Optimal discrete-time Prony series fitting method for viscoelastic materials

Eva Barrientos[1], Fernández Pelayo[1*], Álvaro Noriega[1], María Jesús Lamela-Rey[1], Alfonso Fernández-Canteli[1], Eiji Tanaka[2]

[1] Department of Construction and Manufacturing Engineering, University of Oviedo, Spain.

[2] Department of Orthodontics and Dentofacial Orthopedics, Institute of Biomedical Sciences, Tokushima University Graduate School, Tokushima, Japan.

*Correspondence to: Department of Construction and Manufacturing Engineering, University of Oviedo, Edif. Dep. Oeste. N-7, ZIP: 33204 Gijón, Asturias, Spain.

E-mail address: fernandezpelayo@uniovi.es (F. Pelayo).

Abstract

The most important characteristics of the behavior of viscoelastic materials are the time and temperature dependence of their properties. Viscoelastic models based on Prony series are usually used due to easy implementation in finite element analysis (FEA) codes. The experimental data are fitted to a Prony series using a user-convenience number of terms represented by two coefficients. The time coefficients τ_i are previously fixed in the time scale in order to determine the second parameters of the model. Usually, an homogeneous distribution in logarithmic-time scale is used for τ_i, which produces accurate fittings when a large number of terms in the Prony series are used as well as when the material presents a uniform sigmoidal viscoelastic curve along several decades of time. When short-time curves must be fitted or the relaxation curve shape is not so uniform distributed along time, the homogeneous distribution of time coefficients could be a significant drawback since a large number of coefficients could be needed or even a reasonable fitting with a Prony series model is not possible.

In this study, an optimized τ_i distributed method for fitting master curves of viscoelastic materials based on Prony series model is proposed. The method is based on an optimization algorithm strategy to allocate the time coefficients along the time scale in order to obtain the best fit. The method is validated by using experimental data of temporomandibular joint (TMJ) disc, which is a biological material that presents a short-time and high relaxation rate viscoelastic curve. The results show that the method improves significantly the fitting of the viscoelastic curves when compared with uniform distributed time fittings.

Furthermore, the optimized coefficients are also used to obtain the complex moduli of the material using an analytical conversion, which is compared with the experimental complex moduli curves of the material.

Keywords

Viscoelastic; Prony series; Optimization; Relaxation; Soft materials; Viscoelastic behaviour.

1. Introduction

The viscoelastic behaviour is present in a large number of materials used in both engineering and biomechanical applications. The advantages of viscoelastic materials, such as high-dissipative energy capacity (damping, noise and vibrations reduction or shock impact absorber applications), are due to its mechanical properties which could be say that are between the perfect solid and the perfect fluid (i.e. Newtonian) behaviour (Ferry, 1980; Lakes, 1998; Tschoegl, 2012). Furthermore, the mechanical properties of the viscoelastic materials are, at least, time and temperature dependent (Christensen, 2003; Ferry, 1980). The advantages and mechanical properties of viscoelastic materials can also be seen as drawbacks since, dealing with viscoelastic behaviour, i.e. from a design or calculation point of view, implies taking into account many variables that must be considered in the material characterization as well as in the material model. Many materials, either as natural (wood or biological tissues) or artificial processed (polymers, asphalt pavement or foams) presents viscoelastic behaviour so a better understanding and characterization of these materials are needed. But not only with the objective of improving designs and calculations even to having a better understanding of these materials response, i.e. in biomechanical applications.

To characterize the viscoelastic behaviour, experimental tests are carried out in rheometers or dynamic mechanical analysis (DMA) equipments. After the experimental data is measured, i.e. the relaxation Young’s modulus of the material, a viscoelastic mathematical model is fitted to the experimental curve in order to use the model in further calculations.

Although several models have been used and developed in the last decades(Lakes, 2009; Mainardi, 2010), the generalized Maxwell model is nowadays widely used due to its simplicity. The generalized Maxwell model is usually represented and fitted with a Prony series (Tzikang, 2000), therefore, hereafter, we use in the text the common term Prony series to refer to that viscoelastic model.

On the other hand, the full characterization of a viscoelastic material, most of the times, is not possible due to costs or testing machine limitations. In those cases, analytical or empirical interconversions can be used to complete the characterization of the different moduli of the material (Emri et al., 2005). All the moduli interconversions are fully developed and validated for the Prony series model (Findley et al., 1976; Lakes, 1998; Park and Schapery, 1999; Schapery and Park, 1999; Tschoegl, 2012), being, therefore, easy to implement in finite element analysis (FEA) codes.

With the aim of fitting the experimental data to the Prony series model, several methods have been developed (Cost and Becker, 1970; Emri and Tschoegl, 1993; Park and Kim, 2001; Ramkumar et al., 1997; R. A. Schapery, 1962; Richard A. Schapery, 1962; Tobolsky, 1960; Tobolsky and Murakami, 1959; Tschoegl, 2012; Tschoegl and Emri, 1993). Most of these methods are based on setting a set of discrete times as the first step, followed by the fitting of the rest of the model coefficients. Although several criteria for the allocation of discrete times in the fitting process can be apply (Tschoegl, 2012), usually, its application is not straightforward, so many commercial algorithms (ANSYS, 2013;

Herdy, 2003; SIMULIA, 2007; T.A.Instruments, 2001), use an homogeneous distribution (in logarithmic-time scale) to fit the experimental data. This homogeneous distribution for the discrete times τ_i produces accurate fittings when a large number of terms in the Prony's series can be used as well as when the material presents a uniform sigmoidal viscoelastic curve along several decades of time. When short-time curves must be fitted or the relaxation curve shape is not so uniform distributed along time, the homogeneous distribution of the discrete time coefficients could be a significant disadvantage since a large number of coefficients are usually needed or even a reasonable fitting with a Prony's series model is not possible.

