A geometric and game-theoretic study of the conjunction of possibility measures

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Abstract

In this paper, we study the conjunction of possibility measures when they are interpreted as coherent upper probabilities, that is, as upper bounds for some set of probability measures. We identify conditions under which the minimum of two possibility measures remains a possibility measure. We provide graphical way to check these conditions, by means of a zero-sum game formulation of the problem. This also gives us a nice way to adjust the initial possibility measures so their minimum is guaranteed to be a possibility measure. Finally, we identify conditions under which the minimum of two possibility measures is a coherent upper probability, or in other words, conditions under which the minimum of two possibility measures is an exact upper bound for the intersection of the credal sets of those two possibility measures.

Keywords: possibility measure, conjunction, imprecise probability, game theory, natural extension, coherence

1. Introduction

1.1. Possibility Measures: Why (Not)

Imprecise probability models [34] are useful in situations where there is insufficient information to identify a single probability distribution. Many different kinds of imprecise probability models have been studied in the literature [35]. It has been argued that closed convex sets of probability measures, also called *credal sets*, provide a unifying framework for many—if not most—of these models [34, 23].

A downside of using credal sets in their full generality is that they can be computationally quite demanding, particularly in situations that involve many

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random variables. Therefore, in practice, it is often desirable to work with simpler models whose practicality compensate their limited expressiveness. *Possibility measures* [37, 14, 7, 9] are among the simplest of such models, and present a number of distinct advantages:

- Possibility measures can be easily elicited from experts, either through linguistic assessments [8] or through lower confidence bounds over nested sets [27].
- Possibility distributions provide compact and easily interpretable graphical representations.
- In large models, when exact computations are costly, possibility measures can be simulated straightforwardly through random sets [1] (for example to propagate uncertainty through complex models [2]).
- Lower and upper expectations induced by possibility measures can be computed exactly by Choquet integration [30, Section 7.8].
- When interpreted as sets of probability measures, possibility measures have a limited number of extreme points [24, 21]. Many inference algorithms, for instance many of those used in graphical models, employ extreme point representations: using possibility measures in such algorithms will reduce the computational effort required.

An obvious disadvantage of using a family of simpler models is that the family may not be rich enough to allow certain standard operations. For instance, multivariate joint models obtained from possibilistic marginals are usually not possibility distributions [25], hence outer-approximating possibility measures have been proposed [31, 12] to allow one to use the practical advantages of such models.

1.2. Formulation of the problem

In this paper, we focus exclusively on the *conjunction* of two models, that is, the intersection of two credal sets. The conjunction is of interest, for instance, when possibility measures have been elicited from different experts, and we want to know which probability measures are compatible with the assessments of all experts simultaneously. As such, the conjunction is a combination rule that aggregates pieces of information consisting of several inputs to the same problem.

Many combination rules for imprecise probability models are discussed in the literature; see for instance [5, 26, 17, 19**?** , 10]. In this paper, we define the conjunction of two possibility measures as the upper envelope of the set of probability measures that are compatible (i.e., dominated) by both. The following questions arise:

- It may happen that there is no probability measure that is compatible with both possibility measures, in which case the conjunction does not exist. In the language of imprecise probability theory, this means that the conjunction *incurs sure loss*. When does this happen?

- Even when there is at least one probability measure that is compatible with both possibility measures, the upper envelope may not be a possibility measure. In order words, it is not guaranteed that the conjunction on possibility measures is *closed* [17]. When is the conjunction of two possibility measures again a possibity measure? If it is not, can we effectively approximate it by a possibility measure?

- Finally, if the conjunction is a possibility measure, can we express that possibility measure directly in terms of the two possibility measures that we are starting from, without going through their credal sets?

We will answer each of the questions above, using the notions of avoiding sure loss, coherence and natural extensions from the behavioural theory of imprecise probabilities [34]. The main contributions of this paper are:

- From a theoretical viewpoint, we provide sufficient and necessary conditions for the intersection to be again a credal set that can be represented by a possibility measure (Theorems 14 and 16).

- From a practical perspective, we derive from these conditions correction strategies such that the intersection of the corrected models is an outer-approximating possibility distribution (Lemma 21 and Theorem 22).

Interestingly, some of our results can be proven quite elegantly by means of standard results from zero-sum game theory (Theorem 15). This theory also leads us to a graphical method to check the conditions and to apply the correction strategy (Section 4.3).

1.3. Related literature

The literature on the conjunction of possibility measures is somewhat scarce. However, there are quite a few related results that have been proven in the context of evidence theory, which from a formal point of view includes possibility theory as a particular case.

The compatibility of two possibility measures, meaning that the intersection of their associated sets of probabilities is non-empty, was characterised by Dubois and Prade in [16]. Related work for belief functions was done by Chateauneuf in [4].

With respect to the conjunction of two possibility measures again being a possibility measure, a necessary condition is the *coherence* of the minimum of these two possibility measures. This coherence was investigated by Zaffalon and Miranda in [38]. We are not aware of any necessary and sufficient conditions for the conjunction determining a possibility measures, and the only existing results are counterexamples showing that this need not be the case: see [16], and also [4] for the case of belief functions.

A related problem that has received more attention is the connection between conjunction operators of possibility theory and the conjunction operators of evidence theory: for example Dubois and Prade [15] study how Dempster's rule relate to possibilistic conjunctive operators, and Destercke and Dubois [11] relate belief function combinations to the minimum rule of possibility theory.

1.4. Structure of the paper

The paper is organised as follows. Section 2 presents the notation we use and the problem we propose to tackle, namely the properties of the conjunction of two possibility measures. We begin in Section 3 by providing conditions for the intersection of the credal sets associated with two possibility measures to be non-empty, which means that the conjunction of the possibility measures avoids sure loss. Then we investigate in which cases this conjunction is a coherent upper probability, meaning that it is the upper envelope of a credal set (namely, the intersection of the two credal sets determined by the possibility measures).

As we shall see, the coherence of the conjunction of two possibility measures does not guarantee it is a possibility measure itself. We deal with this problem in Section 4, by studying under which conditions the upper probability resulting from the minimum of two possibility measures is again a possibility measure. We also provide a graphical way to check these conditions that we also use to propose some correction strategy, as well as some illustrative and practical examples.

When this conjunction avoids sure loss but is not coherent, we can always consider its natural extension, that corresponds to taking the upper envelope of the intersection of the credal sets, and that is the greatest coherent upper probability that is dominated by the conjunction of the two possibility measures. In Section 5, we consider the problem of establishing when this natural extension is a possibility measure. Section 6 illustrates the usefulness of our results on a medical diagnosis problem. We conclude the paper in Section 7 with some additional comments and remarks.

2. Notation

2.1. Upper Probabilities, Conjunction, Possibility Measures

Consider a possibility space $\mathcal{X}$. In this paper, we assume that $\mathcal{X}$ is finite. $\wp(\mathcal{X})$ denotes the power set (set of all subsets) of $\mathcal{X}$. A function $Q\colon \wp(\mathcal{X}) \to [0,1]$ is called a *probability measure* [20] whenever $Q(A \cup B) = Q(A) + Q(B)$ for all A and $B \subseteq \mathcal{X}$ such that $A \cap B = \emptyset$, and $Q(\mathcal{X}) = 1$. The set of all probability measures is denoted by $\mathbb{P}$.

A function $\overline{P}\colon \wp(\mathcal{X}) \to [0,1]$ is called an *upper probability* [32, 34]. We can interpret $\overline{P}(A)$ behaviourally as a subject's infimum acceptable selling price for the gamble that pays 1 if A obtains, and 0 otherwise [28, 34]. The *credal set* $\mathcal{M}$ induced by $\overline{P}$ is defined as the set of probability measures it dominates,

$$\mathcal{M} \coloneqq \{Q\colon Q \in \mathbb{P} \ \wedge \ (\forall A \subseteq \mathcal{X})(Q(A) \leq \overline{P}(A))\}. \tag{1}$$

We say that $\overline{P}$ *avoids sure loss* when its credal set is non-empty. In this case, the *natural extension* $\overline{E}$ of $\overline{P}$ is defined as the upper envelope of its credal set, that is

$$\overline{E}(A) \coloneqq \max_{Q \in \mathcal{M}} Q(A) \quad \text{for every } A \subseteq \mathcal{X}. \tag{2}$$

An upper probability is called *coherent* if it coincides with its natural extension, that is, if $\overline{P}(A) = \overline{E}(A)$ for all $A \subseteq \mathcal{X}$. As a consequence, if $\overline{P}$ avoids sure loss then its natural extension is the greatest coherent upper probability it dominates. A coherent upper probability $\overline{P}$ is always sub-additive: $\overline{P}(A \cup B) \leq \overline{P}(A) + \overline{P}(B)$ for any disjoint subsets A and B of $\mathcal{X}$.

