their associated matrices, as discussed in the last paragraph of Subsection 3.2. In particular, it is possible to adapt the previous method just by defining $\vec{s}$ as $(s_0, -s_1, s_2, -s_3, \ldots, s_{n-2}, -s_{n-1})$.

If $n$ is odd, there is no way out and we must use the method defined in the previous subsection. As was stated for the first method, the development of an efficient algorithm for the computation of $\varphi$ and associated correlation matrix is left as a future study subject.

Finally, we end the section by providing an illustrative example.

**Example 4.2.** Consider (as in Example 4.1) the cycle graph of order 6 and $\rho_0 = 0.7$. We need to find a value of $\varphi$ such that the following equation holds:

$$\frac{\cosh\left[-\frac{2\pi\varphi}{3}\right]}{\cosh[\pi\varphi]} = 0.7\,.$$

A numerical approximation of the value, using the Newton-Rapshon method, leads to $\varphi \approx 0.42538$. Next, we just need to compute the values of $s_j = \frac{\cosh\left[\left(\frac{2\pi j}{6}-\pi\right)\varphi\right]}{\cosh[\pi\varphi]}$ for $j \in \{0, \ldots, 5\}$, which leads us to:

$$s_0 = 1\,,\ s_1 \approx 0.700\,,\ s_2 \approx 0.541\,,\ s_3 \approx 0.492\,,\ s_4 \approx 0.541\,,\ s_5 \approx 0.7\,.$$

This results in $S = \text{circ}(\vec{s})$, which is the same matrix obtained in Example 4.1.

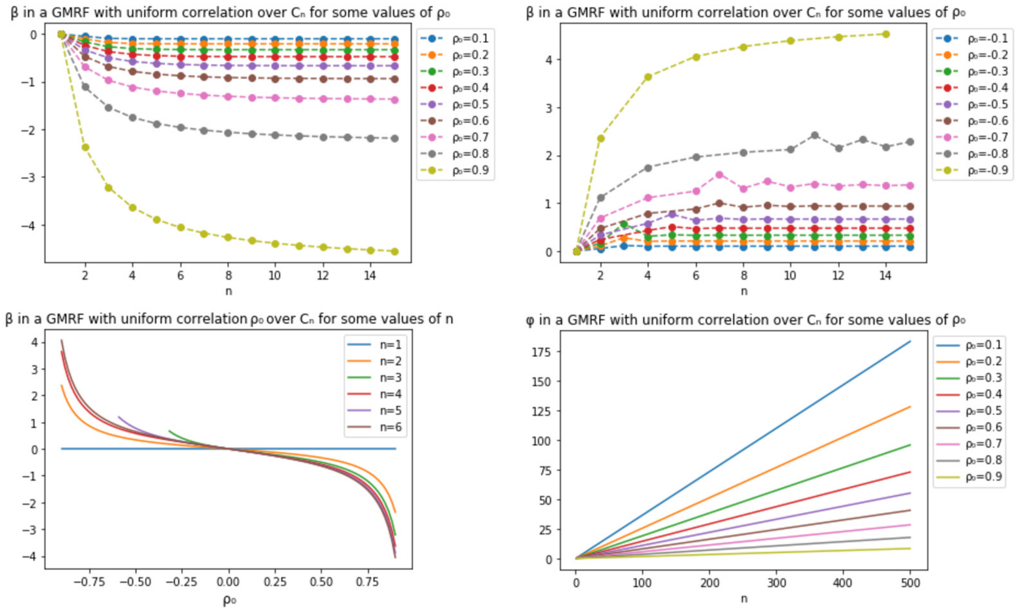


**Fig. 2.** Values of $\beta$ depending on $n$ for positive values of $\rho_0$ (top left), values of $\beta$ depending on $n$ for negative values of $\rho_0$ (top right), values of $\beta$ depending on $\rho_0$ for different values of $n$ (bottom left) and values of $\varphi$ depending on $n$ for positive values of $\rho_0$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

## 5. Asymptotic behavior of a GMRF with uniform correlation over a cycle graph

In this section, we study the asymptotic behavior of GMRFs with uniform correlation over a cycle graph. In particular, we devote our attention to the behavior of $\beta$ and $\varphi$ as $n$ increases. A graphical representation of the values of $\beta$ and $\varphi$ introduced in the previous section can be found in Fig. 2. In the top-right and bottom-left charts, the points corresponding to the combinations of $n$ and $\rho_0$ for which the matrix $S$ does not exist are not drawn. These results hint that it might be possible to obtain some asymptotic expression of $\beta$ and $\varphi$ as $n$ increases. In particular, $\beta$ seems to converge to a fixed value depending on $\rho_0$ and $\varphi$ seems to be linearly dependent on $n$, also depending on $\rho_0$. The convergence of $\beta$ is faster the closer $\rho_0$ is to 0, whereas the slope of $\varphi$ is greater the smaller $\rho_0$ is.

