Joint measurability of mappings induced by a fuzzy random variable

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Abstract

The operation of taking the α-cut of a compact convex fuzzy set is shown to be jointly measurable with respect to both α and the fuzzy set. As a consequence, a number of mappings on product spaces which are induced by a fuzzy random variable are shown to be jointly measurable. Some applications to the relationships between fuzzy random variables and other imprecise random elements are obtained. Finally, a number of conditions are shown to be equivalent to being a fuzzy random variable, at least in the case that the σ-algebra in the sample space is complete, and logical implications and equivalences between them are established in the general case.

Keywords: α-cut mapping; Fuzzy random variable; Product σ-algebra; Random compact set.

1 Introduction

Measurability is a key property in Probability Theory defining which functions can be interpreted as random elements of a set of objects under study. Many papers have dealt with the issue of measurability for fuzzy random variables, which are mathematically more complex than ordinary random variables. In fact, early papers of Puri and Ralescu presented different measurability conditions [20, 19]. The interested reader is referred to papers like [16, 13, 6, 22] for further details on measurability conditions and their interrelationships.

The standard definition of a fuzzy random variable ensures that each α-cut mapping $U \mapsto U_\alpha$ (where U represents a fuzzy set in an appropriate space) is measurable. That is, for fixed U and variable α, the way in which the α-cut U_α changes as α changes is measurable. However, this is insufficient for situations where U is not really fixed, for instance to describe procedures which involve U_α but depend on functions of both U and α.

To fix ideas, consider the paper [24] by Terán and López-Díaz. It studies the problem of approximating a fuzzy set U (e.g., the home range of a wild animal with a safe inner core with high membership and an external ring with smaller membership) by a double sampling procedure. The animal is tagged with a radio transmitter and at fixed intervals a number of measurements of its position are taken. Since the animal moves freely, at the time of measurement it is (1) in a random α-level of its range, and (2) in a random location in that α-level. An estimator of the fuzzy range is then constructed from this random information, which depends simultaneously on the home range U and the sampled α-levels. To ensure that the estimator is a fuzzy random variable, it becomes necessary to require that its dependence on both U and α is *jointly* measurable. That paper also presents an application to breast cancer data where the fuzzy set-valued estimator avoids an obfuscation effect (created by the dependence between two relevant variables) which is suffered by the traditional set-valued estimator.

This paper's first contribution is to show that, for fuzzy random variables having convex values (which are the most common in applications), measurability with respect to α for any fixed U actually ensures joint measurability with respect to U and α simultaneously.

The basic result is Theorem 4.1, which states that the mapping $L : (U, \alpha) \mapsto U_\alpha$ is jointly measurable. This is obtained by combining a measurability result in the theory of Carathéodory functions (with values in a metric space), with the approximation scheme used in [23].

We obtain then a sequence of results which have in common that they yield the joint measurability of mappings defined on product spaces and in many cases involve a fuzzy random variable.

Some of these mappings are related to standard objects and techniques to handle fuzzy random variables, e.g., the support function of a fuzzy random variable X,

$$\widetilde{s}_X : (\omega, \alpha, r) \in \Omega \times [0,1] \times \mathbb{S}^{d-1} \mapsto \sup_{x \in X_\alpha} \langle r, x \rangle$$

whose joint measurability was established by Krätschmer [14] and follows from our results.

Others are very natural but seem not to have been considered in the literature, e.g., the mapping

$$B_X : (\omega, \alpha, x) \in \Omega \times [0,1] \times \mathbb{R}^d \mapsto \begin{cases} 1, & x \in X_\alpha(\omega) \\ 0, & x \notin X_\alpha(\omega), \end{cases}$$

which is shown to be a Bernoulli random variable.

We show that an alternative approach to fuzzy random variables by Castaing *et al.* [5], which looks formally more restrictive, is in fact equivalent to graph measurability. We also show that a fuzzy random variable, when looked at 'vertically' instead of 'horizontally' generates a probabilistic set, thus connecting these two well-known approaches to combining fuzzy and random uncertainty. Another example of a natural result which was missing in the literature is that taking a random α-cut of a fuzzy random variable yields a random set.

Finally, these results will be used to study a number of conditions closely related to being a fuzzy random variable. Some of them are equivalent in general, and the others are so under the assumption that the σ-algebra in the sample space is complete.

The structure of the paper is as follows. Definitions and preliminary notions are presented in Section 2. A number of support results from several sources in the literature are collected in Section 3. Section 4 contains the joint measurability results. Some applications are given in Section 5. Measurability conditions for fuzzy random variables are studied in Section 6. We have found it convenient to close the paper with a short discussion of graph measurability for fuzzy random variables in Section 7.

2 Preliminaries

A mapping $f : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ between two measurable spaces is called *measurable* if $f^{-1}(\mathcal{A}_2) \subseteq \mathcal{A}_1$. If Ω_1 is a product space then f is called *jointly measurable* if it is measurable with respect to the product σ-algebra in Ω_1. Joint measurability is thus a particular case of measurability and the name emphasizes that f is required to be measurable with respect to all its variables, as opposed to *separate measurability* or *section measurability*, i.e., the property that f is measurable as a function of each variable when the values of all the other variables are fixed.

In the sequel, product spaces will always be endowed with the product σ-algebra. A general probability space will be denoted $(\Omega, \mathcal{A}, P)$. The Lebesgue measure in $[0,1]$ will be called λ. The closure of a set A will be denoted by $\operatorname{cl} A$.