The *conjunction* [33] of two upper probabilities $\overline{P}_1$ and $\overline{P}_2$ is defined as

$$\overline{P}(A) \coloneqq \min\{\overline{P}_1(A), \overline{P}_2(A)\} \quad \text{for every } A \subseteq \mathcal{X}. \tag{3}$$

It embodies the behavioural implications of both $\overline{P}_1$ and $\overline{P}_2$. Unfortunately, even if both $\overline{P}_1$ and $\overline{P}_2$ are coherent, the conjunction $\overline{P}$ may not be coherent. One can check that the credal set of the conjunction of $\overline{P}_1$ and $\overline{P}_2$ is the intersection of the credal sets of $\overline{P}_1$ and $\overline{P}_2$ [33]:

$$\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2. \tag{4}$$

If $\mathcal{M}$ is non-empty, $\overline{P}$ can be made coherent through its natural extension.

In this paper, we will be interested in coherent upper probabilities of a very specific form. A function $\pi \colon \mathcal{X} \to [0,1]$ is called a (normalized) *possibility distribution* [37, 14, 7, 18] whenever

$$\max_{x \in \mathcal{X}} \pi(x) = 1. \tag{5}$$

A possibility distribution π induces a *possibility measure* $\Pi \colon \wp(\mathcal{X}) \to [0,1]$ by

$$\Pi(A) \coloneqq \max_{x \in A} \pi(x) \quad \text{for every } A \subseteq \mathcal{X}. \tag{6}$$

A possibility measure is a coherent upper probability [35, p. 37].

2.2. Conjunction of Two Possibility Measures

Consider two possibility distributions π_1 and π_2 that induce possibility measures Π_1 and Π_2, with associated credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$. As just mentioned, the conjunction of these two possibility measures is the upper envelope of $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, and is denoted by $\overline{E}$. Alternatively, $\overline{E}$ is the most conservative (i.e. pointwise largest) coherent upper prevision which is dominated by the upper probability $\overline{P}$ defined by

$$\overline{P}(A) \coloneqq \min\{\Pi_1(A), \Pi_2(A)\} \tag{7}$$

for all events $A \subseteq \mathcal{X}$. Throughout the entire paper, we will use the symbols π_1, π_2, Π_1, Π_2, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}$, $\overline{P}$, and $\overline{E}$, always as defined in this section.

Note that, in general $\overline{P}$ may not avoid sure loss (in which case the conjunction does not exist), or may be incoherent (in which case $\overline{P}$ does not coincide with $\overline{E}$), and even when it is coherent, it may not be a possibility measure itself. In this paper, we investigate in detail each of these cases, by providing necessary and sufficient conditions for $\overline{P}$ to satisfy each of these properties.

3. Avoiding sure loss and coherence

We begin by investigating under which conditions the upper probability determined by the conjunction of two possibility measures avoids sure loss or is coherent. These are the minimal behavioural conditions established by Walley in [34].

3.1. When does $\overline{P}$ avoid sure loss?

It is not difficult to show that $\overline{P}$ does not avoid sure loss in general.

Example 1. *Let $\mathcal{X} = \{1, 2\}$ and*

	1	*2*
π_1	*1*	*0.3*
π_2	*0.5*	*1*

Then any probability measure $Q \in \mathcal{M}_1 \cap \mathcal{M}_2$ must satisfy $Q(\{1\}) \leq 0.5$ and $Q(\{2\}) \leq 0.3$. This is incompatible with $1 = Q(\{1, 2\}) = Q(\{1\}) + Q(\{2\})$, and therefore $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$.

The following theorem, proven by Dubois and Prade [16, Lemma 5], gives a necessary and sufficient condition for the upper probability $\overline{P}$ to avoid sure loss:

Theorem 2. *[16] $\overline{P}$ avoids sure loss if and only if for all $A \subseteq \mathcal{X}$*

$$1 \leq \Pi_1(A) + \Pi_2(A^c). \tag{8}$$

This result was also established for belief functions by Chateauneuf in [4], who refers to the non-empty intersection of the credal sets as the *compatibility* of their associated imprecise probability models; see also [6]. Other characterisations of avoiding sure loss for the conjunction of possibility measures can be found in [16, Propositions 6 and 7].

3.2. When is $\overline{P}$ coherent?

Recall that $\overline{P}$ is coherent if and only if it coincides with its natural extension $\overline{E}$, that is, if and only if it coincides with the upper envelope of its credal set $\mathcal{M}$, as in Eq. (2). The conjunction $\overline{P}$ can be incoherent even if it avoids sure loss, as the following example shows:

Example 3. *Let $\mathcal{X} = \{1, 2, 3\}$ and*

	1	*2*	*3*
π_1	*1*	*0.3*	*0.5*
π_2	*0.5*	*1*	*0.7*

Then every probability measure $Q \in \mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$ must satisfy $Q(A) \leq \overline{P}(A) = \min\{\Pi_1(A), \Pi_2(A)\}$ for all $A \subseteq \mathcal{X}$. In particular,

$$Q(\{1\}) \leq 0.5, \qquad Q(\{2\}) \leq 0.3, \qquad Q(\{3\}) \leq 0.5, \tag{9}$$

$$Q(\{1,2\}) \leq 1, \qquad Q(\{1,3\}) \leq 0.7, \qquad Q(\{2,3\}) \leq 0.5. \tag{10}$$

Since $Q(\{1\}) \leq 0.5$ and $Q(\{2\}) \leq 0.3$ imply that $Q(\{1,2\}) \leq 0.8$, but on the other hand $\overline{P}(\{1,2\}) = 1$, it follows that $\overline{P}$ is incoherent. Still, $\overline{P}$ avoids sure loss because $\mathcal{M}$ contains the probability measure Q with $Q(\{1\}) = 0.5$, $Q(\{2\}) = 0.3$, and $Q(\{3\}) = 0.2$.

Given a credal set $\mathcal{M}$, the upper envelope of the set of expectation operators with respect to the elements of $\mathcal{M}$ is called a *coherent upper prevision.* The conjunction of two coherent upper previsions with respective credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ is coherent if and only if $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex [38, Theorem 6]. From the proof of [38, Theorem 6, (a)⇒(b)⇒(c)], one can easily see that convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$ is still sufficient (but not necessary) for the conjunction of two upper probabilities on events to be coherent. This leads immediately to the following sufficient condition for the coherence of $\overline{P}$:

Proposition 4. *$\overline{P}$ is coherent if $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex.*

The convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$ can be checked in polynomial time [3]. The following example shows that convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$ is not necessary for $\overline{P}$ to be coherent. It simultaneously shows that $\overline{P}$ does not need to be a possibility measure, even if it is coherent.

Example 5. *Let $\mathcal{X} = \{1, 2, 3\}$ and*

	1	*2*	*3*
π_1	*1*	*0.5*	*0.5*
π_2	*0.5*	*1*	*0*

Then $\overline{P}$ is the probability measure with probability mass function $(0.5, 0.5, 0)$. This is not a possibility measure, but it is a coherent upper probability (because it is trivially the upper envelope of itself).

Also, $\mathcal{M}_1 \cup \mathcal{M}_2$ is not convex. Using vector notation for probability mass functions, we have that

$$(0.5, 0.25, 0.25) \in \mathcal{M}_1 \text{ and } (0.25, 0.75, 0) \in \mathcal{M}_2 \tag{11}$$

but their average $(0.375, 0.5, 0.125)$ does not belong to $\mathcal{M}_1 \cup \mathcal{M}_2$, because

$$Q(\{2,3\}) = 0.625 > 0.5 = \Pi_1(\{2,3\}) \text{ and } Q(\{3\}) = 0.125 > 0 = \Pi_2(\{3\}). \tag{12}$$

Indeed, that $Q(A) > \Pi_i(A)$ for some event A implies that $Q \notin \mathcal{M}_i$.

Regarding [38, Theorem 6], let $\overline{P}_1$ and $\overline{P}_2$ denote the upper envelopes of the sets of expectation operators with respect to the credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ in this example. Then the conjunction $\min\{\overline{P}_1, \overline{P}_2\}$ is not equal to the expectation operator associated with $(0.5, 0.5, 0) = \mathcal{M}_1 \cap \mathcal{M}_2$. To see this, consider the gamble f given by $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. For this gamble, $Q(f) = 1.5 < 2 = \min\{\overline{P}_1(f), \overline{P}_2(f)\}$.