From now on, let us introduce a slightly different notation. For clarifying the value of $n$ and $\rho_0$, we will denote by $S(n, \rho_0)$ the correlation matrix of a GMRF with uniform correlation $\rho_0$ over the cycle graph of order $n$. Similarly, we denote by $S^{-1}(n, \rho_0)$ the inverse of the matrix $S(n, \rho_0)$.

We also recall some concepts about convergence of sequences. A sequence $(a_n)_{n\in\mathbb{N}}$ converges to the limit $a \in \mathbb{R}$, denoted by $\lim_{n\to\infty} a_n = a$, if, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n_0 < n \in \mathbb{N}$ it holds that $|a_n - a| < \epsilon$. Given a sequence $(a_n)_{n\in\mathbb{N}}$, a subsequence $(a_{\kappa(n)})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ is the sequence obtained from selecting from $(a_n)_{n\in\mathbb{N}}$ the values associated with the indices indicated by a strictly monotonically increasing mapping $\kappa : \mathbb{N} \to \mathbb{N}$. If a sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, then there exists a convergent subsequence of $(a_n)_{n\in\mathbb{N}}$. If the sequence $(a_n)_{n\in\mathbb{N}}$ is not bounded, then there exists a subsequence $(a_\kappa(n))_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_{\kappa(n)} = \infty$ or $\lim_{n\to\infty} a_{\kappa(n)} = -\infty$. If a sequence $(a_n)_{n\in\mathbb{N}}$ is bounded and any convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to the same limit $a$, then the sequence $(a_n)_{n\in\mathbb{N}}$ converges to the limit $a$. For more details on the convergence of sequences, we refer to [21].

### 5.1. Asymptotic behavior of $\beta$

In this subsection, we study the convergence of $\beta$ as $n$ increases, for a fixed value of $\rho_0$. Note that it is convenient to simultaneously work with $\alpha$ and $\beta$ (see Lemma 4.1), even though we know that they are related by the identity $\alpha + 2\beta\rho_0 = 1$ and thus $\beta$ converges if and only if $\alpha$ converges. Formally, the sequences in which we are interested are defined as follows.

**Definition 5.1.** Let $S(n, \rho_0)$ be the correlation matrix of a GMRF with uniform correlation $\rho_0$ over the cycle graph $C_n$, for any $\rho_0 \in (0, 1)$ and $n \in \mathbb{N}$. The following sequences are defined:

$$\big(\alpha(n, \rho_0)\big)_{n\in\mathbb{N}} = \big(S^{-1}(n, \rho_0)_{1,1}\big)_{n\in\mathbb{N}} ,$$
$$\big(\beta(n, \rho_0)\big)_{n\in\mathbb{N}} = \big(S^{-1}(n, \rho_0)_{1,2}\big)_{n\in\mathbb{N}} .$$

As mentioned previously, the sequences are related through the equality $\alpha(n, \rho_0) + 2\beta(n, \rho_0)\rho_0 = 1$, for any $n \in \mathbb{N}$, and the eigenvalues of $S^{-1}(n, \rho_0)$ have the expression $\lambda_k(n, \rho_0) = 1 + 2\beta(\cos(2\pi k/n) - \rho_0)$ with $k \in \{0, \dots, n-1\}$.

For negative values of $\rho_0$, we define the sequences above just by choosing only the values of $n \in \mathbb{N}$ for which $S(n, \rho_0)$ exists. The existence of the values of $S(n, \rho_0)$ is assured at least for all even values of $n$.

The remainder of the section is devoted to proving that the sequences $(\alpha(n, \rho_0))_{n\in\mathbb{N}}$ and $(\beta(n, \rho_0))_{n\in\mathbb{N}}$ converge to the following limits (see Theorem 5.1):

$$\lim_{n\to\infty} \alpha(n, \rho_0) = \frac{1 + {\rho_0}^2}{1 - {\rho_0}^2} \qquad \text{and} \qquad \lim_{n\to\infty} \beta(n, \rho_0) = -\frac{\rho_0}{1 - {\rho_0}^2} .$$

The result is especially interesting since these values coincide with those of a GMRF with uniform correlation $\rho_0$ over a path graph. This latter model is equivalent to the stationary model $AR(1)$, which has been widely studied in the context of time series, see [9,26], and where, unlike in the case of a GMRF with uniform correlation over a cycle graph, the expression of the correlation matrix and its inverse is straightforward. A similar interpretation of this result follows from the fact that a circle of infinite radius is nothing but a line.

**Proposition 5.1.** *[26] Let $S$ be the correlation matrix of a GMRF with uniform correlation $\rho_0$ over the path graph of order $n$. It holds that:*

1. $(S^{-1})_{1,1} = (S^{-1})_{n,n} = \frac{1}{1-\rho_0{}^2}$,
2. $(S^{-1})_{i,i} = \frac{1+\rho_0{}^2}{1-\rho_0{}^2}$, *for any* $i \in \{2, ..., n-1\}$,
3. $(S^{-1})_{i,i+1} = (S^{-1})_{i+1,i} = -\frac{\rho_0}{1-\rho_0{}^2}$, *for any* $i \in \{1, ..., n-1\}$.