Let $\mathcal{K}_c(\mathbb{R}^d)$ be the space of non-empty convex compact subsets of $\mathbb{R}^d$. Let $\mathcal{F}_c(\mathbb{R}^d)$ be the space of fuzzy subsets of $\mathbb{R}^d$, i.e, mappings $U : \mathbb{R}^d \to [0,1]$, whose α-cuts belong to $\mathcal{K}_c(\mathbb{R}^d)$. The *$\alpha$-cuts* are

$$U_\alpha = \{x \in \mathbb{R}^d : U(x) \geq \alpha\}$$

for $\alpha \in (0,1]$ and

$$U_0 = \operatorname{cl}\{x \in \mathbb{R}^d : U(x) > 0\}.$$

In $\mathcal{K}_c(\mathbb{R}^d)$ we consider the *Hausdorff metric* d_H defined by

$$d_H(K,L) = \max\{\sup_{x\in K} \inf_{y\in L} \|x-y\|, \sup_{y\in L} \inf_{x\in K} \|x-y\|\}$$

for $K, L \in \mathcal{K}_c(\mathbb{R}^d)$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

For each $U \in \mathcal{F}_c(\mathbb{R}^d)$, let $L_U : [0,1] \to \mathcal{K}_c(\mathbb{R}^d)$ be defined by $L_U(\alpha) = U_\alpha$. We denote by $\mathcal{F}_{cc}(\mathbb{R}^d)$ the subspace of all those $U \in \mathcal{F}_c(\mathbb{R}^d)$ such that L_U is continuous.

If $\mathbb{E}$ is a topological space, $\mathcal{B}_{\mathbb{E}}$ will denote its Borel σ-algebra, namely the σ-algebra generated by its open sets. Thus $\mathcal{K}_c(\mathbb{R}^d)$ is endowed with the σ-algebra $\mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)}$.

Definition 2.1. *A measurable mapping $\Gamma : (\Omega, \mathcal{A}) \to (\mathcal{K}_c(\mathbb{R}^d), \mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)})$ will be called a* random compact set*.*

The space $\mathcal{F}_c(\mathbb{R}^d)$ is endowed with the smallest σ-algebra which makes the α-cut mappings

$$L_\alpha : U \in \mathcal{F}_c(\mathbb{R}^d) \mapsto U_\alpha \in \mathcal{K}_c(\mathbb{R}^d)$$

measurable for all $\alpha \in [0,1]$. Equivalently, this condition can be required only for $\alpha \in (0,1]$ or just in a countable dense subset of $[0,1]$. That σ-algebra is called the *levelwise* or *cylindrical σ-algebra* and denoted by σ_L.

Definition 2.2. *Let* $(\Omega, \mathcal{A})$ *be a measurable space. A mapping* $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ *is a* fuzzy random variable *if it is* $(\mathcal{A}, \sigma_L)$*-measurable.*

Therefore, X is a fuzzy random variable if and only if each mapping $X_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ is a random compact set.

Definition 2.3. *Let* (S, Σ) *be a measurable space, and let* Y *and* Z *be topological spaces. A function* $f : S \times Y \to Z$ *is a* Carathéodory function *if:*

1. *For each* $y \in Y$*, the function* $f(\cdot, y) : S \to Z$ *is* $(\Sigma, \mathcal{B}_Z)$*-measurable; and*

2. *For each* $s \in S$*, the function* $f(s, \cdot) : Y \to Z$ *is continuous.*

Definition 2.4. *Let* $K \in \mathcal{K}_c(\mathbb{R}^d)$*. The* support function *of* K *is the mapping*

$$\begin{aligned} s_K : \mathbb{S}^{d-1} &\to \mathbb{R} \\ r &\mapsto \max_{x \in K} \langle r, x \rangle, \end{aligned}$$

where $\mathbb{S}^{d-1}$ *denotes the unit sphere.*

3 Support results

This section collects a few tools which will be used in the sequel. Some recent related results are in [4, 8].

The following classical result can be found in, e.g., [2, Theorem 4.51, p. 153].

Lemma 3.1. *Let* (S, Σ) *be a measurable space,* Y *a separable metrizable space, and* Z *a metrizable space. Then every Carathéodory function* $f : S \times Y \to Z$ *is jointly measurable.*

We provide a short proof for the following result using Lemma 3.1.

Lemma 3.2. *The mapping*

$$\begin{aligned} s : \mathbb{S}^{d-1} \times \mathcal{K}_c(\mathbb{R}^d) &\to \mathbb{R} \\ (r, K) &\mapsto s(r, K) = s_K(r) = \max_{x \in K} \langle r, x \rangle, \end{aligned}$$

is jointly measurable.

Proof. By [17, Theorem H.1, p. 596], $s(\cdot, K)$ is a Lipschitz function for each $K \in \mathcal{K}_c(\mathbb{R}^d)$. Similarly, it follows from, each $r \in \mathbb{S}^{d-1}$. Therefore s is a Carathéodory function. Since $\mathbb{S}^{d-1}$ is a separable metric space and $\mathcal{K}_c(\mathbb{R}^d)$ is a measurable space with its Borel σ-algebra, by Lemma 3.1 the mapping s is jointly measurable. □

We will also need the following relationships between measurability definitions for random compact sets. A proof from Himmelberg's measurability theorem is presented for the reader's convenience.

A mapping on a measurable space whose values are closed sets is called a *random closed set* if it is Effros measurable, i.e., it follows that $\{X \cap G \neq \emptyset\}$ is measurable for every open set G.

Lemma 3.3. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $X : \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ and consider the following properties:*

1. *X is a random compact set,*
2. *X is a random closed set,*
3. *$\operatorname{Gr} X$ is measurable (i.e., $\operatorname{Gr} X \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^d}$).*

Then (1)⇔(2)⇒(3). Moreover, all three conditions are equivalent if $\mathcal{A}$ is complete.

