Unbalanced power flow in distribution systems with embedded transformers using the complex theory in $\alpha\beta 0$ stationary reference frame

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***Abstract*—This paper presents three new contributions to power flow analysis of unbalanced three-phase distribution systems. First, a complex vector based model in $\alpha\beta 0$ stationary reference frame is developed to state the power flow equations using a compact matrix formulation. The proposed model is based on Kirchhoff' s current law (KCL) and Kirchhoff' s voltage law (KVL). Then, a general and exact power transformer model in the $\alpha\beta 0$ reference frame is proposed. Finally, this transformer model is incorporated into the power flow problem. It will be shown that the use of an orthogonal reference frame simplifies the modeling of the distribution network components. In this work, both the network and the power transformer, as well as *PQ* type loads, *PQ* and *PV* type generators and a slack bus are modeled. By using the node incidence matrix instead of the admittance matrix, the information about the grid topology and the grid parameters (including power transformers) is separately organized. As it will be demonstrated, the proposed formulation is ready to incorporate other complex models of loads, generators or storage devices. The model is tested by using the IEEE 4 and the IEEE 123 Node Test Feeders with different transformer connections and balanced and unbalanced lines and loads.**

***Index Terms*—Power flow, three-phase unbalanced power flow, distribution system, unbalanced loads, transformer modeling.**

NOMENCLATURE

Acronyms

AC	Alternating current.
BFS	Backward/Forward sweep.
DC	Direct current.
KCL	Kirchhoff's current law.
KVL	Kirchhoff's voltage law.
PCC	Point of common coupling.
PhSh	Phase shift.
pu	Per unit.
PWM	Pulse width modulation.
VSC	Voltage source converter.

Functions

δ	Delta function (phase shift and connection dependant).

Matrices

$\mathbf{A}$	abc to $\alpha\beta 0$ transformation matrix.
$\mathbf{G}$	Rotation matrix.
$\mathbf{\Gamma}$	Node incidence matrix.
$\mathbf{I}_d$	Identity matrix.
$\mathbf{L}$	Inductance matrix.
$\mathbf{M}$	Power system matrix.
$\mathbf{N}$	Transformer rotation matrix.
$\mathbf{R}$	Resistance matrix.
$\mathbf{T}_{DY}$	Wye to delta transformation matrix.
$\mathbf{T}_R$	Primary to secondary transformation matrix.
$\mathbf{Z}$	Impedance matrix.

Reference frames

$\alpha\beta 0$	Stationary orthogonal reference.
abc	Stationary three phase reference.
$dq0$	Synchronous rotating orthogonal reference.

Subscripts

A, B, C	Transformer primary phases.
a, b, c	Transformer secondary phases.
Br	Branch (line) currents.
α , β , 0	Stationary reference frame components.
r , i	Real and imaginary parts.
G	Generator.
I to IV	Type of transformer rotation matrix.
L	Load.
N	Transformer neutral.
$ph-ph$	Phase to phase.
$trans$	Transformer.

Superscripts

$'$	Ideal transformer primary phases.
$*$	Extended to the whole system.
P, S	Primary, secondary.
PS	Secondary for $\Delta \mathrm{Y}_g$ connection and primary for other connections.
T	Transposed.

Variables

ΔV	Voltage drop.
γ	Transformer rotation angle.
I	Current.
ω	Pulsation.
V	Voltage.
φ	Transformer phase shift.
Z	Transformer impedance per phase.

Vectors

$\mathbf{\Delta V}$	Voltage drop vector (real and/or imaginary parts).
$\mathbf{\Delta v}$	Instantaneous three phase voltage drop vector.
$\mathbf{I}$	Current vector (real and/or imaginary parts).
$\mathbf{i}$	Instantaneous three phase current vector.
$\mathbf{P}$	Active power vector (three components).
$\mathbf{Q}$	Reactive power vector (three components).
$\mathbf{V}$	Voltage vector (real and/or imaginary parts).
$\mathbf{z}$	Whole power system vector (voltages and currents).

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I. INTRODUCTION

UP to the date, many works have been carried out to accurately model unbalanced distribution power systems. Developed models include different components, such as lines and cables, capacitors, loads, voltage regulators and power transformers, see for instance [1], [2].

Due to the increase of distributed generation (DG) in power systems, there have been significant advances in the unbalanced load flow analysis techniques in distribution networks. Most of these advances deal with the inclusion of distributed generators and their controls in power flow calculations. Some related works were presented in [3]–[5], in which single-phase and three-phase voltage source converters (VSC) were modeled. Moreover, recent studies related to DG controls are focused on the use of DG devices working under unbalanced conditions [6]–[8], since it is a common situation in distributed systems. In most of these cases, the control of such devices is implemented in an orthogonal-stationary reference frame. To state the power flow problem, some of these works employed more than one reference frame to solve the different parts of the system [3], [4].