Next we show that the minimum $\overline{P}$ of two possibility measures can be a coherent upper probability that is not even 2-alternating (and thus not a possibility measure, either). Recall that $\overline{P}$ is 2-alternating if $\overline{P}(A) + \overline{P}(B) \leq \overline{P}(A \cup B) + \overline{P}(A \cap B)$ for any $A, B \subseteq \mathcal{X}$.

Example 6. *Let* $\mathcal{X} = \{1, 2, 3, 4\}$ *and*

	1	*2*	*3*	*4*
π_1	*0.3*	*0.4*	*0.6*	*1*
π_2	*0.3*	*0.6*	*0.4*	*1*

It can be shown by linear programming that $\overline{P}$ *is coherent. However, it is not* 2*-alternating: for* $A = \{1, 2\}$ *and* $B = \{1, 3\}$*, it holds that*

$$\overline{P}(A \cup B) + \overline{P}(A \cap B) = \overline{P}(\{1, 2, 3\}) + \overline{P}(\{1\}) = 0.6 + 0.3 = 0.9 \tag{13}$$

$$> \overline{P}(A) + \overline{P}(B) = \overline{P}(\{1, 2\}) + \overline{P}(\{1, 3\}) = 0.8. \tag{14}$$

The following result is rather surprising: we can show that the conjunction $\overline{P}$ of two possibility measures is 2-alternating when $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex; it strengthens Proposition 4.

Proposition 7. $\overline{P}$ *is* 2*-alternating if* $\mathcal{M}_1 \cup \mathcal{M}_2$ *is convex.*

Proof. By [32, Corollary 6.4], to show that $\overline{P}$ is 2-alternating, it suffices to establish that for every $A \subseteq B \subseteq \mathcal{X}$ there is a $Q \in \mathcal{M}$ such that $Q(A) = \overline{P}(A)$ and $Q(B) = \overline{P}(B)$.

Consider $A \subseteq B \subseteq \mathcal{X}$. Because Π_1 is a possibility measure and therefore 2-alternating, there is a $Q_1 \in \mathcal{M}_1$ such that $Q_1(A) = \Pi_1(A)$ and $Q_1(B) = \Pi_1(B)$. Similarly, there is a $Q_2 \in \mathcal{M}_2$ such that $Q_2(A) = \Pi_2(A)$ and $Q_2(B) = \Pi_2(B)$. Now, since $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex, it follows from [38, Theorem 6] that there is an $\alpha \in [0, 1]$ such that $Q \coloneqq \alpha Q_1 + (1 - \alpha) Q_2$ belongs to $\mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}$, and as a consequence Q is dominated by $\overline{P}$:

$$Q(A) \leq \overline{P}(A) \qquad\qquad Q(B) \leq \overline{P}(B). \tag{15}$$

But, by construction of Q, we also have that that

$$\begin{aligned} Q(A) &= \alpha Q_1(A) + (1 - \alpha) Q_2(A) \\ &\geq \min\{Q_1(A), Q_2(A)\} = \min\{\Pi_1(A), \Pi_2(A)\} = \overline{P}(A) \end{aligned} \tag{16}$$

$$\begin{aligned} Q(B) &= \alpha Q_1(B) + (1 - \alpha) Q_2(B) \\ &\geq \min\{Q_1(B), Q_2(B)\} = \min\{\Pi_1(B), \Pi_2(B)\} = \overline{P}(B). \end{aligned} \tag{17}$$

Concluding, $Q(A) = \overline{P}(A)$ and $Q(B) = \overline{P}(B)$, so $\overline{P}$ is 2-alternating. $\square$

To see that the convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$ does not guarantee that $\overline{P}$ is a possibility measure, consider the following example:

Example 8. *Let $\mathcal{X} = \{1, 2\}$ and*

	1	*2*
π_1	*0.5*	*1*
π_2	*1*	*0.5*

Then $\overline{P}$ is the probability measure determined by the probability mass function $(0.5, 0.5)$, which is obviously not a possibility measure. However, $\mathcal{M}_1$ is the set of all probability measures Q for which $Q(\{x_1\}) \leq 0.5$, and $\mathcal{M}_2$ is the set of all probability measures Q for which $Q(\{x_1\}) \geq 0.5$, so $\mathcal{M}_1 \cup \mathcal{M}_2$ is the set of all probability measures on $\mathcal{X}$, which is convex.

From the proof of Proposition 7, we see that the convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$ is actually a really strong requirement. Specifically, it requires that, for all $A \subseteq B$,

$$\Pi_1(A) < \Pi_2(A) \implies \Pi_1(B) \leq \Pi_2(B) \tag{18}$$

$$\Pi_1(A) > \Pi_2(A) \implies \Pi_1(B) \geq \Pi_2(B) \tag{19}$$

$$\Pi_1(B) < \Pi_2(B) \implies \Pi_1(A) \leq \Pi_2(A) \tag{20}$$

$$\Pi_1(B) > \Pi_2(B) \implies \Pi_1(A) \geq \Pi_2(A) \tag{21}$$

Indeed, if $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex, following the proof of Proposition 7, taking Eqs. (16) and (17) and noting that $Q_i(A) = \Pi_i(A)$ and $Q_i(B) = \Pi_i(B)$, we know that there is an $\alpha \in [0, 1]$ such that

$$\alpha\Pi_1(A) + (1 - \alpha)\Pi_2(A) = \min\{\Pi_1(A), \Pi_2(A)\} \tag{22}$$

$$\alpha\Pi_1(B) + (1 - \alpha)\Pi_2(B) = \min\{\Pi_1(B), \Pi_2(B)\} \tag{23}$$

So, if $\Pi_1(A) < \Pi_2(A)$, then it must be that $\alpha = 1$ by the first equality, and therefore also $\Pi_1(B) \leq \Pi_2(B)$ by the second equality. The other cases follow similarly.

These implications give us a simple way to check for typical violations of convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$, through the following corollary.

Corollary 9. *If $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex, then for all subsets A, B, and C of $\mathcal{X}$ such that $\Pi_1(A) < \Pi_2(A)$, $\Pi_1(B) > \Pi_2(B)$, and $C \supseteq A \cup B$, we have that $\Pi_1(C) = \Pi_2(C)$.*

In a way, Example 8 is thus showing a very peculiar situation (corresponding to $A = \{x_1\}$, $B = \{x_2\}$, and $C = \{x_1, x_2\}$ in Corollary 9).

One of the advantages of possibility measures over other imprecise probability models is their computational simplicity, that follows from Eq. (6): possibility measures are uniquely determined by their restriction to singletons, called their possibility distributions. Moreover, possibility distributions connect possibility measures with fuzzy sets [37]. The minimum of two possibility distributions was defined by Zadeh as one instance of *fuzzy set intersection*. However, the connection between imprecise probabilities and fuzzy sets by means of possibility measures does not hold under the conjunction operator we are considering in this paper, in the sense that, as we have seen in Example 6, coherent conjunctions of possibility measures need not be determined by their restrictions to singletons. One might wonder if these restrictions suffice to characterise the coherence of $\overline{P}$. Clearly, a necessary condition for the coherence of $\overline{P}$ is that for every $x \in \mathcal{X}$ there is some $Q \in \mathcal{M}_1 \cap \mathcal{M}_2$ such that $Q(\{x\}) = \overline{P}(\{x\})$. However, this condition is not sufficient, as the following example shows.

Example 10. *Let* $\mathcal{X} = \{1, 2, 3\}$ *and*

	1	*2*	*3*
π_1	*0.8*	*0.2*	*1*
π_2	*0.2*	*0.9*	*1*

Then $(0, 0, 1)$ *belongs to* $\mathcal{M}_1 \cap \mathcal{M}_2$*, so* $\overline{P}$ *avoids sure loss. However, it is not coherent because* $\overline{P}(\{1, 2\}) = 0.8 > \overline{P}(\{1\}) + \overline{P}(\{2\}) = 0.4$.