Proof. By [17, Theorem 1.3.14, p. 65], a mapping X with values in $\mathcal{K}(\mathbb{R}^d)$ is Borel measurable with respect to d_H if and only if it is Effros measurable, namely $\{X \cap G \neq \emptyset\}$ is measurable for every open set G. Since $\mathcal{K}_c(\mathbb{R}^d)$ is a closed subset of $\mathcal{K}(\mathbb{R}^d)$, conditions (1) and (2) are equivalent.

Implication (2)⇒(3) is part (3)⇒(6) in [17, Theorem 1.3.3.(i), p. 59], whereas 'Effros measurability is equivalent to (2) if $\mathcal{A}$ is complete' is part (3)⇔(6) in [17, Theorem 1.3.3.(iii), p. 59]. □

Consider the following approximation of fuzzy sets in $\mathcal{F}_c(\mathbb{R}^d)$ by elements of $\mathcal{F}_{cc}(\mathbb{R}^d)$ defined in [23, Lemma 10, p. 352].

Definition 3.1. *For any $\varepsilon > 0$, define $R_\varepsilon : \mathcal{F}_c(\mathbb{R}^d) \to \mathcal{F}_c(\mathbb{R}^d)$ by*

$$(R_\varepsilon U)_\alpha = \int_{[0,1]} U_{(\alpha-\varepsilon)_+ + t[\alpha-(\alpha-\varepsilon)_+]} dt$$

for $\alpha \in (0, 1]$, where $x_+ = \max\{x, 0\}$.

Lemma 3.4. *Each mapping R_ε has the following properties:*

1. *R_ε is (σ_L, σ_L)-measurable for each $\varepsilon > 0$.*
2. *$R_\varepsilon(U) \in \mathcal{F}_{cc}(\mathbb{R}^d)$ for each $\varepsilon > 0$ and all $U \in \mathcal{F}_c(\mathbb{R}^d)$.*
3. *$d_H(R_\varepsilon(U)_\alpha, U_\alpha) \to 0$ as $\varepsilon \to 0$, for all $U \in \mathcal{F}_c(\mathbb{R}^d)$ and $\alpha \in [0, 1]$.*

4 Joint measurability

In this section, in order to skip obvious steps in proofs we will make some identifications between objects that are not identical, strictly speaking, but which will be easily understood by the reader. A typical example is the identification of $(a,(b,c)) \in A \times (B \times C)$, or $((a,b),c)) \in (A \times B) \times C$, with $(a,b,c) \in A \times B \times C$. Those products are isomorphic as measurable spaces and so avoiding the distinction does not affect our measurability proofs.

The following is our starting point. Note that fuzzy random variables are defined by recourse to taking α-cuts, therefore the mapping $L_\alpha : U \mapsto U_\alpha$ is measurable by definition. We show that the operation of taking the α-cut of U is not only measurable with respect to U but *jointly* measurable with respect to both α and U.

Theorem 4.1. *The mapping*

$$\begin{aligned} L : \mathcal{F}_c(\mathbb{R}^d) \times [0,1] &\to \mathcal{K}_c(\mathbb{R}^d) \\ (U,\alpha) &\mapsto L(U,\alpha) = U_\alpha \end{aligned}$$

is jointly measurable.

Proof. Set

$$\begin{aligned} L_n : \mathcal{F}_c(\mathbb{R}^d) \times [0,1] &\to \mathcal{K}_c(\mathbb{R}^d) \\ (U,\alpha) &\mapsto L_n(U,\alpha) = (R_{1/n}U)_\alpha. \end{aligned}$$

Fix $\alpha \in [0,1]$ and $n \in \mathbb{N}$. We have $L_n(\cdot,\alpha) = L_\alpha \circ R_{1/n}$. The mapping $R_{1/n}$ is (σ_L, σ_L)-measurable by Lemma 3.3 and, by definition, L_α is $(\mathcal{A}, \sigma_L)$-measurable. Hence $L(\cdot,\alpha)$ is measurable.

In its turn, for any fixed $U \in \mathcal{F}_c(\mathbb{R}^d)$ the mapping $L_n(U,\cdot)$ is continuous because $R_{1/n}(U) \in \mathcal{F}_{cc}(\mathbb{R}^d)$. Therefore, each L_n is a Carathéodory function and, by Lemma 3.1, jointly measurable, i.e., $(\sigma_L \otimes \mathcal{B}_{[0,1]}, \mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)})$-measurable.

By Lemma 3.4, $L_n(U,\alpha) \to L(U,\alpha)$ in the Hausdorff metric for each U,α. Being the pointwise limit of a sequence of Borel measurable functions with values in a metric space, L is measurable [10, Theorem 4.2.2, p. 125]. $\square$

A fuzzy random variable X induces naturally the mapping

$$\begin{aligned} L_X : \Omega \times [0,1] &\to \mathcal{K}_c(\mathbb{R}^d) \\ (\omega,\alpha) &\mapsto L_X(\omega,\alpha) = X_\alpha(\omega). \end{aligned}$$

With Theorem 4.1 we deduce

Corollary 4.2. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable. Then L_X is jointly measurable.*

Proof. For each $(\omega, \alpha) \in \Omega \times [0, 1]$,

$$L_X(\omega, \alpha) = X_\alpha(\omega) = [X(\omega)]_\alpha = L(X(\omega), \alpha) = (L \circ (X, \pi))(\omega, \alpha),$$

where $\pi : \mathcal{F}_c(\mathbb{R}^d) \times [0, 1] \to [0, 1]$ is the projection onto the second component. Since both π and X are measurable, (X, π) is measurable with respect to the product σ-algebra $\mathcal{A} \times \mathcal{B}_{[0,1]}$. Thus L_X is measurable. □

Also naturally induced by a fuzzy random variable is the 0-1 mapping

$$B_X : \Omega \times [0, 1] \times \mathbb{R}^d \to \mathbb{R}$$
$$(\omega, \alpha, x) \mapsto B_X(\omega, \alpha, x) = \begin{cases} 1, & x \in X_\alpha(\omega) \\ 0, & x \notin X_\alpha(\omega). \end{cases}$$