These short viscoelastic curves with different relaxation ratios are usually obtained for soft-like materials, such as rubbery-like materials, acoustic isolated foams and almost most of the soft tissue biological materials (Barrientos et al., 2016; Fernández et al., 2013; Lamela et al., 2011; Pioletti et al., 1998; Provenzano et al., 2001; Tanaka et al., 2014).

Moreover, it must be therefore taken into account that a simpler viscoelastic model with a reduced number of terms is preferred in order to solve complex calculations, e.g. finite element (FE) calculations. Therefore, although a good fitting could be obtained with a large number of terms, a reduced model with fewer terms or parameters, but with the same accuracy, can lead a reduction of the computational time as well as a better compression of the material model.

In the present study, a new optimized discrete times method for fitting Prony's coefficients is proposed. The method is based on an optimization algorithm strategy to best allocate the time coefficients along the time scale. The method is validated for fitting the experimental relaxation curve of the temporomandibular Joint (TMJ) disc.

Furthermore, the time optimized allocate Prony series coefficients are used to determine the complex moduli (Emri and Tschoegl, 1993; Tschoegl and Emri, 1993) of the TMJ disc. Then, the analytical complex moduli is compared with the experimental complex moduli of the TMJ disc.

2 Viscoelasticity

Viscoelastic materials can be understood like those materials whose properties are somewhere between elastic solids and fluids, i.e. Newtonian fluids. Although its behavior is more complex, this point of view allows an easier understanding of the viscoelastic mathematical models for this kind of materials. Both elastic solid and Newtonian fluid behaviour are each one represented by springs and dashpots, respectively. The combination of these two elements allows building linear viscoelastic models. The simplest model for the relaxation curve is the Maxwell model. The phenomenon of relaxation is produced in a viscoelastic material when is subjected to a constant strain. Under applied constant strain, the stress in the material is dismissing close to zero when the material would be completely relaxed. This behaviour is represented in the Maxwell model with a spring element connected in series with a dashpot element (see Figure 1) (Ferry, 1980; Findley et al., 1976; Lakes, 1998; Tschoegl, 2012).

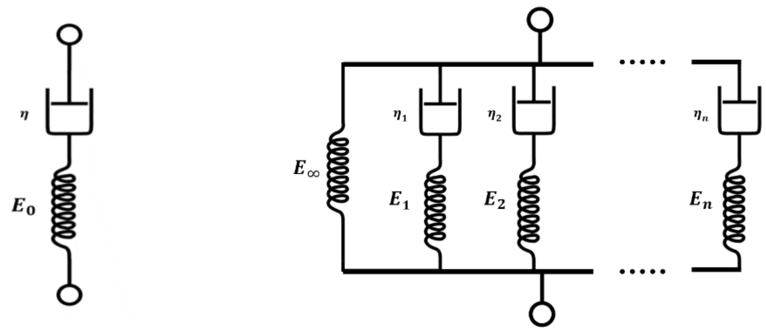

Figure 1. Individual and Generalized Maxwell Models.

When complex relaxation curves have to be fitted, the results obtained with the Maxwell model can be not satisfactory so, in general, the generalized Maxwell model is used. This model is composed of a convenience number of individual Maxwell elements in parallel (see Figure 1). To fit the experimental data with the generalized model, this is usually represented by means of Prony series where each term of the series is identified with one of the individual Maxwell models. The Prony series for the generalized Maxwell model is (Tzikang, 2000):

$$E(t) = E_0\left[1 - \sum_{i=1}^{n_t} e_i\left(1 - \exp\left(-\frac{t}{\tau_i}\right)\right)\right] \quad (1)$$

where E_0 is the instantaneous modulus of the material, n_t the number of Maxwell terms and (e_i, τ_i) the Prony coefficients (e_i is the i'th prony constant for the i'th prony retardation time constant τ_i). The Prony coefficients can be understood as: e_i is the $E(t)$ percentage change in each term of the Prony series whereas τ_i is the discrete time at which the term of Prony series intersects the curve of experimental data.