One can easily check that both $(0.2, 0.2, 0.6)$ *and* $(0, 0, 1)$ *are in* $\mathcal{M}_1 \cap \mathcal{M}_2$*, and that* $(0.2, 0.2, 0.6)$ *achieves the upper bound for* $\{1\}$ *and* $\{2\}$*, and* $(0, 0, 1)$ *achieves the upper bound for* $\{3\}$*. We have thereby shown that* $\overline{P}(\{x\}) = \max_{Q \in \mathcal{M}_1 \cap \mathcal{M}_2} Q(\{x\})$ *for all* $x \in \mathcal{X}$*.*

The following graph summarises the implications between conditions established in this section:

$$\begin{tikzcd}
& \overline{P}\text{ possibility} \arrow[d, Rightarrow] & \\
\mathcal{M}_1 \cup \mathcal{M}_2\text{ convex} \arrow[r, Rightarrow] & \overline{P}\text{ 2-alternating} \arrow[r, Rightarrow] & \overline{P}\text{ coherent} \arrow[d, Rightarrow] \\
& & \begin{array}{c} \forall x \in \mathcal{X}: \\ \overline{P}(\{x\}) = \max_{Q \in \mathcal{M}} Q(\{x\}) \end{array}
\end{tikzcd}$$

The examples in this section show that the converses of these implications do not hold in general. To see that there is no implication between the convexity of $\mathcal{M}_1 \cup \mathcal{M}_2$ and $\overline{P}$ being a possibility measure, consider Example 8 above as well as Example 11 later on.

4. When is $\overline{P}$ a possibility measure?

Next, we are going to study under which conditions the conjunction $\overline{P}$ of two possibility measures is again a possibility measure. We shall begin by providing a simple sufficient (yet not necessary) condition, followed by more advanced necessary and sufficient conditions. One of them will establish a link with game theory, along with a corresponding method for graphical verification.

4.1. Sufficient conditions

Clearly, $\overline{P}$ is a possibility measure (and therefore also coherent) when $\pi_1(x) \leq \pi_2(x)$ for all $x \in \mathcal{X}$, or equivalently, when $\Pi_1(A) \leq \Pi_2(A)$ for all $A \in \mathcal{X}$, since then $\mathcal{M}(\overline{P}) = \mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}_1$. This condition means that the possibility measure Π_1 is more *specific* [36, 13] than Π_2. However, this is not the only case in which the conjunction of possibility measures is again a possibility measure, as the following example shows.

Example 11. *Let $\mathcal{X} = \{1, 2, 3\}$ and*

	1	*2*	*3*
π_1	*1*	*0.5*	*0.7*
π_2	*1*	*0.6*	*0.6*

Then

$$\overline{P}(\{1\}) = 1,\ \overline{P}(\{2\}) = 0.5,\ \overline{P}(\{3\}) = 0.6 \tag{24}$$

$$\overline{P}(\{1,2\}) = 1,\ \overline{P}(\{1,3\}) = 1,\ \overline{P}(\{2,3\}) = 0.6. \tag{25}$$

Thus, $\overline{P}$ is a possibility measure, even though $\pi_1(2) < \pi_2(2)$ and $\pi_1(3) > \pi_2(3)$. We can also note that, in this case, $\mathcal{M}_1 \cup \mathcal{M}_2$ is not convex: $\Pi_1(\{2\}) < \Pi_2(\{2\})$, $\Pi_1(\{3\}) > \Pi_2(\{3\})$, and yet $\Pi_1(\{2,3\}) = 0.7 \neq 0.6 = \Pi_2(\{2,3\})$; now use Corollary 9.

In the example, the possibility distributions π_1 and π_2 follow the same order, in the sense that $\pi_i(2) \leq \pi_i(3) \leq \pi_i(1)$ for both $i = 1$ and $i = 2$. This ordering condition turns out to be sufficient for the conjunction of the two possibility measures to be again a possibility measure:

Theorem 12. *$\overline{P}$ is a possibility measure whenever there is an ordering $x_1, \dots, x_n$ of the elements of $\mathcal{X}$ such that for both $i = 1$ and $i = 2$ we have that*

$$\pi_i(x_1) \leq \pi_i(x_2) \leq \cdots \leq \pi_i(x_n). \tag{26}$$

Proof. Consider $A \subseteq \mathcal{X}$ and let $j(A) := \max\{j \in \{1, \dots, n\} \colon x_j \in A\}$. Then, by Eq. (26), $\Pi_i(A) = \pi_i(x_{j(A)})$, and so

$$\overline{P}(A) = \min\{\Pi_1(A), \Pi_2(A)\} = \min\{\pi_1(x_{j(A)}), \pi_2(x_{j(A)})\} \tag{27}$$

$$= \overline{P}(\{x_{j(A)}\}) = \max_{x_i \in A} \overline{P}(\{x_i\}) \tag{28}$$

where the last equality follows from

$$\overline{P}(\{x_1\}) \leq \overline{P}(\{x_2\}) \leq \cdots \leq \overline{P}(\{x_n\}), \tag{29}$$

which also follows from Eq. (26). Thus, $\overline{P}$ is a possibility measure. □

Equivalently, this means that $\overline{P}$ is a possibility measure when π_1 and π_2 are comonotone functions. To see that this sufficient condition is not necessary, simply note that it may not hold when $\pi_1 \leq \pi_2$:

Example 13. *Let $\mathcal{X} = \{1, 2, 3\}$ and*

	1	2	3
π_1	1	0.9	0.8
π_2	1	0.5	0.6

Then $\Pi_2 \leq \Pi_1$, so $\overline{P} = \min\{\Pi_1, \Pi_2\} = \Pi_2$ is a possibility measure. However, π_1 and π_2 are not comonotone because $\pi_1(2) > \pi_1(3)$ and $\pi_2(2) < \pi_2(3)$.

4.2. Sufficient and necessary conditions

Next we give a necessary and sufficient condition for $\overline{P}$ to be a possibility measure. It will allow us to make a link with game theory.

Theorem 14. *$\overline{P}$ is a possibility measure Π if and only if*

$$\min\left\{\max_{x\in A}\pi_1(x), \max_{x\in A}\pi_2(x)\right\} = \max_{x\in A}\min\{\pi_1(x), \pi_2(x)\} \tag{30}$$

for all non-empty $A \subseteq \mathcal{X}$. In such a case, $\overline{E}$ coincides with $\overline{P}$, and whence, $\overline{E}$ is a possibility measure as well.

Proof. Note that the left hand side is $\overline{P}(A)$.

"if". If the equality holds, then $\overline{P}$ is a possibility measure, and therefore is coherent. Whence, $\overline{E} = \overline{P}$, and so $\overline{E}$ is a possibility measure too.

"only if". On the one hand, by the definition of $\overline{P}$,

$$\overline{P}(A) = \min\{\Pi_1(A), \Pi_2(A)\} = \min\left\{\max_{x\in A}\pi_1(x), \max_{x\in A}\pi_2(x)\right\} \tag{31}$$

On the other hand, if $\overline{P}$ is a possibility measure, its possibility distribution must be $\pi(x) = \overline{P}(\{x\}) = \min\{\pi_1(x), \pi_2(x)\}$, and so,

$$\overline{P}(A) = \max_{x\in A}\min\{\pi_1(x), \pi_2(x)\}. \tag{32}$$

Combining both equalities, we arrive at the desired equality. □

Theorem 14 has a very nice game-theoretic interpretation. Consider a zero-sum game with two players, where player 1 can choose α from $\{1, 2\}$ and player 2 can choose β from $\{1, \dots, n\}$, with the following payoffs to player 1:

	$\beta = 1$	$\dots$	$\beta = n$
$\alpha = 1$	a_{11}	$\dots$	a_{1n}
$\alpha = 2$	a_{21}	$\dots$	a_{2n}

This table with payoffs to player 1 is called the *payoff matrix*. For example, if $(\alpha, \beta) = (2, 3)$, then player 1 gains a_{23} and player 2 loses a_{23}. A pair (α, β) is called *pure strategy*.

A pure strategy $(\hat{\alpha}, \hat{\beta})$ is said to be in *equilibrium* if it does not benefit either player to change his choice if the other does not change his choice [22, p. 62–64]:

$$a_{\hat{\alpha}\hat{\beta}} = \max_{\alpha} a_{\alpha\hat{\beta}} = \min_{\beta} a_{\hat{\alpha}\beta} \tag{33}$$

Theorem 15. *$\overline{P}$ is a possibility measure Π if and only if for all non-empty $A \subseteq \mathcal{X}$, the zero-sum game with choices $\alpha \in \{1, 2\}$ and $\beta \in A$, and payoffs $a_{\alpha\beta} := -\pi_\alpha(\beta)$, has a pure equilibrium strategy.*

Proof. "if". If the zero-sum game associated with $A \subseteq \mathcal{X}$ has a pure equilibrium strategy $(\hat{\alpha}, \hat{\beta})$, then [22, p. 67]

$$a_{\hat{\alpha}\hat{\beta}} = \max_\alpha \min_\beta a_{\alpha\beta} = \min_\beta \max_\alpha a_{\alpha\beta}. \tag{34}$$

But $a_{\alpha\beta} := -\pi_\alpha(\beta)$, so this is precisely Equation (30).