Proposition 4.3. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable. Then, B_X is a Bernoulli random variable.*

Proof. Since B_X has only two different values, it is enough to prove that $B_X^{-1}(\{1\})$ is measurable. But

$$B_X^{-1}(\{1\}) = \{(\omega, \alpha, x) \in \Omega \times [0, 1] \times \mathbb{R}^d : x \in X_\alpha(\omega)\}$$
$$= \{(\omega, \alpha, x) \in \Omega \times [0, 1] \times \mathbb{R}^d : x \in L_X(\omega, \alpha)\} = \operatorname{Gr} L_X.$$

By Corollary 4.2, L_X is a random compact set. Hence by Lemma 3.3, its graph $\operatorname{Gr} L_X$ is a measurable set. □

Measurability of fuzzy random variables with convex values, as is well known, can also be approached using support functions. Consider the *support mapping*

$$\widetilde{s} : \mathbb{S}^{d-1} \times [0, 1] \times \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}$$
$$(r, \alpha, U) \mapsto \widetilde{s}(r, \alpha, U) = \max_{x \in U_\alpha} \langle r, x \rangle.$$

Proposition 4.4. *The support mapping $\widetilde{s}$ is jointly measurable.*

Proof. Let $\phi : \mathbb{S}^{d-1} \times [0, 1] \times \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{S}^{d-1} \times \mathcal{K}_c(\mathbb{R}^d)$ be the mapping given by $\phi(r, \alpha, U) = (r, L(U, \alpha))$. By Theorem 4.1, L is measurable, therefore ϕ is measurable because each component is so. Since $\widetilde{s}$ is the composition $s \circ \phi$, it is measurable as well.

□

The support function allows us to define the following mappings:

$$\begin{aligned}\text{mid} : \mathbb{S}^{d-1} \times [0,1] \times \mathcal{F}_c(\mathbb{R}^d) &\to \mathbb{R} \\ (r, \alpha, U) &\mapsto \text{mid}(r, \alpha, U) = \frac{s(r, \alpha, U) - s(-r, \alpha, U)}{2},\end{aligned}$$

$$\begin{aligned}\text{spr} : \mathbb{S}^{d-1} \times [0,1] \times \mathcal{F}_c(\mathbb{R}^d) &\to \mathbb{R} \\ (r, \alpha, U) &\mapsto \text{spr}(r, \alpha, U) = \frac{s(r, \alpha, U) + s(-r, \alpha, U)}{2}.\end{aligned}$$

For any $r \in \mathbb{S}^{d-1}$, projecting U over the line $\{tr : t \in \mathbb{R}\}$ yields a one-dimensional fuzzy set $\pi_r(U)$ with $\pi_r(U)(t)$ being the membership degree of tr in the projection. With that notation, $\text{mid}(r, \alpha, U)$ and $\text{spr}(r, \alpha, U)$ are the midpoint and the radius (or spread) of $\pi_r(U)_\alpha$. These generalized midpoint and spread were introduced by Trutschnig *et al.* [26].

An alternative representation is obtained by considering the infimum and the supremum instead of the midpoint and the spread:

$$\begin{aligned}\inf : \mathbb{S}^{d-1} \times [0,1] \times \mathcal{F}_c(\mathbb{R}^d) &\to \mathbb{R} \\ (r, \alpha, U) &\mapsto \inf(r, \alpha, U) = -s(-r, \alpha, U)\end{aligned}$$

$$\begin{aligned}\sup : \mathbb{S}^{d-1} \times [0,1] \times \mathcal{F}_c(\mathbb{R}^d) &\to \mathbb{R} \\ (r, \alpha, U) &\mapsto \sup(r, \alpha, U) = s(r, \alpha, U)\end{aligned}$$

Clearly, the two functions $\text{mid}(\cdot, \cdot, U)$ and $\text{spr}(\cdot, \cdot, U)$, or the two functions $\inf(\cdot, \cdot, U)$ and $\sup(\cdot, \cdot, U)$ identify U.

Corollary 4.5. *The functions* inf*,* sup*,* mid *and* spr *are jointly measurable.*

Proof. It follows from Proposition 4.4. □

From a fuzzy random variable X, we can easily construct its support mapping and prove its measurability now [14, Lemma 4].

Proposition 4.6. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable. Then*

$$\begin{aligned}\widetilde{s}_X : \Omega \times [0,1] \times \mathbb{S}^{d-1} &\to \mathbb{R} \\ (\omega, \alpha, r) &\mapsto \widetilde{s}_X(\omega, \alpha, r) = \max_{x \in X_\alpha(\omega)} \langle r, x \rangle\end{aligned}$$

is jointly measurable.

Proof. For a fixed $(\omega, \alpha, r) \in \Omega \times [0,1] \times \mathbb{S}^{d-1}$ we have $\widetilde{s}_X(\omega, \alpha, r) = \widetilde{s}(r, L_X(\omega, \alpha))$. Reasoning like in the proof of Proposition 4.4 we conclude that $\widetilde{s}_X$ is measurable. □

Corollary 4.7. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable. Then the mappings*

$$\mathrm{mid}_X : (\omega, r, \alpha) \mapsto \mathrm{mid}(X(\omega), r, \alpha) \in \mathbb{R},$$

$$\mathrm{spr}_X : (\omega, r, \alpha) \mapsto \mathrm{spr}(X(\omega), r, \alpha) \in \mathbb{R},$$

$$\inf_X : (\omega, r, \alpha) \mapsto \inf(X(\omega), r, \alpha) \in \mathbb{R},$$

$$\sup_X : (\omega, r, \alpha) \mapsto \sup(X(\omega), r, \alpha) \in \mathbb{R}$$

are jointly measurable.