Once the relaxation modulus of the material has been fitted, the Prony coefficients, (e_i, τ_i), can be used to obtain by interconversion the components of the complex modulus $E^*(\omega)$, i.e., storage modulus $E'(\omega)$ and loss modulus $E''(\omega)$ (Emri et al., 2005; Park and Schapery, 1999; Schapery and Park, 1999; Tschoegl, 2012):

$$E'(\omega) = E_\infty + \sum_{i=1}^{n} \frac{\tau_i^2\omega^2 e_i}{\tau_i^2\omega^2 + 1} \quad (2)$$

$$E''(\omega) = \sum_{i=1}^{n} \frac{\tau_i^2\omega^2 e_i}{\tau_i^2\omega^2 + 1} \quad (3)$$

2.1 Model fitting with a homogeneous distribution of discrete times

For each number of the serie's terms (which vary from 1 and n_t, being n_t a user convenience number of terms), a Prony series will be built, assuming that τ_i are uniformly spaced in the logarithmic time-space, in the following sequence:

$$\tau_i = \tau_{min} + \frac{\tau_{max} - \tau_{min}}{n_t + 1} i \quad (4)$$

being

$$\tau_{min} = \log_{10} \min(\tau_k) \quad (5)$$

and

$$\tau_{max} = \log_{10} \max(\tau_k) \quad (6)$$

where $k = 1, \cdots, r$ being r the number of experimental data building the master curve. Then, the following optimization problem is proposed:

$$\begin{gathered} \min S_{err} = \sum_{k=1}^{k=r} \left(log_{10}E(t_k) - log_{10}E_{Prony}(t_k, \vec{\tau}, \vec{e}) \right)^2 = f_1(\vec{e}) \\ \forall i = 1,2, \dots, n_t \rightarrow 0 \leq e_i = g_i(\vec{e}) \end{gathered} \tag{7}$$

To solve the optimization problem, it is converted to the following unconstrained optimization problem:

$$\min S_{err} = \sum_{k=1}^{k=r} \left(log_{10}E(t_k) - log_{10}E_{Prony}(t_k, \vec{\tau}, \vec{e}) \right)^2 + \omega \cdot \sum_{i=1}^{n_t} g_i^* = f_1^*(\vec{e}) \tag{8}$$

In this modified formulation, we have transformed any inequality constraint into a minimization problem by the following way:

$0 \leq g(\vec{v})$ is equivalent to $\min g^*(\vec{v})$

being: (9)

$$g^*(\vec{v}) = \begin{cases} 0 \;\; if \;\; 0 \leq g(\vec{v}) \\ -g(\vec{v}) \; if \; g(\vec{v}) < 0 \end{cases}$$

To solve this issue, Matlab® software is used (MathWorks, 2016) where function "*fminunc*" is applied to solve the optimization problem. This function uses a quasi-Newton method with cubic line search procedure where the method uses the BFGS (Broyden, 1970; Fletcher, 1970; Goldfarb, 1970; Shanno, 1970) formula for updating the approximation of Hessian matrix.

2.2 Model fitting with an optimal distribution of discrete times

In this case, for each number of the series terms n_t, the following optimization problem is solved:

$$\begin{gathered} \min \mathrm{S_{err}} = \sum_{\mathrm{k=1}}^{\mathrm{k=r}} \left(\log_{10}\mathrm{E}(\mathrm{t_k}) - \log_{10}\mathrm{E_{Prony}}(\mathrm{t_k}, \vec{\tau}) \right)^2 = \mathrm{f}_2^*(\vec{\tau}) \\ \tau_\mathrm{i} \in [\tau_{\min} \;\; \tau_{\max}] \end{gathered} \tag{10}$$

where $k = 1, \cdots, r$ being r the number of experimental data building the master curve. Internally, in each evaluation of the objective function, τ_i , a similar problem to the one raised with homogeneous distribution optimization is solved. To solve this problem, the function "*fmincon*", implemented in Matlab® is used (MathWorks, 2016). This command uses a gradient-based method using an interior-point approach (Byrd et al., 2000; Waltz et al., 2006).

3 Experimental Data

To check the proposed optimized fitting method, experimental data from a previous work (Barrientos et al., 2016) was used. The data consist on the viscoelastic curves for the temporomandibular joint (TMJ) disc. Both relaxation and complex modulus were obtained using a DMTA (T.A. Instruments) equipment and 10 specimens (Barrientos et al., 2016). An example of the test set-up is presented in Figure 2. This biological material presents short relaxation curves with a relatively high relaxation rate. The relaxation modulus, as well as the storage and loss components (real and imaginary parts, respectively, of the complex modulus), are presented in Figures 3 and 4. In the figures are presented the mean values together with the standard deviation.

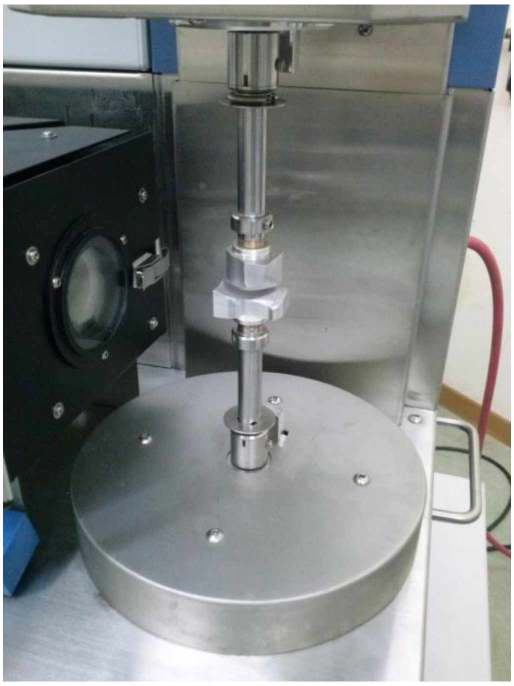

Figure 2: Experimental test set-up.