"only if". If $\overline{P}$ is a possibility measure, then Equation (30) can be rewritten as

$$\max_\alpha \min_\beta a_{\alpha\beta} = \min_\beta \max_\alpha a_{\alpha\beta}. \tag{35}$$

This means that the zero-game has a pure equilibrium strategy, for example

$$\hat{\alpha} := \operatorname*{argmax}_\alpha \min_\beta a_{\alpha\beta} \qquad\qquad \hat{\beta} := \operatorname*{argmin}_\beta a_{\hat{\alpha}\beta} \tag{36}$$

□

Although Theorem 15 is in essence nothing more but a rephrasing of Theorem 14, it highlights an interesting fact: we can use any method for solving $2 \times n$ zero-sum games in order to determine whether our conjunction $\overline{P}$ is a possibility measure.

The traditional way of finding pure equilibrium strategies goes by removing dominated options from the game, until only a single strategy remains. For $2 \times n$ games, this is a particularly simple process: it suffices first to remove columns that are not optimal for player 2, and then to check whether, in the payoff matrix that remains, one of the rows dominates the other. For example, consider the following 2×4 game with the following payoff to player 1:

	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$
$\alpha = 1$	3	2	2	4
$\alpha = 2$	0	3	1	0

We can remove the column $\beta = 2$ because its payoff is higher than the payoff of column $\beta = 3$ regardless of α—remember that the column player wants to minimize the payoff. We can also remove the column $\beta = 4$ because its payoff is higher than the payoff of column $\beta = 1$ regardless of α. No further columns can be removed. Now, in the remaining payoff matrix, the row $\alpha = 2$ can be removed because its payoff is lower than the payoff of row $\alpha = 1$—remember that the row player wants to maximize the payoff. So, the row player will play $\alpha = 1$. In the remaining row $\alpha = 1$, clearly $\beta = 3$ achieves the minimum payoff for player 2. This game therefore has a pure equilibrium strategy, namely $(\hat{\alpha}, \hat{\beta}) = (1, 3)$.

The two sufficient conditions provided in Section 4.1 follow immediately from Theorem 15. Indeed, let $A = \{a_1, a_2, \ldots, a_m\} \subseteq \mathcal{X}$. By Theorem 15, we need to consider the payoff matrix

	$\beta = 1$	...	$\beta = k$	...	$\beta = m$
$\alpha = 1$	$-\pi_1(a_1)$	...	$-\pi_1(a_k)$	...	$-\pi_1(a_m)$
$\alpha = 2$	$-\pi_2(a_1)$	...	$-\pi_2(a_k)$	...	$-\pi_2(a_m)$

- If $\pi_1(x) \leq \pi_2(x)$ for all $x \in \mathcal{X}$, then clearly $-\pi_1(x) \geq -\pi_2(x)$ for every $x \in A$, regardless of A. Therefore the first row of the payoff matrix will dominate the second row. As player 1 aims to maximize his payoff, $\alpha = 1$ will achieve his optimal strategy, regardless of what player 2 does. Consequently, the second row can be eliminated, and the pure equilibrium is reached for $\alpha = 1$ and $\beta = \operatorname{argmin}_{k \in \{1,\dots,m\}}\{-\pi_1(a_k)\}$.

- If there is an ordering $x_1, \dots, x_n$ of the elements of $\mathcal{X}$ such that $\pi_i(x_j) \leq \pi_i(x_{j+1})$ for all $i \in \{1, 2\}$ and $j \in \{1, \dots, n-1\}$ then, without loss of generality, we may assume that the elements $a_1, \dots, a_m$ of A are ordered reversely, that is, $-\pi_i(a_k) \leq -\pi_i(a_{k+1})$ for all $i \in \{1, 2\}$ and $k \in \{1, \dots, m-1\}$. But then the first column is dominated by all other columns. As player 2 aims to minimize his payoff, $\beta = 1$ will achieve his optimal strategy, regardless of what player 1 does. Consequently, all columns other than the first can be eliminated, and the pure equilibrium strategy is reached for $\alpha = \operatorname{argmax}_{i \in \{1,2\}}\{-\pi_i(a_1)\}$ and $\beta = 1$.

It is important to note that not every $2 \times n$ game has a pure equilibrium. For example, consider the 2×2 zero-sum game with the following payoff matrix:

	$\beta = 1$	$\beta = 2$
$\alpha = 1$	1	0
$\alpha = 2$	0	1

Luce and Raiffa [22, Appendices 3 and 4] discuss two very nice graphical ways of representing and solving $2 \times n$ zero-sum games. Both methods are particularly suited also to determine whether there are pure equilibrium points. Without going into too much detail, their first method makes it easy to identify whether player 1 has a pure equilibrium strategy, whilst their second method makes it easy to identify whether player 2 has a pure equilibrium strategy. Because player 2 must have a pure equilibrium strategy whenever player 1 has a pure one, the first method is most straightforward for our purpose.

First, we draw all lines $f_\beta(p) \coloneqq p a_{1\beta} + (1-p) a_{2\beta}$, for $p \in [0, 1]$ and all $\beta \in A$. We then determine the lower envelope $f_A(p)$ of these lines:

$$f_A(p) \coloneqq \min_{\beta \in A} f_\beta(p). \tag{37}$$

Note that f_A will be a concave function. If f_A is monotone (i.e. has its maximum at $p = 0$ or $p = 1$), then there is a pure equilibrium point.

A further substantial gain can be made by recognising that the monotonicity of a concave function $f_A(p)$ between $p = 0$ and $p = 1$ is uniquely determined by $f'_A(0)$ and $f'_A(1)$: f is monotone if and only if $f'_A(0) f'_A(1) \geq 0$. Because $f_A(p)$ is piece-wise linear, it suffices therefore to look at the left-most line and right-most line only: the lower envelope is monotone if and only if these lines are sloped in the same direction. Consequently, for application to Theorem 15, it suffices to look at pairs of lines. In fact, it suffices to look at pairs of intersecting lines, because if the lines do not intersect, then the lower envelope is linear and so guaranteed to be monotone.

We have thus reached the following rather surprising result, for which we also give a simple proof that does not rely on zero-sum games:

Theorem 16. *$\overline{P}$ is a possibility measure if and only if*

$$\min_{i\in\{1,2\}}\left(\max_{j\in\{1,2\}}\pi_i(x_j)\right)=\max_{j\in\{1,2\}}\left(\min_{i\in\{1,2\}}\pi_i(x_j)\right) \tag{38}$$

for all $\{x_1,x_2\}\subseteq\mathcal{X}$. In such a case, $\overline{E}$ coincides with $\overline{P}$, and whence, $\overline{E}$ is a possibility measure as well.

Proof. First, note that Eq. (38) is equivalent to saying that

$$\overline{P}(\{x_1,x_2\})=\max\{\overline{P}(\{x_1\}),\overline{P}(\{x_2\})\} \tag{39}$$

for every $\{x_1,x_2\}\subseteq\mathcal{X}$. We show that this is indeed equivalent to $\overline{P}$ being a possibility measure

'if'. Consider any non-empty $A\subseteq\mathcal{X}$. Let

$$x_1\coloneqq\operatorname*{argmax}_{x\in A}\pi_1(x),\qquad\qquad x_2\coloneqq\operatorname*{argmax}_{x\in A}\pi_2(x). \tag{40}$$

Since Π_1 and Π_2 are possibility measures, it immediately follows that

$$\Pi_1(A)=\Pi_1(\{x_1\})=\Pi_1(\{x_1,x_2\}), \tag{41}$$
$$\Pi_2(A)=\Pi_2(\{x_2\})=\Pi_2(\{x_1,x_2\}). \tag{42}$$

Consequently,

$$\overline{P}(A)=\min\{\Pi_1(A),\Pi_2(A)\} \tag{43}$$
$$=\min\{\Pi_1(\{x_1,x_2\}),\Pi_2(\{x_1,x_2\})\} \tag{44}$$
$$=\overline{P}(\{x_1,x_2\}) \tag{45}$$

and now applying Eq. (38),

$$=\max\{\overline{P}(\{x_1\}),\overline{P}(\{x_2\}) \tag{46}$$
$$\le\max_{x\in A}\overline{P}(\{x\}) \tag{47}$$

The converse inequality follows by monotonicity of $\overline{P}$—indeed, both Π_1 and Π_2 are monotone, so their minimum must be monotone too. Specifically, for every $x\in A$ we have that $\Pi_1(A)\ge\pi_1(x)$ and $\Pi_2(A)\ge\pi_2(x)$, so

$$\overline{P}(A)=\min\{\Pi_1(A),\Pi_2(A)\}\ge\min\{\pi_1(x),\pi_2(x)\}=\overline{P}(\{x\}) \tag{48}$$

and therefore $\overline{P}(A)\ge\max_{x\in A}\overline{P}(\{x\})$. Thus, $\overline{P}(A)=\max_{x\in A}\overline{P}(\{x\})$ and as a consequence it is a possibility measure.