5 Some consequences

In this section, we will explore some applications of the former results as regards the connections between fuzzy random variables and some related results, like random sets and probabilistic sets.

We start by considering an alternative approach to fuzzy random variables introduced by Castaing *et al.* [5] (and used by, e.g., [21]). That paper, as is at times the case when non-fuzzy researchers consider fuzzy sets (e.g., [1]), introduces its own non-standard terminology in which a fuzzy set with non-empty closed convex α-sets is called a *fuzzy convex upper semicontinuous variable.*

Definition 5.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. A* random fuzzy convex upper semicontinuous variable *is a $(\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^d}, \mathcal{B}_{[0,1]})$-measurable mapping $Y : \Omega \times \mathbb{R}^d \to [0,1]$ such that, for each $\omega \in \Omega$, the mapping $Y(\omega, \cdot) : \mathbb{E} \to [0,1]$ is a fuzzy convex upper semicontinuous variable.*

Notice that Castaing *et al.* required $(\Omega, \mathcal{A}, P)$ to be complete, which is not necessary for the definition to make sense.

Proposition 5.1. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable. Then,*

$$\begin{aligned} C_X : \Omega \times \mathbb{R}^d &\to [0,1] \\ (\omega, x) &\mapsto C_X(\omega, x) = X(\omega)(x) \end{aligned}$$

is a random fuzzy convex upper semicontinuous variable.

Proof. We have $C_X^{-1}([0,\infty)) = \Omega \times \mathbb{R}^d$. For $\alpha \in (0,1]$,

$$C_X^{-1}([\alpha,\infty)) = \{(\omega,x) \in \Omega \times \mathbb{R}^d : C_X(\omega,x) \geq \alpha\}$$

$$= \{(\omega,x) \in \Omega\times\mathbb{R}^d : X(\omega)(x) \geq \alpha\} = \{(\omega,x) \in \Omega\times\mathbb{R}^d : x \in X_\alpha(\omega)\} = \operatorname{Gr} X_\alpha,$$

so by Lemma 3.3, C_X is measurable. It remains to show that for each $\omega \in \Omega$, the mapping $C_X(\omega,\cdot) : \mathbb{R}^d \to [0,1]$ is a fuzzy convex upper semicontinuous variable. Notice that $X(\omega)$ belongs to $\mathcal{F}_c(\mathbb{R}^d)$, so it satisfies all conditions. □

We consider now the probabilistic sets introduced by Hirota [12], one of the early notions to combine probabilistic uncertainty with fuzzy sets.

Definition 5.2. *A* probabilistic set *A on $\mathbb{R}^d$ is a function*

$$\begin{aligned} H : \mathbb{R}^d \times \Omega &\to [0,1] \\ (x,\omega) &\mapsto H(x,\omega) \end{aligned}$$

such that $H(x,\cdot)$ is measurable for each $x \in \mathbb{R}^d$.

Apparently, it has not been realized that a fuzzy random variable induces naturally a probabilistic set. At least we have not found this observation in the sources which discuss both types of objects, see [15, Chapter 7], [3, Section 7.4.1], [18, Section 6.5], [9, Section 1.5.6].

Corollary 5.2. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable. Then,*

$$\begin{aligned} H_X : \mathbb{R}^d \times \Omega &\to [0,1] \\ (x,\omega) &\mapsto H_X(\omega,x) = X_\omega(x) \end{aligned}$$

is a probabilistic set.

Proof. By Proposition 5.1, C_X is jointly measurable, so it follows that $H_X(x,\cdot) = C_X(\cdot,x)$ is measurable for each $x \in \mathbb{R}^d$. □

Another consequence of Theorem 4.1 is that the randomly chosen α-cut of a fuzzy random variable is a random compact set. While this is a very natural result, to the best of our knowledge it had not been established in the literature.

Corollary 5.3. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a fuzzy random variable and $\xi : \Omega \to [0,1]$ a random variable. Then*

$$\begin{aligned} X_\xi : \Omega &\to \mathcal{K}_c(\mathbb{R}^d) \\ \omega &\mapsto X_{\xi(\omega)}(\omega) \end{aligned}$$

is a random compact set.

Proof. For any $\omega \in \Omega$,

$$X_\xi(\omega) = L_X(\omega, \xi(\omega)) = (L_X \circ (id, \xi))(\omega),$$

where $id : \Omega \to \Omega$ is the identity function. Since id and ξ are measurable, it turns out that (id, ξ) is measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}_{[0,1]}$. By Corollary 4.2, L_X is measurable. Therefore X_ξ is a composition of Borel measurable mappings, whence it is measurable. □

A further application of Corollary 5.3 is the following.

Proposition 5.4. *Let* $X : (\Omega, \mathcal{A}, P) \to \mathcal{F}_c(\mathbb{R}^d)$ *be a fuzzy random variable. For any density function* φ *on* $[0,1]$*, the mapping*

$$\begin{aligned} \mathbb{P} : \mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)} &\to [0,1] \\ \mathbf{A} &\mapsto \mathbb{P}(\mathbf{A}) = \int_{[0,1]} P(X_\alpha \in \mathbf{A}) \varphi(\alpha) \mathrm{d}\alpha \end{aligned}$$

is well defined and is a probability measure.