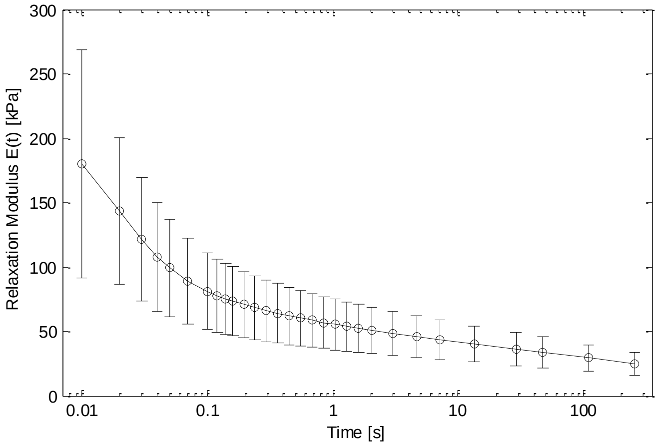

Figure 3. Relaxation modulus for the whole TMJ disc (Barrientos et al. 2016).

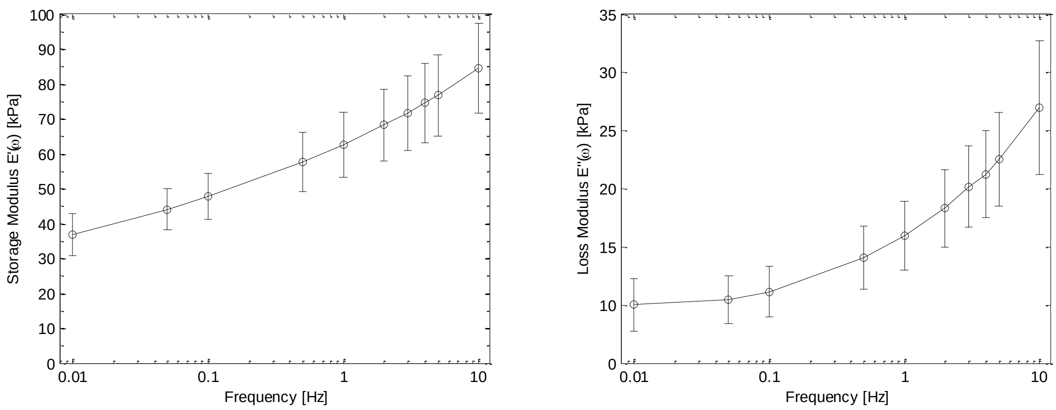

Figure 4. Storage (E’) and loss (E’’) moduli for the whole TMJ disc (Barrientos et al. 2016)

4 Results

4.1 Prony series relaxation fitting: optimal and homogeneous distributions.

The mean value of the TMJ disc experimental curves was fitted with both homogeneous and optimal distributions using from 1 to 10 terms in the Prony series model. The errors obtained in each model with the homogeneous as well as with the optimal distributions are presented in Figure 5. It can be seen that the optimal distribution always has an error lower than in the homogeneous distribution. About 5 terms, the errors between the homogeneous and optimal distribution decreases but remains always more favorable for the optimal distribution. For the case analyzed, the optimal distributed model with

4 terms can be considered the best model since produces almost the same errors that the 9 or 10 terms models. The fact that, in this case, the optimal distributed model with 4 terms produces an accurate fitting depends on the time spam and relaxation rate of the experimental curve to fit. So, it would be recommended for other experimental curves to perform several fittings using in each one a different number of terms. This is due to the fact that each fitting must be analyzed independently and, therefore, there is not a direct relation between the number of terms used and the accuracy obtained. This fact can be observed in figure 4 between the optimal 4 and 5 terms models.

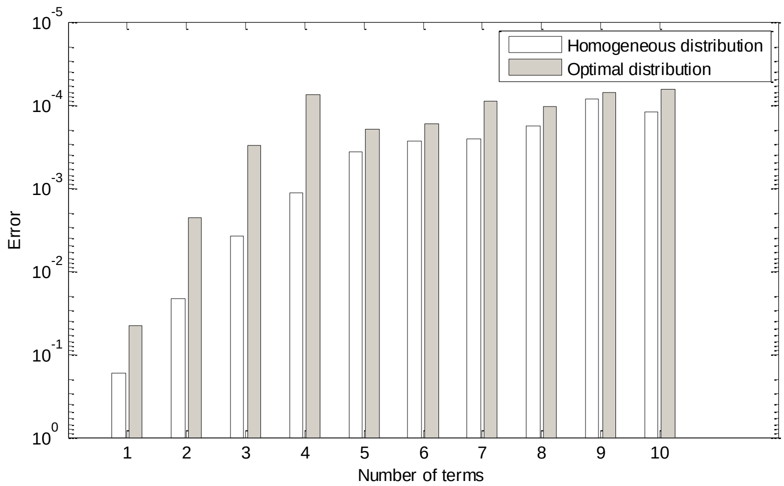

Figure 5. Evolution of the error with the number of terms of the Prony series for the TMJ disc.

The numerical errors obtained in the fittings for both the homogenous and optimal discrete times distributions are presented in Table 1.

Table 1. Errors obtained in the fittings.