'only if'. If $\overline{P}$ is a possibility measure then $\overline{P}(A)=\max_{x\in A}\overline{P}(\{x\})$ for all non-empty $A\subseteq\mathcal{X}$, and in particular also for all $A=\{x_1,x_2\}$. Eq. (39) follows. $\square$

4.3. Examples

The verification of Theorem 15 entails looking at every pair of lines f_β and f_γ, and checking:

- whether f_β and f_γ intersect for some $0 < p < 1$, that is, whether $f_\beta(p) = f_\gamma(p)$ for some $0 < p < 1$;
- if so, whether f_β and f_γ have the same slope.

If for all intersecting pairs, both lines have the same slope, then the conditions of Theorem 15 are satisfied, and the conjunction will be a possibility measure.

Let us first provide an example, inspired by Sandri *et al.* [27], where the conditions hold.

Example 17. *Two economists provide their opinion about the value ($\mathcal{X} = \{1, \ldots, 9\}$) of a future stock market:*

	1	2	3	4	5	6	7	8	9
π_1	1	0.95	0.95	0.8	0.7	0.2	0.3	0.1	0.05
π_2	1	0.8	0.6	0.7	0.6	0.6	0.3	0.4	0.1

which are pictured as f_β for $\beta \in \{1, \ldots, 9\}$ in Figure 1. We actually pictured $-f_\beta$, to make it easier to relate the lines to the possibility distributions. It can be checked that the conditions required by Theorem 15 hold for every pair. This means that the merged opinion $\overline{P}$ of the two economists can be represented as a possibility distribution. Figure 1 makes verification even easier: there are only three intersecting pairs, namely (f_3, f_4), (f_6, f_7), and (f_7, f_8), and in each pair, both lines have the same slope. Consequently, $\overline{P}$ is a possibility measure induced by the possibility distribution

	1	2	3	4	5	6	7	8	9
π	1	0.8	0.6	0.7	0.6	0.2	0.3	0.1	0.05

When π_1 and π_2 do not satisfy the conditions of Theorem 15, our graphical verification technique also allows us to heuristically adjust π_1 and π_2 into new possibility distributions that do satisfy the conditions of Theorem 15. The next example illustrates this heuristic procedure.

Example 18. *Two economists provide the following opinions:*

	1	2	3	4	5	6	7	8
π_1	1	0.9	0.7	0.6	0.5	0.4	0.3	0.1
π_2	0.8	0.2	1	0.6	0.1	0.2	0.3	0.9

The left hand side of Figure 2 depicts our graphical method. Many pairs of intersecting lines have opposite slopes, for instance (f_8, f_2). Therefore, the conditions of Theorem 15 are not satisfied. Interestingly, there is no $x \in \mathcal{X}$ such that $\pi_1(x) = \pi_2(x) = 1$—this is a necessary condition for $\overline{P}$ to be a possibility measure; see proof of Lemma 21(c) further on.

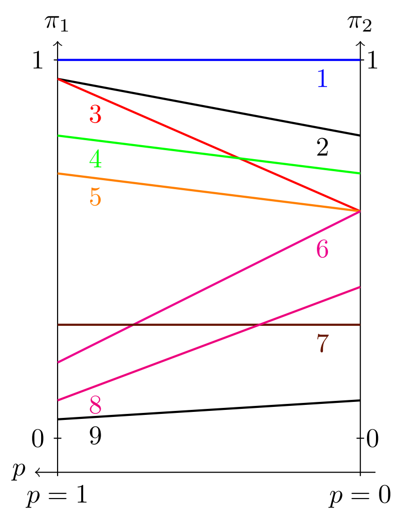

Figure 1: Example 17 game-theoretic figure.

A possible adjustment that allows to satisfy the conditions of Theorem 15, can be done for example by modifying f_1, f_2 and f_8, so that f_1 and f_2 become positively slopped, and so that f_8 no longer intersects with f_5—of course, conservative adjustments should only be done by moving lines upwards. The right hand side of Figure 2 shows the adjusted lines dashed. They result in the following adjusted possibility distributions:

	1	2	3	4	5	6	7	8
π'_1	1	0.9	0.7	0.6	0.5	0.4	0.3	0.5
π'_2	1	0.9	1	0.6	0.1	0.2	0.3	0.9

The resulting adjusted conjunction is:

	1	2	3	4	5	6	7	8
π'	1	0.9	0.7	0.6	0.1	0.2	0.3	0.5

It is clear that any upward adjustment implies a loss of information. In general, there is no unique adjustment minimizing this loss. In any case, upward adjustment ensures that the obtained result will be consistent with the initial information, as it will give an outer approximation.

If there is an element x such that $\pi_1(x) = \pi_2(x) = 1$, then adjustments can also be done downwards, in which case the obtained approximation would be an inner approximation.

5. When is $\overline{E}$ a possibility measure?

The above condition for $\overline{P}$ to be a possibility measure is obviously sufficient for $\overline{E}$ to be a possibility measure. However, the condition is not necessary, as

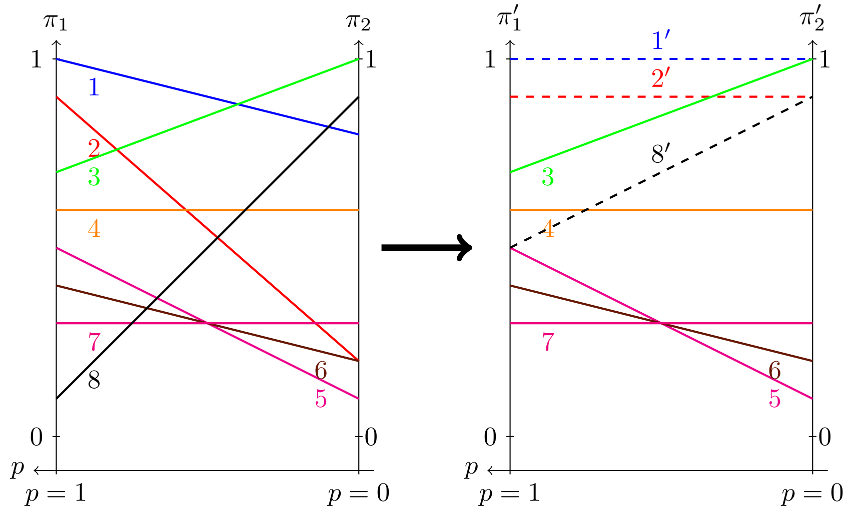

Figure 2: Example 18 game-theoretic figure and adjustment (in dashed).

shown by the next example:

Example 19. *Let*

	1	*2*	*3*
π_1	*1*	*1*	*0*
π_2	*1*	*0*	*1*

The credal set of the conjunction is the singleton $\mathcal{M} = \{Q\}$ for which $Q(\{1\}) = 1$ (and zero elsewhere), because this is the only probability measure that satisfies $Q(\{x\}) \leq \overline{P}(\{x\})$ for all x. Whence, the natural extension $\overline{E}$ of $\overline{P}$ is obviously a possibility measure.

Nevertheless, $\overline{P}$ is not *a possibility measure. Indeed,*

$$\min_{i \in \{1,2\}} \left(\max_{j \in \{2,3\}} \pi_i(j) \right) = \max_{j \in \{2,3\}} \left(\min_{i \in \{1,2\}} \pi_i(j) \right) \tag{49}$$

as the left hand side is one, and the right hand side is zero. This is because $\overline{P}$ is not a coherent upper probability, since $\overline{P}(\{2,3\}) = 1 > \overline{P}(\{2\}) + \overline{P}(\{3\})$.

Indeed, when $\overline{P}$ is coherent then it coincides with $\overline{E}$, and therefore in that case $\overline{E}$ is a possibility measure if and only if $\overline{P}$ is. Below, we state a number of necessary conditions for $\overline{E}$ to be a possibility measure. So far, we failed to identify a condition that is both sufficient *and* necessary.