For proving that both sequences converge, firstly, we prove that the sequences are bounded.

**Proposition 5.2.** *The sequences $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ and $(\beta(n,\rho_0))_{n\in\mathbb{N}}$ are bounded.*

**Proof.** Suppose that the sequence $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ is not bounded. In such case, there exists a subsequence $(\alpha(\kappa(n),\rho_0))_{n\in\mathbb{N}}$ that diverges to $\pm\infty$. We distinguish two cases.

- If $\lim_{n\to\infty} \alpha(\kappa(n),\rho_0) = -\infty$, then $\alpha(k(n_0),\rho_0) < 0$ for a $n_0 \in \mathbb{N}$ and any $n > n_0$. This contradicts the fact that $S^{-1}(k(n_0+1),\rho_0)$ is positive definite.
- If $\lim_{n\to\infty} \alpha(\kappa(n),\rho_0) = \infty$, we distinguish two subcases.
  (a) If $\rho_0 > 0$, since $\beta(n,\rho_0) = \frac{1-\alpha(n,\rho_0)}{2\rho_0}$, the eigenvalue $\alpha(\kappa(n),\rho_0) + 2\beta(\kappa(n),\rho_0) = \frac{1}{\rho_0} + \alpha(\kappa(n),\rho_0)\left(1 - \frac{1}{\rho_0}\right)$ tends to $-\infty$.
  (b) If $\rho_0 < 0$, we define $\ell_{k(n)} = \frac{\kappa(n)}{2}$ if $\kappa(n)$ is even and $\ell_{k(n)} = \frac{\kappa(n)-1}{2}$ if $\kappa(n)$ is odd, which are the indices of the smallest eigenvalues of $S^{-1}(\kappa(n),\rho_0)$. It follows that the eigenvalue $\lambda_{\ell_{k(n)}} = \alpha(n,\rho_0) + 2\beta(n,\rho_0)\cos(2\pi\ell_{k(n)}/\kappa(n)) = \frac{\cos(2\pi\ell_{k(n)}/\kappa(n))}{\rho_0} + \alpha(\kappa(n),\rho_0)\left(1 - \frac{\cos(2\pi\ell_{k(n)}(n)/\kappa(n))}{\rho_0}\right)$ tends to $-\infty$.
  In both cases, the contradiction follows from the positive definiteness of $S^{-1}(\kappa(n),\rho_0)$ for any $n \in \mathbb{N}$.

We end by noting that, if $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ is bounded, then $(\beta(n,\rho_0))_{n\in\mathbb{N}}$ is also bounded. □

In the following, we study a convergent subsequence of $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$. In particular, we prove that the eigenvalues of the inverse of the correlation matrix do not tend to zero as $n$ increases.

**Proposition 5.3.** *Let $(\alpha(\kappa(n),\rho_0))_{n\in\mathbb{N}}$ and $(\beta(\kappa(n),\rho_0))_{n\in\mathbb{N}}$ be two convergent subsequences of $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ and $(\beta(n,\rho_0))_{n\in\mathbb{N}}$. It holds that*

$$\lim_{n\to\infty} \alpha(\kappa(n),\rho_0) = a(\rho_0)\,,\ \lim_{n\to\infty} \beta(\kappa(n),\rho_0) = b(\rho_0)\,,\ \textit{with } a(\rho_0) > |2b(\rho_0)|\,.$$

**Proof.** Firstly, we prove that $a(\rho_0) > 0$. Since all eigenvalues are positive, it trivially follows that $a(\rho_0) \geq 0$. Suppose that $a(\rho_0) = 0$. Since $\beta(n,\rho_0) = \frac{1-\alpha(n,\rho_0)}{2\rho_0}$, it holds that $b(\rho_0) = (2\rho_0)^{-1}$. Therefore, the eigenvalue $\lambda_0 = \alpha(\kappa(n), \rho_0) + 2\beta(\kappa(n), \rho_0)$ tends to $0 + {\rho_0}^{-1}$, which is strictly negative for any $\rho_0 < 0$. We recall the definition $\ell_{k(n)} = \frac{\kappa(n)}{2}$ if $\kappa(n)$ is even and $\ell_{k(n)} = \frac{\kappa(n)-1}{2}$ if $\kappa(n)$ is odd. The eigenvalue $\lambda_{\ell_{k(n)}} = \alpha(n, \rho_0) + 2\beta(n, \rho_0)\cos(2\pi\ell_{k(n)}/\kappa(n))$ tends to $0 - {\rho_0}^{-1}$, which is strictly negative for any $\rho_0 > 0$. The contradiction follows from the fact that no eigenvalue can tend to a limit smaller than 0 because this would imply the existence of $n_0 \in \mathbb{N}$ such that $S^{-1}(n_0, \rho_0)$ has a negative eigenvalue, which contradicts that $S^{-1}(n_0, \rho_0)$ is positive definite.