	Distribution	Number of terms in the series fitting									
		1	2	3	4	5	6	7	8	9	10
Error [%]	Homogeneous	17.0851	2.1464	0.3801	0.1133	0.0369	0.0275	0.0256	0.0181	0.0086	0.0123
	Optimal	4.5438	0.2262	0.0306	0.0075	0.0195	0.0167	0.0090	0.0106	0.0071	0.0065

In Figure 6, the fitting of Prony series with increasing number of terms (from 1 to 5) can visually be compared for the homogeneous distribution (left) and the optimal distribution (right). From figure 6, it can be inferred that the optimal distribution fitting converges faster than the homogeneous one. As a rough comparison, 6-7 terms are needed in the homogeneous distribution for obtaining a similar error than the 3 terms optimal distribution (see Figure 5).

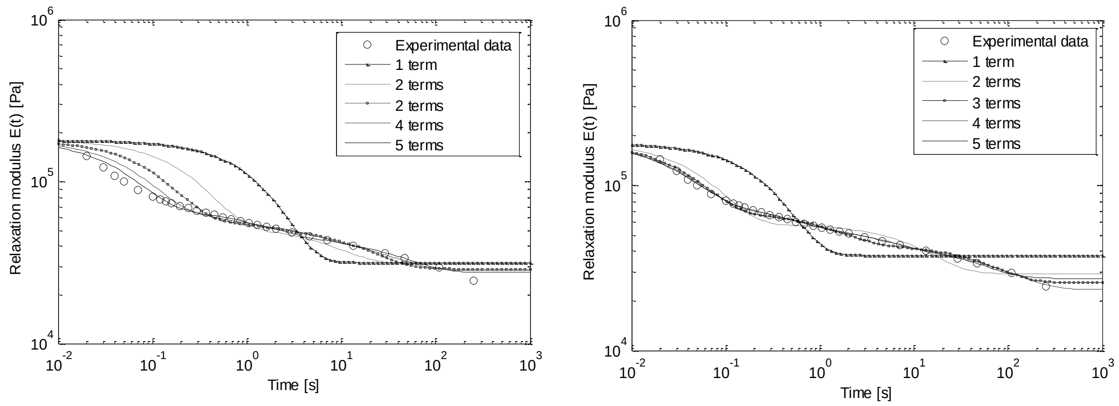

Figure 6. Relaxation curve fittings for both homogeneous (left) and optimal (right) distributions.

4.2 Interconversion between relaxation and complex moduli: homogeneous and optimal distributions.

Once the relaxation curves are fitted with the user-convenience terms. The Prony series coefficients can be used to determine the corresponding complex modulus or, that is the same, its real and imaginary components: the storage and loss moduli, respectively (see Eqs. (2) and (3)). Although the optimal time distributions fit the relaxation experimental data with lower errors than the homogeneous time distributions (for the same number of terms), the fact that these optimal discrete times predicts with higher accuracy the complex moduli is not a straightforward step.

In figures 6 and 7, the relaxation-complex modulus interconversions are presented using the Prony coefficients obtained for 2 and 4 terms, respectively. In the figures are presented both experimental storage (E’) and loss (E’’) moduli together with its corresponding predicted curves.

In both cases, it can be seen that the predicted curves with the optimal distribution present a better accuracy than those obtained with the homogeneous distribution. Same results are obtained with other comparisons with a different number of terms.

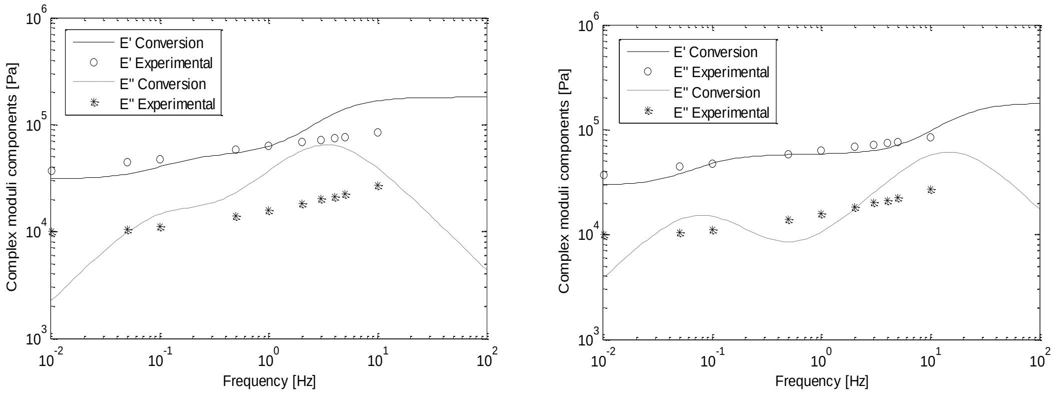

Figure 7: Complex modulus interconversion for 2 Prony series terms with homogeneous distribution (left) and optimal distribution (right).