Lemma 20. *If $\overline{E}$ is a possibility measure Π, then there is an $x \in \mathcal{X}$ such that $\pi(x) = \pi_1(x) = \pi_2(x) = 1$.*

Proof. If $\overline{E}$ is a possibility measure, then $\overline{E}(\{x\}) = \pi(x) = 1$ for at least one $x \in \mathcal{X}$. For any such x,

$$1 = \overline{E}(\{x\}) \leq \min\{\pi_1(x), \pi_2(x)\}, \tag{50}$$

whence, it can only be that $\pi_1(x) = \pi_2(x) = 1$ for such x. □

Of course, if $\overline{E}$ is a possibility measure and $\overline{P}$ is not coherent, then $\overline{E}$ and $\overline{P}$ will not coincide on all events. We shall show next that they are always guaranteed to coincide on the singletons. In order to see this, note that if $\overline{P}$ is a possibility measure, then its possibility distribution is given by

$$\pi(x) \coloneqq \min\{\pi_1(x), \pi_2(x)\} = \overline{P}(\{x\}). \tag{51}$$

We denote the possibility measure determined by this distribution by

$$\Pi(A) \coloneqq \max_{x \in A} \pi(x). \tag{52}$$

We can establish the following.

Lemma 21. *The following statements hold.*

(a) $\overline{P} \geq \Pi$.

(b) Π *is normed if and only if there is an* $x \in \mathcal{X}$ *such that* $\overline{P}(\{x\}) = 1$. *In that case,* $\overline{P}$ *avoids sure loss and* $\overline{P} \geq \overline{E} \geq \Pi$.

(c) $\overline{P}$ *is a possibility measure if and only if* $\overline{P} = \Pi$.

(d) $\overline{E}$ *is a possibility measure if and only if* $\overline{E} = \Pi$.

Proof. (a) Consider any $A \subseteq \mathcal{X}$. Observe that, for any $x \in A$,

$$\max_{x' \in A} \pi_1(x') \geq \pi_1(x) \geq \pi(x), \tag{53}$$

$$\max_{x' \in A} \pi_2(x') \geq \pi_2(x) \geq \pi(x). \tag{54}$$

Whence,

$$\overline{P}(A) = \min\left\{\max_{x' \in A} \pi_1(x'), \max_{x' \in A} \pi_2(x')\right\} \geq \pi(x) \tag{55}$$

for all $x \in A$. We immediately arrive at the desired inequality.

(b) Π is normed if and only if there is some $x \in \mathcal{X}$ such that $\pi(x) = \overline{P}(\{x\}) = 1$. In that case, the degenerate probability measure on x belongs to $\mathcal{M}_1 \cap \mathcal{M}_2$, and as a consequence $\overline{P}$ avoids sure loss. Moreover, Π is then a coherent upper probability that is dominated by $\overline{P}$, whence Π must also be dominated by the natural extension $\overline{E}$ of $\overline{P}$, because $\overline{E}$ is the point-wise largest coherent upper probability that is dominated by $\overline{P}$ [34, 3.1.2(e)].

(c) If $\overline{P}$ is a possibility measure, then $\overline{P}(\{x\}) = 1$ for some $x \in \mathcal{X}$. Consequently, by (b)

$$\overline{P}(A) \geq \overline{E}(A) \geq \Pi(A) \text{ for all } A \subseteq \mathcal{X}. \tag{56}$$

Because $\overline{P}(\{x\}) = \min\{\pi_1(x), \pi_2(x)\} = \Pi(\{x\})$ for all $x \in \mathcal{X}$, it follows that also

$$\overline{P}(\{x\}) = \overline{E}(\{x\}) = \Pi(\{x\}) \text{ for all } x \in \mathcal{X}. \tag{57}$$

Because both $\overline{P}$ and Π are possibility measures, they are uniquely determined by their restriction to singletons, and therefore $\overline{P} = \Pi$. The converse implication is trivial.

(d) Similarly, if $\overline{E}$ is a possibility measure, then $\overline{E}(\{x\}) = 1$ for some $x \in \mathcal{X}$. Because $\overline{P} \geq \overline{E}$, it can only be that also $\overline{P}(\{x\}) = 1$ for that same x. Consequently, by (b), Eq. (56) must hold here as well. Again, because $\overline{P}(\{x\}) = \min\{\pi_1(x), \pi_2(x)\} = \Pi(\{x\})$ for all $x \in \mathcal{X}$, it follows that Eq. (57) holds here too. Because both $\overline{E}$ and Π are possibility measures, they are uniquely determined by their restriction to singletons, and therefore $\overline{E} = \Pi$. (Note that $\overline{P}$ does not always coincide with Π in this case because $\overline{P}$ may not be a possibility measure; see Example 19.) Again, the converse implication is trivial. □

To see that Π need not be normed for $\overline{P}$ to avoid sure loss (or even to be coherent), it suffices to consider Example 5. However, for $\overline{P}$ to be a possibility measure, Π need to be normed, as we can deduce from Lemma 21(c).

Lemma 21(a) also indicates that taking the minimum between two possibility distributions π_1 and π_2, which is the most conservative conjunctive operator in possibility theory, will always provide an inner approximation of $\overline{P}$ when $\overline{P}$ is not a possibility measure. In a way, our heuristic method for adjusting possibility distributions to ensure that the conjunction is a possibility measure provides an even more conservative conjunctive operator, which in addition also ensures coherence unlike the plain minimum operator.

The next result shows that Example 19 hinges on π_1 and π_2 not being strictly positive.

Theorem 22. *Let π_1 and π_2 be two strictly positive possibility distributions. Then $\overline{E}$ is a possibility measure if and only if $\overline{P}$ is a possibility measure.*

Proof. 'if'. If $\overline{P}$ is a possibility measure, then $\overline{P}$ is coherent, and therefore coincides with its natural extension. So, $\overline{E}$ will be a possibility measure as well.

'only if'. If $\overline{E}$ is a possibility measure then, by Lemma 21(d), $\overline{E} = \Pi$, with π and Π defined as in Eqs. (51) and (52). In particular, there is some $x^* \in \mathcal{X}$ such that $\overline{E}(\{x^*\}) = \overline{P}(\{x^*\}) = \pi_1(x^*) = \pi_2(x^*) = 1$.

Assume ex-absurdo that $\overline{P}$ is not a possibility measure. By Theorem 16, there must be $\{x_1, x_2\} \subseteq \mathcal{X}$ such that

$$\min_{i \in \{1,2\}} \left(\max_{j \in \{1,2\}} \pi_i(x_j) \right) \neq \max_{j \in \{1,2\}} \left(\min_{i \in \{1,2\}} \pi_i(x_j) \right) \tag{58}$$

This inequality can only hold if the matrix

$$\begin{bmatrix} \pi_1(x_1) & \pi_1(x_2) \\ \pi_2(x_1) & \pi_2(x_2) \end{bmatrix} \tag{59}$$

has neither dominating rows nor dominating columns, or in other words, we must have either

$$\begin{array}{ccc} \pi_1(x_1) & < & \pi_1(x_2) \\ \wedge & & \vee \\ \pi_2(x_1) & > & \pi_2(x_2) \end{array} \quad \text{or} \quad \begin{array}{ccc} \pi_1(x_1) & > & \pi_1(x_2) \\ \vee & & \wedge \\ \pi_2(x_1) & < & \pi_2(x_2) \end{array} \tag{60}$$

Without loss of generality, we can assume that the first situation holds, as we can always swap x_1 and x_2. From these strict inequalities, it follows that

$$\max\{\pi_1(x_1), \pi_2(x_2)\} = \max_{j\in\{1,2\}} \left(\min_{i\in\{1,2\}} \pi_i(x_j) \right) = \max\{\overline{E}(\{x_1\}), \overline{E}(\{x_2\})\}, \tag{61}$$

where last equality follows from Lemma 21(d). So, if we can show that

$$\overline{E}(\{x_1, x_2\}) > \max\{\pi_1(x_1), \pi_2(x_2)\}, \tag{62}$$

then we have established a contradiction. By Eqs. (1) and (2), it suffices to show that there is a $Q \leq \overline{P}$ such that

$$Q(\{x_1, x_2\}) > \max\{\pi_1(x_1), \pi_2(x_2)\}. \tag{63}$$

Now, a probability measure Q which is zero everywhere except on $\{x_1, x_2, x^*\}$ satisfies $Q \leq \overline{P}$ if and only if all of the following inequalities are satisfied:

$$Q(\{x_1\}) \leq \pi_1(x_1) \tag{64}$$

$$Q(\{x_2\}) \leq \pi_2(x_2) \tag{65}$$

$$Q(\{x_1\}) + Q(\{x_2\}) \leq \min\{\pi_1(x_2), \pi_2(x_1)\} \tag{66}$$

Indeed, consider any $A \subseteq \mathcal{X}$.

(a) If $A \cap \{x_1, x_2, x^*\} = \emptyset$ then $Q(A) = 0$, and no constraints are required.