Secondly, we prove that $a(\rho_0) > |2b(\rho_0)|$. Since $|\alpha(\kappa(n), \rho_0)| > |2\beta(\kappa(n), \rho_0)|$ due the positiveness of the eigenvalues of $S^{-1}(\kappa(n), \rho_0)$, it holds that $a(\rho_0) \geq |2b(\rho_0)|$. Suppose that $a(\rho_0) = |2b(\rho_0)|$. It either holds that $b(\rho_0) = \frac{a(\rho_0)}{2}$ or $b(\rho_0) = \frac{-a(\rho_0)}{2}$. The identity $S(n, \rho_0)S^{-1}(n, \rho_0) = I_n$ implies that:

$$\alpha(\kappa(n), \rho_0)\rho_d(\kappa(n), \rho_0) + \beta(\kappa(n), \rho_0)\left(\rho_{d-1}(\kappa(n), \rho_0) + \rho_{d+1}(\kappa(n), \rho_0)\right) = 0\,,$$

where $\rho_d(\kappa(n), \rho_0)$ is Pearson's correlation coefficient between two variables that are at distance $d > 0$ in the graph. The limit of these Pearson's correlation coefficients exists as a result of the expression above and the fact that the limits of $\alpha(\kappa(n), \rho_0)$ and $\beta(\kappa(n), \rho_0)$ exist and $\rho_0(\kappa(n), \rho_0) = 1$ and $\rho_1(\kappa(n), \rho_0) = \rho_0$ for any $n \in \mathbb{N}$. Denoting $\overline{\rho_d} = \lim_{n\to\infty} \rho_d(n, \rho_0)$, we arrive to the following recurrence relation: $a(\rho_0)\overline{\rho_d} + b(\rho_0)\left(r_{d-1}(\rho_0) + r_{d+1}(\rho_0)\right) = 0$, for $d > 0$ with $r_0 = 1$ and $r_1 = \rho_0$. We distinguish two cases:

- Case $b(\rho_0) = \frac{-a(\rho_0)}{2}$. The solution of the recurrence relation is $\overline{\rho}_d = d(\rho_0 - 1) + 1$. Since $(\rho_0 - 1)$ is negative, if we consider $d_0 > \frac{2}{1-\rho_0}$, then it follows that $r_{d_0} < -1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $\rho_{d_0}(n_0, \rho_0) < -1$, which contradicts the fact that $S(n_0, \rho_0)$ is a correlation matrix.
- Case $b(\rho_0) = \frac{a(\rho_0)}{2}$. The solution of the recurrence relation can be proved to be $\overline{\rho}_d = (-1)^{d+1}(d(\rho_0+1)-1)$. Since $(\rho_0+1)$ is positive, if we consider $d_0 > \frac{2}{\rho_0+1}$, then it follows that $|r_{d_0}| > 1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $|\rho_{d_0}(n_0, \rho_0)| > 1$, which contradicts the fact that $S(n_0, \rho_0)$ is a correlation matrix. □

In the following, we study the limit of Pearson's correlation coefficient between variables at a fixed distance as the order of the cycle graph tends to infinity (see Proposition 5.4). Next, we prove that the obtained expression tends to zero as the distance between the variables tends to infinity (see Proposition 5.5).

**Proposition 5.4.** *Let $(\alpha(\kappa(n), \rho_0))_{n\in\mathbb{N}}$ and $(\beta(\kappa(n), \rho_0))_{n\in\mathbb{N}}$ be two convergent subsequences of $(\alpha(n, \rho_0))_{n\in\mathbb{N}}$ and $(\beta(n, \rho_0))_{n\in\mathbb{N}}$. It holds that*

$$\overline{\rho_d} = \lim_{n\to\infty} \rho_d(\kappa(n), \rho_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(td)}{a(\rho_0) + 2b(\rho_0)\cos(t)}\, dt\,,$$

*where $\rho_d(\kappa(n), \rho_0)$ denotes Pearson's correlation coefficient between two variables whose associated nodes are at distance $d$ on the cycle graph.*

**Proof.** From Proposition 3.2, it follows that the inverse of the correlation matrix is $S^{-1}(n, \rho_0) = \alpha(n, \rho_0) I_n + \beta(n, \rho_0) A_{C_n}$. Bear in mind that the eigenvalues of $S^{-1}(n, \rho_0)$ are $\lambda_k(n, \rho_0) = 1 + 2\beta(\cos(2\pi k/n) - \rho_0)$ with $k \in \{0, \ldots, n-1\}$. From Proposition 2.1, it follows that Pearson's correlation coefficient between two variables whose associated nodes are at distance $d$ on the cycle graph equals:

$$\rho_d(n, \rho_0) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{e^{-2\pi i d \frac{k}{n}}}{\lambda_k(n, \rho_0)} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{e^{-2\pi i d \frac{k}{n}}}{\alpha(\rho_0, n) + 2\beta(\rho_0, n) \cos(2\pi k/n)} \,.$$

Replacing the values of $\alpha(n, \rho_0)$ and $\beta(n, \rho_0)$ by their limits does not introduce any indeterminacy in the expression, since Proposition 5.3 assures us that the eigenvalues of $S^{-1}(\kappa(n), \rho_0)$ do not tend to 0.