Proof. Let ξ be a random variable in $([0,1], \lambda)$ with density function φ. Consider the mapping

$$\begin{aligned} \widehat{X} : \Omega \times [0,1] &\to \mathcal{F}_c(\mathbb{R}^d) \\ (\omega, \alpha) &\mapsto \widehat{X}(\omega, \alpha) = X(\omega). \end{aligned}$$

Since for any $A \in \sigma_L$ we have $\widehat{X}^{-1}(A) = X^{-1}(A) \times [0,1]$, clearly $\widehat{X}$ is a fuzzy random variable. Similarly, define the random variable

$$\begin{aligned} \widehat{\xi} : \Omega \times [0,1] &\to [0,1] \\ (\omega, \alpha) &\mapsto \widehat{\xi}(\omega, \alpha) = \xi(\alpha). \end{aligned}$$

By Corollary 5.3, it follows that $\widehat{X}_{\widehat{\xi}} : \Omega \times [0,1] \to \mathcal{K}_c(\mathbb{R}^d)$ is a random set. Then its induced distribution $(P \otimes \lambda)_{\widehat{X}_{\widehat{\xi}}}$ is a probability distribution. It suffices to show that $(P \otimes \lambda)_{\widehat{X}_{\widehat{\xi}}} = \mathbb{P}$.

Let $\mathbf{A} \in \mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)}$ Then, by the Fubini theorem,

$$(P \otimes \lambda)_{\widehat{X}_{\widehat{\xi}}}(\mathbf{A}) = \int_{\Omega \times [0,1]} I_{\{\widehat{X}_{\widehat{\xi}} \in \mathbf{A}\}} d(P \otimes \lambda) = \int_\Omega \int_{[0,1]} I_{\{\widehat{X}_{\widehat{\xi}} \in \mathbf{A}\}} d\lambda \, dP$$

For a fixed $\omega \in \Omega$, let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(\alpha) = I_{\{\alpha \in [0,1] : X_\alpha(\omega) \in \mathbf{A}\}}.$$

Since $I_{\{\widehat{X}_{\widehat{\xi}}\in\mathbf{A}\}}(\omega,\cdot)$ is a random variable, it follows that

$$\{\alpha\in[0,1]:X_{\xi(\alpha)}(\omega)\in\mathbf{A}\}=\{\alpha\in[0,1]:\widehat{X}_{\widehat{\xi}}(\omega,\alpha)\in\mathbf{A}\}$$
$$=\left(I_{\{\widehat{X}_{\widehat{\xi}}\in\mathbf{A}\}}(\omega,\cdot)\right)^{-1}(\{1\})$$

is measurable, therefore f is a random variable. Then

$$\int_{[0,1]} I_{\{\widehat{X}_{\widehat{\xi}}\in\mathbf{A}\}}(\omega,\cdot)d\lambda=\int_{[0,1]} I_{\{\widehat{X}_{\widehat{\xi}}\in\mathbf{A}\}}(\omega,\alpha)d\alpha$$
$$=\int_{[0,1]} f(\xi(\alpha))d\alpha=\int_{[0,1]} f(\alpha)\varphi(\alpha)d\alpha,$$

where the last term is the expectation of $f(\xi)$ with respect to the probability measure λ. Therefore,

$$(P\otimes\lambda)_{\widehat{X}_{\widehat{\xi}}}(\mathbf{A})=\int_\Omega\int_{[0,1]} I_{\{\alpha\in[0,1]:X_\alpha(\omega)\in\mathbf{A}\}}(\alpha)\varphi(\alpha)d\alpha\, dP$$
$$=\int_{[0,1]} P(\{\omega\in\Omega:X_\alpha(\omega)\in\mathbf{A}\})\varphi(\alpha)d\alpha$$
$$=\int_{[0,1]} P(X_\alpha(\omega)\in\mathbf{A})\varphi(\alpha)d\alpha=\mathbb{P}(\mathbf{A})$$

□

6 Measurability of fuzzy random variables

We will now take advantage of the previous results to present a number of conditions which are either equivalent or closely related (equivalent under mild conditions) to being a fuzzy random variable.

Definition 6.1. *The* endograph *of a fuzzy set $U\in\mathcal{F}_c(\mathbb{R}^d)$ is the following subset of $\mathbb{R}^d\times[0,1]$:*

$$\operatorname{end}U=\{(x,\alpha)\in\mathbb{R}^d\times[0,1]:U(x)\geq\alpha\}.$$

Definition 6.2. *The* sendograph *(supported endograph) of U is*

$$\operatorname{send}U=\{(x,\alpha)\in\mathbb{R}^d\times[0,1]:x\in U_0,U(x)\geq\alpha\}.$$

Theorem 6.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a mapping. Then the following conditions are equivalent:*

1. *X is a fuzzy random variable,*
2. *L_X is a random compact set,*
3. *$\widetilde{s}_X$ is a random variable,*
4. *$\widetilde{s}_X(\cdot, \alpha, r)$ is a random variable for each $\alpha \in [0, 1]$ and $r \in S^{d-1}$,*
5. *$\operatorname{send} X$ is a random compact set,*
6. *$\operatorname{end} X$ is a random closed set.*

Proof. Step 1. (1)⇔(2):

Implication (1)⇒(2) is the same as Corollary 4.2, while the converse follows from the fact that, if L_X is measurable, then each $L(\cdot, \alpha) = X_\alpha$ is measurable.

Step 2. (1)⇒(3)⇒(4)⇒(1):

The first implication is Proposition 4.6. The second one is trivial. Implication (4)⇒(1) is known and follows from the fact that the measurability of each support functional of a $\mathcal{K}_c(\mathbb{R}^d)$-valued mapping implies it is a random compact set [17, Proposition 1.3.8, p. 61].

Step 3. (1)⇒(5)⇒(6)⇒(1):

Implication (1)⇔(5) was proved by Trutschnig [25, Theorem 22].

(5)⇒(6): Let us show that $\operatorname{end} X$ is Effros measurable. Let G be an open set of $\mathbb{R}^d \times [0, 1]$, then

$$\begin{aligned}\{\operatorname{end} X \cap G \neq \emptyset\} &= \{(\operatorname{send} X \cup (\mathbb{R}^d \times \{0\})) \cap G \neq \emptyset\} \\ &= \{\operatorname{send} X \cap G \neq \emptyset\} \cup \{(\mathbb{R}^d \times \{0\})) \cap G \neq \emptyset\}.\end{aligned}$$

By Lemma 3.3, the first term of the union is measurable, as is the second one since it is either Ω or $\emptyset$. By the arbitrariness of G, $\operatorname{end} X$ is Effros measurable, i.e., a random closed set.