(b) If $x^* \in A \cap \{x_1, x_2, x^*\}$ then $\overline{P}(A) = 1$, and no constraints are required.

(c) If $A \cap \{x_1, x_2, x^*\} = \{x_1\}$ then $Q(A) = Q(\{x_1\})$. Clearly, $Q(\{x_1\}) \leq \overline{P}(A)$ for all such A if and only if

$$Q(\{x_1\}) \leq \overline{P}(\{x_1\}) = \min\{\pi_1(x_1), \pi_2(x_1)\} = \pi_1(x_1). \tag{67}$$

This is precisely Eq. (64).

(d) If $A \cap \{x_1, x_2, x^*\} = \{x_2\}$ then $Q(A) = Q(\{x_2\})$. Clearly, $Q(\{x_2\}) \leq \overline{P}(A)$ for all such A if and only if

$$Q(\{x_2\}) \leq \overline{P}(\{x_2\}) = \min\{\pi_1(x_2), \pi_2(x_2)\} = \pi_2(x_2). \tag{68}$$

This is precisely Eq. (65).

(e) If $A \cap \{x_1, x_2, x^*\} = \{x_1, x_2\}$ then we obtain $Q(A) = Q(\{x_1, x_2\})$. Clearly, $Q(\{x_1, x_2\}) \leq \overline{P}(A)$ for all such A if and only if

$$Q(\{x_1, x_2\}) \leq \overline{P}(\{x_1, x_2\}) = \min\{\Pi_1(\{x_1, x_2\}), \Pi_2(\{x_1, x_2\})\} \tag{69}$$
$$= \min\{\pi_1(x_2), \pi_2(x_1)\} \tag{70}$$

where the last equality follows from Eq. (60) (left case). This is precisely Eq. (66).

So, we are done if we can construct a probability measure Q on $\{x_1, x_2, x^*\}$ which simultaneously satisfies Eqs. (63), (64), (65), and (66).

Also note that we always have $x^* \neq x_1$ and $x^* \neq x_2$ (and obviously also $x_1 \neq x_2$), because Eq. (60) (left case) implies that $\pi_1(x_1) < 1$ and $\pi_2(x_2) < 1$, so $\{x_1, x_2, x^*\}$ always contains exactly three elements.

We consider two cases.

1. If $\pi_1(x_1) + \pi_2(x_2) \leq \min\{\pi_1(x_2), \pi_2(x_1)\}$, then the probability measure Q with

$$Q(\{x_1\}) \coloneqq \pi_1(x_1), \quad Q(\{x_2\}) \coloneqq \pi_2(x_2), \quad Q(\{x^*\}) \coloneqq 1 - (\pi_1(x_1) + \pi_2(x_2)) \tag{71}$$

clearly satisfies Eqs. (64), (65), and (66). We also have that

$$Q(\{x_1, x_2\}) = Q(\{x_1\}) + Q(\{x_2\}) = \pi_1(x_1) + \pi_2(x_2) > \max\{\pi_1(x_1), \pi_2(x_2)\} \tag{72}$$

because both $\pi_1(x_1)$ and $\pi_2(x_2)$ are strictly positive by assumption, so Eq. (63) is satisfied as well, finishing the proof for this case.

2. If $\pi_1(x_1) + \pi_2(x_2) > \min\{\pi_1(x_2), \pi_2(x_1)\}$, then the probability measure Q with

$$Q(\{x_1\}) \coloneqq \pi_1(x_1), \tag{73}$$
$$Q(\{x_2\}) \coloneqq \min\{\pi_1(x_2), \pi_2(x_1)\} - \pi_1(x_1), \tag{74}$$
$$Q(\{x^*\}) \coloneqq 1 - (\min\{\pi_1(x_2), \pi_2(x_1)\}) \tag{75}$$

clearly satisfies Eqs. (64), (65), and (66). We also have that

$$Q(\{x_1, x_2\}) = \min\{\pi_1(x_2), \pi_2(x_1)\} > \max\{\pi_1(x_1), \pi_2(x_2)\} \tag{76}$$

where the strict inequality follows from Eq. (60) (left case), so Eq. (63) is satisfied as well, finishing the proof for this case. $\square$

6. Exampe: a simple medical diagnosis problem

To conclude this paper, we illustrate our results on a medical diagnosis problem, inspired by Palacios *et al.* [**?**].

Consider $\mathcal{X} = \{d, h, n\}$ where d, h, and n stand for dyslexic, hyperactive and no problem, respectively. As is explained by Palacios *et al.* [**?**], it may be difficult for physicians to recognize between dyslexia and hyperactivity of children, yet it is important to provide reliable information.

Let us now assume that the available information is expressed by means of possibility distributions: these may be the result of a classification process [**?**] or of an elicitation procedure. We wish to provide a joint summary of these distributions which is still representable as a possibility distribution, for instance because we want to use it in methods tailored for possibility distributions, or because it is easier to present possibility distributions to physicians.

Example 23. *Two physicians provide the following possibility distributions:*

	d	h	n
π_1	*1*	*0.5*	*0.2*
π_2	*1*	*0.3*	*0.4*

The two physicians actually agree that dyslexia is quite possible, but they are not in agreement on the possibility of the other two options.

The conjunction $\overline{P} := \min\{\Pi_1, \Pi_2\}$ *avoids sure loss: for example, the probability measure* Q *with* $Q(\{d\}) = 1$ *is dominated by* $\overline{P}$*. It can be verified that* $\overline{P}$ *is coherent. Interestingly, the condition of Proposition 4 is not satisfied: no convex combination of the probability measures determined by the mass functions* $(0.5, 0.3, 0.2) \in \mathcal{M}_1$ *and* $(0.6, 0, 0.4) \in \mathcal{M}_2$ *belongs to* $\mathcal{M}_1 \cup \mathcal{M}_2$*.*

The natural extension $\overline{E}$ *of* $\overline{P}$*, which is the upper envelope of the credal set* $\mathcal{M}_1 \cap \mathcal{M}_2$*, coincides with* $\overline{P}$ *in this example, because* $\overline{P}$ *happens to be coherent:*

$$\overline{E}(\{d\}) = 1 \qquad \overline{E}(\{h\}) = 0.3 \qquad \overline{E}(\{n\}) = 0.2 \tag{77}$$

$$\overline{E}(\{h, n\}) = 0.4 \qquad \overline{E}(\{d, h\}) = \overline{E}(\{d, n\}) = \overline{E}(\{d, h, n\}) = 1. \tag{78}$$

However, $\overline{E}$ *is not a possibility measure because*

$$\overline{E}(\{h, n\}) = 0.4 > \max\{\overline{E}(\{h\}, \overline{E}(\{n\})\} = 0.3. \tag{79}$$

The graphical procedure summarized at the beginning of Section 4.3 suggests a possible correction of π_2 *for the conjunction to become a possibility measure:*

	d	h	n
π_2'	*1*	*0.4*	*0.4*

By Theorem 12, the conjunction of π_1 *and* π_2' *is then a possibility measure with possibility distribution*

	d	h	n
π	*1*	*0.4*	*0.2*

which is still quite informative.

7. Conclusions

In this paper, we have characterized in different ways the conjunction of two possibility measures. In particular, we have addressed the following questions:

1. When does the conjunction avoid sure loss?
2. When is the conjunction coherent?
3. When is the conjunction again a possibility measure?
4. When is the natural extension of the conjunction again a possibility measure?

For each of these, we have provided both sufficient and necessary conditions. We demonstrated through many examples that these conditions remain quite restrictive; this seems to be the price to pay for working with possibility distributions.

From a practical point, one result that we find particularly interesting is the game-theoretic characterization of the conditions under which the conjunction is again a possibility measure. Indeed, this characterization offers a very simple and convenient graphical verification method. It can also be used in practice to heuristically adjust possibility distributions to ensure that their conjunction remains a possibility distribution.

It is not too difficult to extend some of our results to the conjunction of more than two possibility measures, by noting that the conjunction can be taken in a pairwise sequential manner. Note nevertheless that these pairwise conjunctions being possibility measures is sufficient, but not necessary, for the conjunction of all the possibility measures to be a possibility measure. For some other results, such as Theorem 16, some adjustments should be made.

As for future lines of research, we would like to point out a few. It would be interesting to study under what conditions possibility measures are closed under other combination rules, such as those discussed in [16, 26, 29]. We could also study if the results can be extended to infinite possibility spaces; although clearly the game-theoretic interpretation may prove problematic in this respect. Finally, many other imprecise probability models, such as belief functions, probability boxes, and so on, might benefit from similar studies.

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