$$\begin{aligned} \overline{\rho_d} &= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{e^{-2\pi i d \frac{k}{n}}}{\alpha(\rho_0, n) + 2\beta(\rho_0, n) \cos(2\pi k/n)} \\ &= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{e^{-2\pi i d \frac{k}{n}}}{a(\rho_0) + 2b(\rho_0) \cos(2\pi k/n)} \,. \end{aligned}$$

By using Riemann's left approximation, [10], we obtain:

$$\overline{\rho_d} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-itd}}{a(\rho_0) + 2b(\rho_0) \cos(t)} \, dt \,,$$

and, from Euler's formula, [13], we obtain:

$$\overline{\rho_d} = \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{\cos(td)}{a(\rho_0) + 2b(\rho_0)\cos(t)} dt + i \int_0^{2\pi} \frac{\sin(td)}{a(\rho_0) + 2b(\rho_0)\cos(t)} \, dt \right) \,.$$

Since $\sin(d(t+\pi)) = -\sin(d(t-\pi))$ and $\cos(d(t+\pi)) = \cos(d(t-\pi))$, the function in the latter addend is an odd function over the axis $t = \pi$, and, therefore, the latter integral equals zero. $\square$

**Proposition 5.5.** *Let $(\alpha(\kappa(n), \rho_0))_{n \in \mathbb{N}}$ and $(\beta(\kappa(n), \rho_0))_{n \in \mathbb{N}}$ be two convergent subsequences of $(\alpha(n, \rho_0))_{n \in \mathbb{N}}$ and $(\beta(n, \rho_0))_{n \in \mathbb{N}}$. It holds that*

$$\lim_{d \to \infty} \lim_{n \to \infty} \rho_d(\ \kappa(n), \rho_0) = 0 \,,$$

*where $\rho_d(\kappa(n), \rho_0)$ denotes Pearson's correlation coefficient between two variables that are at distance $d$ on the cycle graph.*

**Proof.** We consider the result obtained in Proposition 5.4 and study the limit as $d$ tends to infinity:

$$\lim_{d\to\infty}\lim_{n\to\infty}\rho_d(n,\rho_0) = \lim_{d\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(td)}{a(\rho_0)+2b(\rho_0)\cos(t)}dt\,.$$

Note that $\{e_d\}_{d=0}^{\infty}$, with $e_0(t) = \frac{1}{\sqrt{2}}$, $e_d(t) = \sin(\frac{d+1}{2}t)$ if $d$ is odd and $e_d(t) = \cos(\frac{d}{2}t)$ if $d$ is even, form an orthonormal sequence in the space of all piecewise continuous functions on $[0, 2\pi]$ (see Theorem 4.4 in [40]) and, according to Riemman-Lebesgue's Theorem (see Theorem 4.2 in [40]), it holds that

$$\lim_{d\to\infty}\int_0^{2\pi} f(t)\,e_d(t)\,dt = 0\,,$$

for any $f$ that is piecewise continuous on $[0, 2\pi]$. The result follows from the (piecewise) continuity of $f(t) = \frac{1}{a(\rho_0)+2b(\rho_0)\cos(t)}$ on $[0, 2\pi]$. $\square$

Bear in mind that the order of the limits is important, since the distance between the nodes associated with two of the variables must be smaller than the order of the cycle graph. More specifically, it is not possible to consider the limit $\lim_{d\to\infty}\rho_d(n, \rho_0)$ since $\rho_d(n, \rho_0)$ is only defined for $d \leq d_{\max}$, where $d_{\max}$ is the maximum distance between two nodes on a cycle graph, i.e., $d_{\max} = \frac{n}{2}$ if $n$ is even and $d_{\max} = \frac{n-1}{2}$ if $n$ is odd.

The next proposition provides a value for the limits of both sequences. As a result, we conclude that, aside of the choice of the marginal distributions, the conditional distribution of any finite path of a GMRF with uniform correlation over a cycle graph is asymptotically equivalent to that of a GMRF with uniform correlation over a path graph.

**Proposition 5.6.** *Let $(\alpha(\kappa(n), \rho_0))_{n\in\mathbb{N}}$ and $(\beta(\kappa(n), \rho_0))_{n\in\mathbb{N}}$ be two convergent subsequences of $(\alpha(n, \rho_0))_{n\in\mathbb{N}}$ and $(\beta(n, \rho_0))_{n\in\mathbb{N}}$. It holds that*

$$\lim_{n\to\infty}\alpha(\kappa(n),\rho_0) = \frac{1+{\rho_0}^2}{1-{\rho_0}^2} \qquad \textit{and} \qquad \lim_{n\to\infty}\beta(\kappa(n),\rho_0) = -\frac{\rho_0}{1-{\rho_0}^2}\,.$$