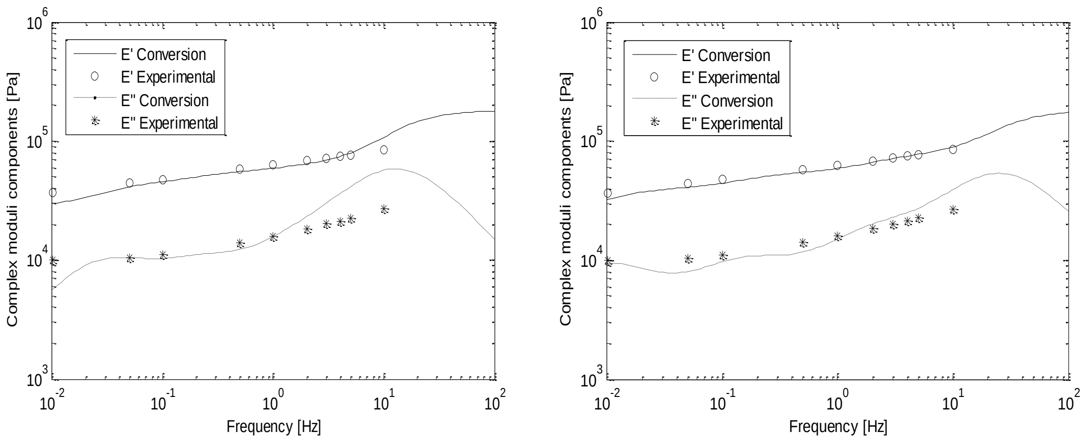

Figure 8: Complex modulus interconversion for 4 Prony series terms with homogeneous distribution (left) and optimal distribution (right).

Conclusions

The generalized Maxwell model, represented by a Prony series is one of the most used viscoelastic models for fitting relaxation experimental data. Although many commercial applications or finite element codes include this model, a homogeneous discrete time distribution, in logarithmic scale, is usually implemented. However, for short time viscoelastic curves or higher relaxations rate, a large number of terms in the Prony series can be needed to obtain accurate results. In most of the cases, a limited number of terms can only be used in the implemented models so the optimization of the fittings can improve the calculations reducing the number of terms used as well as the complexity of the viscoelastic model.

In this work, it has been proposed and validated an optimal fitting method for viscoelastic relaxation curves. The optimization process does not previously fix the discrete times of the Prony series, τ_i, being the time coefficients part of the fitting process. This allows the allocation of the time coefficients of the model to produce the best fit for each number of terms selected.

The method has been validated for fitting the experimental relaxation curve of the temporomandibular Joint (TMJ) disc. From the results, it can be concluded that for the same number of terms, the optimal distribution presents always lower errors. The errors between the optimal and homogeneous distributions are closer when the number of terms used in the fitting process increase being, nevertheless, always lower for the optimal distributions.

For the analyzed case, the model with 4 optimal distributed terms can be considered the best fit with an error of 0.0075%. On the other hand, the best fit for the homogeneous distribution occurs for 9 terms with an error of 0.0086%. Therefore, it can be concluded that the proposed method improves significantly the fitting of the viscoelastic curves when compare with uniform distributed time fittings.

Furthermore, time optimized allocate Prony series coefficients were used to determine the complex moduli disc of the TMJ disc. The results show that the optimized fitted model can be also used successfully for the interconversion between the relaxation modulus (time domain viscoelastic properties) and the complex moduli (frequency domain viscoelastic properties). From the results, it is shown that the optimal time distribution predicts with higher accuracy the complex modulus when compared with the homogeneous time distribution, independently of the number of terms used in the interconversion.

Acknowledgments

This research was supported in part by Grants-in-Aid 26293436 (E.T.) for Science Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan. The funder had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript. The authors would like to acknowledge the support of the CajAstur Fellowship-University of Oviedo 2011 program.

References

ANSYS, 2013. ANSYS Mechanical User's Guide.

Barrientos, E., Pelayo, F., Tanaka, E., Lamela-Rey, M.J., Fernández-Canteli, A., 2016. Dynamic and stress relaxation properties of the whole porcine temporomandibular joint disc under compression. J. Mech. Behav. Biomed. Mater. 57, 109–115. doi:10.1016/j.jmbbm.2015.12.003

Broyden, C.G., 1970. The Convergence of a Class of Double-rank Minimization Algorithms 1. General Considerations. IMA J. Appl. Math. 6, 76–90. doi:10.1093/imamat/6.1.76

Byrd, R.H., Gilbert, J.C., Nocedal, J., 2000. A trust region method based on interior point techniques for nonlinear programming. Math. Program. 89, 149–185. doi:10.1007/PL00011391

Christensen, R.M., 2003. Theory of Viscoelasticity: Second Edition, Second Edition edition. ed. Dover Publications, Mineola, N.Y.

Cost, T.L., Becker, E.B., 1970. A multidata method of approximate Laplace transform inversion. Int. J. Numer. Methods Eng. 2, 207–219. doi:10.1002/nme.1620020206

Emri, I., Tschoegl, N.W., 1993. Generating line spectra from experimental responses. Part I: Relaxation modulus and creep compliance. Rheol. Acta 32, 311–322. doi:10.1007/BF00434195

Emri, I., Von Bernstorff, B.S., Cvelbar, R., Nikonov, A., 2005. Re-examination of the approximate methods for interconversion between frequency-and time-dependent material functions. J. Non-Newton. Fluid Mech. 129, 75–84.

Fernández, P., Jesús Lamela, M., Ramos, A., Fernández-Canteli, A., Tanaka, E., 2013. The region-dependent dynamic properties of porcine temporomandibular joint disc under unconfined compression. J. Biomech. 46, 845–848. doi:10.1016/j.jbiomech.2012.11.035

Ferry, J.D., 1980. Viscoelastic Properties of Polymers. John Wiley & Sons.