(6)⇒(1): Let $G \subseteq \mathbb{R}^d$ be open. For each $\alpha \in (0, 1]$,

$$\{X_\alpha \cap G \neq \emptyset\} = \{\operatorname{end} X \cap (G \times \{\alpha\}) \neq \emptyset\}.$$

Since $G \times \{\alpha\}$ is not open in the product topology of $\mathbb{R}^d \times [0, 1]$, a little work is needed. We write G as a countable union of compact sets $\bigcup_n K_n$ (for instance, closed balls). Then

$$\{\operatorname{end} X \cap (G \times \{\alpha\}) \neq \emptyset\} = \{\operatorname{end} X \cap (\bigcup_n K_n \times \{\alpha\}) \neq \emptyset\}$$

$$= \bigcup_n \{\mathrm{end}\, X \cap (K_n \times \{\alpha\}) \neq \emptyset\}.$$

Since $K_n \times \{\alpha\}$ is compact and $\mathrm{end}\, X$ is a random closed set, the event $\{\mathrm{end}\, X \cap (K_n \times \{\alpha\}) \neq \emptyset\}$ is measurable by [11, Theorem 3.2.(i)]. Thence $\{X_\alpha \cap G \neq \emptyset\} \in \mathcal{A}$ for any open G, namely X_α ($\alpha > 0$) is a random closed set. By Lemma 3.3, it is a random compact set. There follows that X_0 is so as well, since it is the pointwise d_H-limit of the sequence $X_{1/n}$. Thus X is a fuzzy random variable. □

In connection to Theorem 6.1, we should mention that the reader can find other equivalent conditions in earlier papers. For instance, Krätschmer [13] (see also [14]) proved that being a fuzzy random variable is equivalent to being a measurable function with respect to the Borel σ-algebra of some topologies in the space of fuzzy sets. Another family of results includes those in [16, 22] which characterize fuzzy random variables as those functions which can be approximated by a sequence of certain better behaved functions. Theorem 6.1 complements those results by expressing the property of being a fuzzy random variable in terms of the measurability of functions taking on simpler values (random sets or random variables).

Theorem 6.2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be a mapping. Consider the following conditions:*

(1) X is a fuzzy random variable,

(2) B_X is a random variable,

(3) $\mathrm{Gr\,send}\, X \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{[0,1]}$.

(4) $\mathrm{Gr}\, X_\alpha \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^d}$ *for each* $\alpha \in [0, 1]$,

(5) $\mathrm{Gr\,end}\, X \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{[0,1]}$,

(6) $\mathrm{Gr}\, X_\alpha \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}^d}$ *for each* $\alpha \in (0, 1]$,

(7) C_X is a random fuzzy convex upper semicontinuous variable.

Then the following implications hold true:
(1)⇒(2)⇔(3)⇒(4)⇒(6)⇔(7),
(3)⇒(5)⇒(6).

Proof. Proof of the first chain. Implication (1)⇒(2) is Proposition 4.3. To prove (2)⇔(3), observe

$$\mathrm{Gr\,send}\, X = \{(\omega, x, \alpha) \in \Omega \times \mathbb{R}^d \times [0, 1] : (x, \alpha) \in \mathrm{send}\, X(\omega)\}$$

$$= \{(\omega, x, \alpha) \in \Omega \times \mathbb{R}^d \times [0,1] : x \in X_0(\omega), X(\omega)(x) \geq \alpha\}$$
$$= \{(\omega, x, \alpha) \in \Omega \times \mathbb{R}^d \times [0,1] : x \in X_\alpha(\omega)\}.$$

This set is identical with the event $B_X^{-1}(\{1\}) \subseteq \Omega \times [0,1] \times \mathbb{R}^d$ except for the order of the spaces Ω, $\mathbb{R}^d$ and $[0,1]$. Recall that B_X is 0-1 valued, whence it is a random variable if and only if $B_X^{-1}(\{1\})$ is measurable, which happens if and only if (3) holds.

To prove (3)⇒(4) notice that, for each $\alpha \in [0,1]$,

$$\operatorname{Gr} X_\alpha \times \{\alpha\} = \{(\omega, x, \alpha) \in \Omega \times \mathbb{R}^d \times \{\alpha\} : x \in X_\alpha(\omega)\}$$
$$= (\operatorname{Gr} \operatorname{send} X) \cap (\Omega \times \mathbb{R}^d \times \{\alpha\}).$$

Thence $\operatorname{Gr} X_\alpha \times \{\alpha\}$ or, equivalently, $\operatorname{Gr} X_\alpha$ is a measurable set.

Implication (4)⇒(6) is trivial. Equivalence (6)⇔(7) is just the proof of Proposition 5.1.

Proof of the second chain. To prove (3)⇒(5), one easily checks

$$\operatorname{Gr} \operatorname{end} X = (\operatorname{Gr} \operatorname{send} X) \cup (\Omega \times \mathbb{R}^d \times \{0\}),$$

from which the desired implication follows. In its turn, implication (5)⇒(6) is analogous to implication (3)⇒(4) above. □

If the probability space is assumed to be complete, then all conditions above turn out to be equivalent. That assumption is mild in the sense that it is usually not considered problematic in Probability Theory to assume completeness and it is often used as a blanket assumption to avoid some burdensome details in proofs. Those details involve, for instance, routinely modifying a function in a set of probability 0 to ensure that a certain null (possibly non-measurable) set becomes measurable, or to make the function a random variable instead of just measurable except for a null set. Since all null sets are measurable in a complete σ-algebra, completing the σ-algebra is a way to avoid those situations while ensuring that all random variables with respect to the original σ-algebra are still random variables with respect to the completion.