**Proof.** Let $\vec{X}(\kappa(n))$ be a GMRF with uniform correlation $\rho_0$ over $C_{\kappa(n)}$. Without loss of generality consider the variances of the variables to be equal to 1. The conditional distribution given the value of one of the variables is a GMRF over a path graph of length $\kappa(n) - 1$. Let $C$ be a path of length $k + 2$ (for a certain $k < \kappa(n) - 2$) and $v$ a node that is at distance at least $d > 0$ to any node of $C$. The conditional distribution of $\vec{X}_C$ given $\vec{X}_v$ has the following covariance matrix [35] (see Example 2.1):

$$\Sigma(\kappa(n), \rho_0)_{C|v} = S(\kappa(n), \rho_0)_C - S(\kappa(n), \rho_0)_{Cv}\, \rho_{vv}\, S(\kappa(n), \rho_0)_{vC}\,.$$

From Proposition 5.5, it follows that

$$\lim_{d\to\infty}\lim_{n\to\infty} S(\kappa(n), \rho_0)_{C|v} = \lim_{d\to\infty}\lim_{n\to\infty} S(\kappa(n), \rho_0)_C\,,$$

since the elements of $S(\kappa(n), \rho_0)_{Cv}$ and $S(\kappa(n), \rho_0)_{vC}$ tend to zero as $n$ increases (see Proposition 5.5).

In other words, $\vec{X}_C$ tends to be independent of $\vec{X}_v$, therefore the conditional distribution of $\vec{X}_C$ given $\vec{X}_v$ tends to be the same as the distribution of $\vec{X}_C$. In addition, since the conditional distribution is a GMRF over the path graph, the marginal distribution tends to be a GMRF over the path graph. In particular, the marginal distribution tends to be a GMRF with uniform correlation over the path graph because the whole distribution has uniform correlation.

Let $c$ be the path formed by the nodes of $C$ with incidence equal to 2. From the equivalence of the pairwise and global Markov property, it follows that

$$\lim_{n\to\infty} S(\kappa(n), \rho_0)_{c|(C\setminus c)\cup\{v\}} = \lim_{n\to\infty} S(\kappa(n), \rho_0)_{c|-c}.$$

From the expression of the inverse of the correlation matrix of a GMRF over the path graph described in Proposition 5.1, and since $\vec{X}_{C|v}$ tends to be a GMRF with uniform correlation over the path graph:

- $\lim_{n\to\infty}(S(\kappa(n), \rho_0)_{c|-c})^{-1}_{i,i} = \frac{1+{\rho_0}^2}{1-{\rho_0}^2}$
- $\lim_{n\to\infty}\,, (S(\kappa(n), \rho_0)_{c|-c})^{-1}_{i,i+1} = (S(\kappa(n), \rho_0)_{c|(V\setminus c)})^{-1}_{i+1,i} = \frac{-\rho_0}{1-{\rho_0}^2}$.

Bearing in mind that $\Sigma_{c|-c} = ((\Sigma^{-1})_c)^{-1}$, it holds that $(S(\kappa(n), \rho_0)^{-1})_{i,i} = \alpha(\kappa(n), \rho_0)$ and $(S(\kappa(n), \rho_0))^{-1}_{i,i+1} = (S(\kappa(n), \rho_0))^{-1}_{i+1,i}$ for any $n \in \mathbb{N}$. It is concluded:

$$\lim_{n\to\infty} \alpha(\kappa(n), \rho_0) = a(\rho_0) = \frac{1+{\rho_0}^2}{1-{\rho_0}^2} \text{ and } \lim_{n\to\infty} \beta(\kappa(n), \rho_0) = b(\rho_0) = \frac{-\rho_0}{1-{\rho_0}^2}. \quad \square$$

It has been proved that any convergent subsequence of $(\alpha(n, \rho_0))_{n\in\mathbb{N}}$ converges to the same limit and any convergent subsequence of $(\beta(n, \rho_0))_{n\in\mathbb{N}}$ converges to the same limit. In addition, it can be seen that these limits are continuous as a function of $\rho_0$ in $(-1, 1)$. Finally, we only have to prove that the sequences $(\alpha(n, \rho_0))_{n\in\mathbb{N}}$ and $(\beta(n, \rho_0))_{n\in\mathbb{N}}$ converge to the aforementioned limits. The result follows from the fact that the sequences are bounded.

**Theorem 5.1.** *Let $S(n,\rho_0)$ be the correlation matrix of a GMRF with uniform correlation $\rho_0$ over the cycle graph $C_n$. It holds that $S(n,\rho_0)^{-1} = \alpha(n,\rho_0)I_n + \beta(n,\rho_0)A_{C_n}$ with:*

$$\lim_{n\to\infty} \alpha(n,\rho_0) = \frac{1+{\rho_0}^2}{1-{\rho_0}^2} \text{ and } \lim_{n\to\infty} \beta(n,\rho_0) = \frac{-\rho_0}{1-{\rho_0}^2}\,.$$

**Proof.** From Proposition 5.6, it follows that any convergent subsequence of $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ converges to the same limit and that this limit is $\frac{1+{\rho_0}^2}{1-{\rho_0}^2}$. From Proposition 5.2, it follows that $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ is bounded, therefore, it is concluded that the limit of $(\alpha(n,\rho_0))_{n\in\mathbb{N}}$ is $\frac{1+{\rho_0}^2}{1-{\rho_0}^2}$. Analogously, it is concluded that the limit of $(\beta(n,\rho_0))_{n\in\mathbb{N}}$ is $\frac{-\rho_0}{1-{\rho_0}^2}$. □

As a corollary, the expression of the limit of Pearson's correlation coefficient between two variables is provided.