Findley, W.N., Lai, J.X., Onaran, K., 1976. Creep and Relaxation of Nonlinear Viscoelastic Materials. Dover Publications.

Fletcher, R., 1970. A new approach to variable metric algorithms. Comput. J. 13, 317–322. doi:10.1093/comjnl/13.3.317

Goldfarb, D., 1970. A family of variable-metric methods derived by variational means. Math. Comput. 24, 23–26. doi:10.1090/S0025-5718-1970-0258249-6

Herdy, M., 2003. Introductory theory manual ViscoData & ViscoShift.

Lakes, R.S., 2009. Viscoelastic Materials. Cambridge University Press.

Lakes, R.S., 1998. Viscoelastic Solids. CRC Press.

Lamela, M.J., Prado, Y., Fernández, P., Fernández-Canteli, A., Tanaka, E., 2011. Non-linear Viscoelastic Model for Behaviour Characterization of Temporomandibular Joint Discs. Exp. Mech. 51, 1435–1440. doi:10.1007/s11340-011-9465-4

Mainardi, F., 2010. Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press.

MathWorks, 2016. MATLAB Documentation R2016b [WWW Document]. URL https://es.mathworks.com/help/matlab/ (accessed 2.25.17).

Park, S.W., Kim, Y.R., 2001. Fitting Prony-Series Viscoelastic Models with Power-Law Presmoothing. J. Mater. Civ. Eng. 13, 26–32. doi:10.1061/(ASCE)0899-1561(2001)13:1(26)

Park, S.W., Schapery, R.A., 1999. Methods of interconversion between linear viscoelastic material functions. Part I—a numerical method based on Prony series. Int. J. Solids Struct. 36, 1653–1675. doi:10.1016/S0020-7683(98)00055-9

Pioletti, D.P., Rakotomanana, L.R., Benvenuti, J.-F., Leyvraz, P.-F., 1998. Viscoelastic constitutive law in large deformations. J. Biomech. 31, 753–757. doi:10.1016/S0021-9290(98)00077-3

Provenzano, P., Lakes, R., Keenan, T., Vanderby, R., 2001. Nonlinear Ligament Viscoelasticity. Ann. Biomed. Eng. 29, 908–914. doi:10.1114/1.1408926

Ramkumar, D.H.S., Caruthers, J.M., Mavridis, H., Shroff, R., 1997. Computation of the linear viscoelastic relaxation spectrum from experimental data. J. Appl. Polym. Sci. 64, 2177–2189. doi:10.1002/(SICI)1097-4628(19970613)64:11<2177::AID-APP14>3.0.CO;2-1

Schapery, R.A., 1962. A Simple Collocation Method for Fitting Viscoelastic Models to Experimental Data. GALCIT SM 61-23A Calif. Inst. Technol. Pasadena CA.

Schapery, R.A., 1962. Approximate methods of transform inversion for viscoelastic stress analysis, in: Proceedings Fourth US National Congress of Applied Mechanics. pp. 1075–1085.

Schapery, R.A., Park, S.W., 1999. Methods of interconversion between linear viscoelastic material functions. Part II—an approximate analytical method. Int. J. Solids Struct. 36, 1677–1699. doi:10.1016/S0020-7683(98)00060-2

Shanno, D.F., 1970. Conditioning of quasi-Newton methods for function minimization. Math. Comput. 24, 647–656. doi:10.1090/S0025-5718-1970-0274029-X

SIMULIA, 2007. ABAQUS Analysis User’s Manual. SIMULIA Dassault Systemes.

T.A.Instruments, 2001. RSA3 UserManual. USA.

Tanaka, E., Pelayo, F., Kim, N., Lamela, M.J., Kawai, N., Fernández-Canteli, A., 2014. Stress relaxation behaviors of articular cartilages in porcine temporomandibular joint. J. Biomech. 47, 1582–1587. doi:10.1016/j.jbiomech.2014.03.007

Tobolsky, A.V., 1960. Properties and structure of polymers.

Tobolsky, A.V., Murakami, K., 1959. Existence of a sharply defined maximum relaxation time for monodisperse polystyrene. J. Polym. Sci. 40, 443–456. doi:10.1002/pol.1959.1204013714

Tschoegl, N.W., 2012. The Phenomenological Theory of Linear Viscoelastic Behavior: An Introduction. Springer Science & Business Media.

Tschoegl, N.W., Emri, I., 1993. Generating line spectra from experimental responses. Part II: Storage and loss functions. Rheol. Acta 32, 322–327. doi:10.1007/BF00434196

Tzikang, C., 2000. Determining a Prony Series for a Viscoelastic Material From Time Varying Strain Data. NASA Langley Technical Report Server.

Waltz, R.A., Morales, J.L., Nocedal, J., Orban, D., 2006. An interior algorithm for nonlinear optimization that combines line search and trust region steps. Math. Program. 107, 391–408. doi:10.1007/s10107-004-0560-5

Appendix A.

Table A1. Prony series fitting parameters for both homogeneous and optimal distributions.