Theorem 6.3. *Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. For a mapping from Ω to $\mathcal{F}_c(\mathbb{R}^d)$, all twelve different conditions in Theorems 6.1 and 6.2 are equivalent.*

Proof. There suffices to prove (6)⇒(1) in Theorem 6.2 under the additional assumption of completeness, since that implies that all conditions in Theorem 6.2 are equivalent to being a fuzzy random variable; Theorem 6.1 already states that the remaining conditions are equivalent.

Under (6), Lemma 3.3 implies that X_α is a random compact set for all $\alpha \in (0, 1]$. As already mentioned in the proof of Theorem 6.1, that implies that X_0 is a random compact set as well, whence X is a fuzzy random variable. □

7 Discussion: on the definition of a fuzzy random variable

In this paper, there appear two conditions which have been used in the literature as definitions of the term 'fuzzy random variable'. Both of them define a fuzzy random variable X by the property that each α-cut mapping X_α be a random compact set. But the latter can be understood as X_α being measurable with respect to the Borel σ-algebra of the metric space $(\mathcal{K}_c(\mathbb{R}^d), d_H)$ (function measurability), or as $\operatorname{Gr} X_\alpha$ being in the product σ-algebra of $\Omega \times \mathbb{R}^d$ (graph measurability).

It has become usual to refer to the former condition as a fuzzy random variable in 'Puri and Ralescu's sense'. Their original paper [20], though, does use the latter (it must be emphasized that they use still other conditions in other papers, e.g., [19]). Notice that, more broadly, all conditions in Section 6 can be split into two kinds with a family resemblance: those in Theorem 6.1 involve measurability of certain functions and those in Theorem 6.2 (except for (2)) involve measurability of certain graphs.

In particular, implication (1)⇒(7) in Theorem 6.2 raises the question whether the definition of a random fuzzy convex upper semicontinuous variable in [5] may be more convenient than the usual definition of a fuzzy random variable. Graph measurability is a weaker condition and it is more elementary (it does not involve spaces of sets). It is elegant and some of its theorems are beautiful.

Despite appearing in the original definition of a fuzzy random variable, graph measurability ended up being abandoned by the fuzzy community for a long time. To the best of our knowledge, no written record of the plausible reasons exists. This discussion aims at providing one.

Graph measurability is strongly associated to conceiving a set-valued mapping as a one-to-many relation. To understand why, consider the fact that a function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable if and only if its graph is a Borel subset of $\mathbb{R} \times \mathbb{R}$. Whereas both copies of $\mathbb{R}$ play asymmetric roles in function measurability, graph measurability is entirely and surprisingly symmetric. This almost begs for an explanation in the form of a theory in which that asymmetry can be reverted as

$$f : \mathbb{R} \to \mathbb{R} \text{ measurable} \Leftrightarrow \operatorname{Gr} f \text{ measurable} \Leftrightarrow f^{-1} : \mathbb{R} \to \mathcal{P}(\mathbb{R}) \text{ measurable.}$$

Taking preimages is the prototype of a one-to-many relation. This mindset is underlined by the titles of some of the classical papers [7, 11].

The usual interpretations of fuzzy data do not fit well with this semantics, whence the definition of fuzzy random variables via graph measurability is not semantically forceful. Indeed, if a man chosen at random is recorded as being 'tall' and that word is modelled by a fuzzy set, it will typically be understood that 'tall' corresponds to a generalized non-numerical value (called the ontic view) or that it represents a state of limited knowledge about his height, an unknown real number (called the epistemic view). But it will not be interpreted that this man has all possible heights to some partial degree indicated by the fuzzy set 'tall' (the fuzzy analog of the one-to-many semantics).

From the technical point of view, graph measurability comes with a number of obstacles for the development of a theory of fuzzy random variables.

Let $X, Y : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ be fuzzy-valued mappings. If they are measurable as functions between measurable spaces, one automatically has notions like the induced distribution ($P_X = P \circ X^{-1}$) or independence ($P_{(X,Y)} = P_X \otimes P_Y$). Graph measurability does not lend itself easily to a transposition of those basic probability notions.

Further, if $f : \mathcal{F}_c(\mathbb{R}^d) \to \mathcal{F}_c(\mathbb{R}^d)$ is measurable and X is a fuzzy random variable in the sense of d_H-Borel measurability, then $f(X)$ is a composition of measurable functions and thus a fuzzy random variable. If X is graph measurable, is the transformed variable $f(X)$ graph measurable? Even assuming that f is Zadeh's extension of a continuous function $g : \mathbb{R}^d \to \mathbb{R}^d$, a particularly favourable case, we have

$$\begin{aligned} \operatorname{Gr} f(X)_\alpha = \{(\omega, x) : x \in f(X(\omega))_\alpha\} &= \{(\omega, g(x)) : x \in X_\alpha(\omega)\} \\ &= (id, g)(\operatorname{Gr} X_\alpha) \end{aligned}$$

where id denotes the identity mapping in Ω. But the measurable image of a Borel measurable set is not necessarily measurable.

Moreover, graph measurability does not necessarily imply that very useful mappings like $\widetilde{s}(r, \alpha, X)$ are random variables. Compare Proposition 4.6 which states that $\widetilde{s}_X = \widetilde{s}(\cdot, \cdot, X(\cdot))$ is jointly measurable in its three arguments if X satisfies Definition 2.2.

It can also be mentioned that graph measurability equally creates problems in the theory of random sets, where assuming it is not enough to ensure the measurability of some interesting events or to imply key results like the measurable selection theorem.

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