**Corollary 5.1.** *Let $S(n,\rho_0)$ be the correlation matrix of a GMRF with uniform correlation $\rho_0$ over the cycle graph $C_n$. Pearson's correlation coefficient between two variables that are at distance $d$ in the graph satisfies that:*

$$\overline{\rho_d} = \lim_{n\to\infty} \rho_d(\kappa(n),\rho_0) = {\rho_0}^d\,.$$

*5.2. Asymptotic behavior of $\varphi$*

The bottom right chart in Fig. 2 hints that, as $n$ increases, $\varphi$ seems to behave as a linear function of $n$ with a slope only dependent on the value of $\rho_0$. Therefore, we will study the convergence of the quantity $\overline{\varphi}$ defined as $\overline{\varphi} = \frac{\varphi}{n}$ as $n$ increases. Formally, we define the following sequence.

**Definition 5.2.** Consider $\rho_0 \in (0,1)$. The sequence $(\overline{\varphi}(n,\rho_0))_{n\in\mathbb{N}}$ is the sequence of positive terms that satisfy:

$$\frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\overline{\varphi}(n,\rho_0)n\right]}{\cosh\left[\pi\overline{\varphi}(n,\rho_0)n\right]} = \rho_0\,,$$

for any $n \in \mathbb{N}$.

We note that the previous sequence is well-defined (see Proposition 4.2). The next result determines the limit of the sequence and completes the study of the asymptotic behavior of GMRFs over cycle graphs.

**Proposition 5.7.** *The sequence $(\overline{\varphi}(n,\rho_0))_{n\in\mathbb{N}}$ converges to $-\frac{\log(\rho_0)}{2\pi}$.*

**Proof.** We recall that the following equation holds for any $n \in \mathbb{N}$:

$$\frac{\cosh\left[\left(\frac{2\pi}{n}-\pi\right)\overline{\varphi}(n,\rho_0)n\right]}{\cosh\left[\pi\overline{\varphi}(n,\rho_0)n\right]} = \frac{e^{\left(\frac{2\pi}{n}-\pi\right)\overline{\varphi}(n,\rho_0)n} + e^{-\left(\frac{2\pi}{n}-\pi\right)\overline{\varphi}(n,\rho_0)n}}{e^{\pi\overline{\varphi}(n,\rho_0)n} + e^{-\pi\overline{\varphi}(n,\rho_0)n}}$$

$$= \frac{e^{(\frac{2\pi}{n}-2\pi)\overline{\varphi}(n,\rho_0)n} + e^{-2\pi\overline{\varphi}(n,\rho_0)}}{1+e^{-2\pi\overline{\varphi}(n,\rho_0)n}} = \rho_0\,,$$

thus the limit equals $\rho_0$ too. It also holds that $\overline{\varphi}(n,\rho_0) > 0$ for any $n \in \mathbb{N}$.

Firstly, we will prove that $(\overline{\varphi}(n,\rho_0))_{n\in\mathbb{N}}$ is bounded. Suppose that it is not bounded. Therefore, there exists a subsequence $(\overline{\varphi}(\kappa(n),\rho_0))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\overline{\varphi}(\kappa(n),\rho_0) = \infty$. This implies that

$$\lim_{n\to\infty} \frac{e^{(\frac{2\pi}{\kappa(n)}-2\pi)\overline{\varphi}(\kappa(n),\rho_0)n} + e^{-2\pi\overline{\varphi}(n,\rho_0)}}{1+e^{-2\pi\overline{\varphi}(\kappa(n),\rho_0)\kappa(n)}} = 0 \neq \rho_0\,,$$

which leads to a contradiction.

Consider a convergent subsequence $(\overline{\varphi}(\kappa(n),\rho_0))_{n\in\mathbb{N}}$ that converges to the limit $\lim_{n\to\infty}\overline{\varphi}(\kappa(n),\rho_0) = s(\rho_0)$. Since $(\overline{\varphi}(\kappa(n),\rho_0))_{n\in\mathbb{N}}$ is convergent, there are two possibilities regarding the convergence of $(\overline{\varphi}(\kappa(n),\rho_0)\kappa(n))_{n\in\mathbb{N}}$. Note that the latter sequence correspond to the values of $\varphi$ for a fixed $\rho_0$. It either converges to a limit $\lim_{n\to\infty}\overline{\varphi}(\kappa(n),\rho_0)\kappa(n) = d(\rho_0) \in \mathbb{R}$ or it holds that $\lim_{n\to\infty}\overline{\varphi}(\kappa(n),\rho_0)\kappa(n) = \infty$.