Number of terms	$\left[E_0 \quad e_1 \quad \tau_1 \quad \cdots \quad e_n \quad \tau_n\right]$	$\left[E_0 \quad e_1 \quad \tau_1 \quad \cdots \quad e_n \quad \tau_n\right]$
1	[180298.480000000 0.824136899437915 1.59394160783480]	[180298.480000000 0.790641298834234 0.322619191008081]
2	[180298.480000000 0.707284956635547 0.293978070709744 0.120770268412372 8.64231077866853]	[180298.480000000 0.676862461948996 0.0685651171281151 0.161502682209828 13.1397290138809]
3	[180298.480000000 0.673376703487496 0.126251399040010 0.0427478277226709 1.59394160783480 0.123798872722651 20.1237362001974]	[180298.480000000 0.617563686189283 0.0463869557979109 0.138900430634949 1.50008759822847 0.100956721878062 65.7721611364916]
4	[180298.480000000 0.633319663251058 0.0760306945400917 0.0522217650402256 0.578066662786207 0.0649518678646969 4.39508107411245 0.0924974124245412 33.4161073308003]	[180298.480000000 0.573375258226682 0.0384396848204708 0.122309844341341 0.492517002572728 0.0818891526813639 6.34991278396463 0.0926709562646039 106.481558689669]
5	[180298.480000000 0.608604338867814 0.0542197440818547 0.0303225376319685 0.293978070709744 0.101624474480116 1.59394160783480 0.0163492061424714 8.64231077866853 0.0911651105378306 46.8583888066941]	[180298.480000000 0.555613598284982 0.0360419070273322 0.122044868675654 0.367926427644592 0.0776018004654576 3.44521016251314 2.89973123637766e-14 8.46034177690573 0.0931430220571901 47.1953714976349]
6	[180298.480000000 0.508707441498211 0.0425870761741618 0.151309350688599 0.181365909333705 0.00672291282557246 0.772384395066749 0.0943870052690803 3.28935937263134 -1.78713787475290e-16 14.0084200968128 0.0920410067233453 59.6577665674181]	[180298.480000000 0.596564191679323 0.0428293235109045 0.113621692730670 0.754783504886141 0.0235777953423940 5.72612215145064 0.0321425785981521 6.69917402670398 4.63150568244530e-14 13.9636166715441 0.0864555822538190 60.0149005687220]
7	[180298.480000000 0.408219752351044 0.0355318725252387 0.235327241728579 0.126251399040010 0.0141788152005874 0.448594870654164 0.0510141503794219 1.59394160783480 0.0589523788855325 5.66357311549749 3.15990723170900e-16	[180298.480000000 0.546937236520348 0.0364867850655664 0.0385071970807493 0.271652071485464 0.0660473613409926 0.248609442252288 0.0588192209865351 1.30811026892141 0.0567704318759862 5.54205993040787 -1.14727563386985e-15

	20.1237362001974 0.0900085651950930 71.5034043697626]	20.0788535236374 0.0908857077109137 71.9091369435042]
8	[180298.480000000 0.369990955721492 0.0308630118804967 0.240850502920411 0.0952525521386189 0.0458910141995189 0.293978070709744 0.0188356293543053 0.907304887038130 0.0622974554542280 2.80021620679337 0.0332005677334223 8.64231077866854 -4.06792655116561e-16 26.6727745571547 0.0905142744188743 82.3202174507317]	[180298.480000000 0.589658272922427 0.0432669875222498 0.0303775737583543 0.600112293532432 0.0629286716178740 0.537097786007722 0.00831289587941676 0.780621602225845 0.0326988236074771 2.51791282224508 0.0500052612492851 8.50820603043783 1.25562906426457e-12 26.7087835598032 0.0874363808398120 82.3417865879124]
9	[180298.480000000 0.376344297113462 0.0275736637064206 0.211202644168666 0.0760306945400917 0.0321358387691986 0.209644484454298 0.0432347021656302 0.578066662786207 0.0563927105582293 1.59394160783480 0.0285816881683271 4.39508107411245 0.0281739078871121 12.1188489923801 1.16638469715991e-14 33.4161073308003 0.0884982688808037 92.1404524345236]	[180298.480000000 0.280987293878332 0.0318536667293212 0.265159986799160 0.0423189949558151 0.0896025805541497 0.230252599397086 0.0384312876093378 0.615302475996150 0.0322037787237573 1.64288706474620 0.0403619543960961 4.42464404784351 0.0317649841934840 12.1033969967161 1.08396364800756e-14 33.4194270703965 0.0851108596901352 92.1407522031724]
10	[180298.480000000 0.302657616979265 0.0251448773463986 0.230851656981111 0.0632264869410725 0.0983945553297901 0.158982229097338 0.0171243013736513 0.399758873086139 0.0349530439569653 1.00518880329232 0.0443509236939499 2.52753496742656 0.0405200111308422 6.35545580157655 0.00433176004831986 15.9807159807243 2.94682762349771e-16 40.1833151279606 0.0967305453309271 101.040455047234]	[180298.480000000 0.258058957462417 0.0432890549439342 0.296899712797504 0.0329483246902424 0.0419647103995531 0.182137570185088 0.0436229754722881 0.378282411998683 0.0619516809958969 0.854724969422352 7.05574502841438e-05 2.52434116774581 0.0715741436824561 6.31832276371012 0.00214432685348868 15.9060488163711 0.000319792703734464 40.1359070960688 0.0916967123211249 101.052342008355]