If $\lim_{n\to\infty}\overline{\varphi}(\kappa(n),\rho_0)\kappa(n) = d(\rho_0)$, then $s(\rho_0) = 0$ and consequently

$$\lim_{n\to\infty} \frac{e^{(\frac{2\pi}{\kappa(n)}-2\pi)\overline{\varphi}(\kappa(n),\rho_0)\kappa(n)} + e^{-2\pi\overline{\varphi}(n,\rho_0)}}{1+e^{-2\pi\overline{\varphi}(\kappa(n),\rho_0)\kappa(n)}} = \frac{e^{-2\pi d(\rho_0)} + e^{2\pi s(\rho_0)}}{1+e^{-2\pi d(\rho_0)}} = 1 \neq \rho_0\,,$$

which leads to a contradiction.

We conclude that $\lim_{n\to\infty}\overline{\varphi}(\kappa(n),\rho_0)\kappa(n) = \infty$. We compute the latter limit for this case:

$$\lim_{n\to\infty} \frac{e^{(\frac{2\pi}{\kappa(n)}-2\pi)\overline{\varphi}(\kappa(n),\rho_0)\kappa(n)} + e^{-2\pi\overline{\varphi}(n,\rho_0)}}{1+e^{-2\pi\overline{\varphi}(\kappa(n),\rho_0)\kappa(n)}} = e^{-2\pi s(\rho_0)}\,.$$

It follows that $e^{-2\pi s(\rho_0)} = \rho_0$, therefore $s(\rho_0) = -\frac{\log(\rho_0)}{2\pi}$ regardless of the chosen convergent subsequence. Since $(\overline{\varphi}(n,\rho_0))_{n\in\mathbb{N}}$ is bounded and any convergent subsequence converges to the same limit $-\frac{\log(\rho_0)}{2\pi}$, it holds that $(\overline{\varphi}(n,\rho_0))_{n\in\mathbb{N}}$ converges to $-\frac{\log(\rho_0)}{2\pi}$. $\square$

We conclude that, for large values of $n$, $\varphi$ behaves as $-\frac{\log(\rho_0)n}{2\pi}$. This result coincides with the results illustrated in the bottom right side of Fig. 2, where the slope of the line seems to increase as $\rho_0$ is closer to 0.

As a corollary of Proposition 5.7, the asymptotic value of Pearson's correlation coefficient between two variables at distance $d$ can be computed when $\rho_0 \geq 0$. This result is equivalent to that of Corollary 5.1 for the case of positive correlation.

**Corollary 5.2.** *Let $S(n,\rho_0)$ be the correlation matrix of a GMRF with uniform correlation $\rho_0 \geq 0$ over the cycle graph $C_n$. Pearson's correlation coefficient between two variables that are at distance $d$ in the graph satisfies that:*

$$\overline{\rho_d} = \lim_{n \to \infty} \rho_d(\kappa(n), \rho_0) = {\rho_0}^d .$$

## 6. Conclusions

A characterization of the correlation matrix of a GMRF with uniform correlation over a cycle graph by using circulant matrices is provided. Based on this characterization, a method to compute the correlation matrix of a GMRF with uniform correlation over a cycle graph is also provided. Special interest is devoted to the study of the asymptotic behavior of the non-null values of the inverse of the correlation matrix, $\alpha$ and $\beta$, which are only dependent on $\rho_0$. The obtained expressions provide a reasonable approximation of $\alpha$ and $\beta$ for not-too-large values of $n$. The convergence speed depends on the value of $\rho_0$ as can be seen in Fig. 2. In particular, the closer $\rho_0$ is to 0, the faster the convergence seems to be. All these results point out that a GMRF over a cycle graph asymptotically behaves like a GMRF over a path graph as the order of the graph increases.

In addition, a study of the relationship between a GMRF with positive uniform correlation over a cycle graph and the stationary GMP over the circle has been addressed. As a result of this relationship, an alternative method to compute the correlation matrix is proposed. This second method seems to be faster than the first one, even though it cannot be applied if $\rho_0 < 0$ and $n$ is odd. The asymptotic behavior has also been studied.

From the results of this paper the following question arises: Is the structure of a GMRF with uniform correlation over a cycle graph useful to determine the structure of a GMRF with uniform correlation over some families of graphs containing cycles? For instance, the structure of graphs constructed such that any connected component is a cycle follows trivially from the results of this paper. In addition, the complementary of cycle graphs, which share the same automorphism group, can be characterized by using similar methods than those developed in this paper. On the contrary, the case of the Cartesian product of cycle graphs is not immediate since the correlation matrix will then be block circulant rather than circulant. Still, we believe that most results of this paper can be adapted to such case by studying the automorphism group and considering the properties of block circulant matrices. However, the study of GMRFs over the Cartesian product of cycle graphs and possibly over some other cycle-based graphs is left as a future study subject.

## Declaration of competing interest

There is no competing interest.

## Data availability

No data was used for the research described in the article.

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