In the literature the authors have found that there is no previous work combining the traditional power flow problem with the new distributed technologies in a unique reference frame. To include these emerging technologies into distribution system power flow analyses (where the presence of transformers is really important), an unbalanced power flow model in $\alpha\beta0$ stationary reference frame, capable of handling distributed generators and loads, and power transformers in an easy way would be very useful. This is the aim of the present work.

The main characteristics of a distribution system that must be taken into account for power flow calculations can be summarized as follows [3], [9], [10]:

- Radially or weakly meshed topology.
- Extremely large number of nodes and branches with a wide X/R ratio.
- Unbalanced operation basically derived from untransposed lines and unbalanced loading formed by the users.
- Presence of DGs and distributed storage systems.

The first two features may cause the traditional power flow algorithms based on Newton-Raphson or Gauss-Seidel approaches to fail due to ill-conditioning problems [10]. However, some authors still defend power flow algorithms with a solver derived from Newton-Raphson. For instance in [11] a network-based distributed slack bus model was proposed and good results were obtained in a 394-bus radial system.

Most authors agree that the special characteristics of distribution systems require different algorithms based on the application of KCL and KVL. This is the case of the Backward/Forward sweep (BFS) sequential methods and some direct methods, as for instance [12], [13]. In [14] the BFS method was applied to develop a general four-wire distribution network approach. The Kron's reduction to merge the neutral and the ground into the phases was not employed, thus neutral and ground currents and voltages had to be explicitly represented. In [15] the BFS method was applied using the Kron's reduction. In this case, the mutual coupling was simulated through equivalent branch voltage sources or current injections. More recently, the authors in [9] propose a modified BFS algorithm to solve weakly meshed distribution systems. They break the meshes and apply a compensation technique based on current injection.

Some of the most sophisticated analyses were presented in [3], [4]. In these works, a sequential power flow analysis, which combines a BFS approach for single phase laterals with sequence-components for three phase networks, is proposed. The authors apply steady-state sequence components to different kind of DGs based on VSC. They also consider different control modes and operational constraints, including phase current limit, modulation index or voltage limit at the point of common coupling (PCC). They solve the power flow problem in the sequence component frame, but they need the $dq0$ synchronous frame to calculate the internal parameters and operational limits of DGs. In that case it would have been very helpful a unique reference frame to model the whole system.

The application of the $dq0$ reference frame to the power flow formulation is not new [16], and it has been recently used in microgrids steady-state modeling [12], and in unified AC/DC power flow analysis [13]. However, in those cases the $dq0$ reference frame is applied to balanced systems, so the $dq0$ components are constant in steady state analysis. This is not a valid assumption when working with unbalanced systems.

In [5], the $dq0$ reference frame was employed to solve the power flow problem in unbalanced three-phase power systems containing PWM converters. However in that case, the zero and negative sequences caused pulsating terms to appear, so the obtained expressions were quite intricate.

Regarding the unbalanced three phase power flow problem, the use of the $\alpha\beta0$ reference frame and a complex vector model are proposed in this paper. The use of this reference frame includes the benefits provided by an orthogonal reference frame, avoiding the pulsating terms derived from the existence of sequence components.

Related to the power transformer modeling for power-flow analysis, most authors have developed models based on the primitive admittance matrix and the nodal admittance matrix [1], [17], [18]. A number of authors proposed methods to incorporate such models into the BFS problem [19]–[22]. In [23] the primitive admittance based model was formulated in sequence components. In [24] a positive sequence component model of a power transformer with different connections was described and tested with a Newton based power flow solver.

In [25] an exact method satisfying KVL, KCL and the ideal relationship between voltages and currents in both transformer windings was proposed in abc coordinates, but the authors only described the ungrounded wye-delta connection.

As this last one, some of the cited works present a model for a specific connection, while others describe general models valid for several connections. The former lack of standardization, and the latter use quite complex matrices.

As it will be demonstrated in Section III, in this paper the power transformer is modeled as an exact approach in the same $\alpha\beta0$ stationary reference frame employed for the power flow problem. Moreover, the model is stated in a way that, unlike in [25], the formulation does not need to be rewritten for each connection type and phase shift.

Summarizing, the present work states a power flow formulation conjugating the use of the $\alpha\beta 0$ coordinates and the node incidence matrix instead of the admittance matrix. On one hand, the use of the $\alpha\beta 0$ reference frame allows the authors to obtain a general but simple power transformer model, applicable to any connection or phase shift. It must also be remarked that the model is ready to incorporate any other device controlled into the same reference frame, as for instance some DG's designed to be used in unbalanced conditions [6]–[8]. On the other hand, the use of the node incidence matrix, instead of the admittance matrix, will prevent some serious drawbacks [26]:

- The admittance matrix merges together all parallel lines and shunt devices. It is not possible unequivocally go back to the line and transformer parameters. With the proposed method, the information regarding the system parameters and topology is separately organized.
- Any change in the system topology or parameters requires rebuilding the whole admittance matrix. In this proposal, the transformer connection or phase shift, as well as any other system parameter, can be independently modified without restoring the node incidence matrix.

To the authors' knowledge there are no previous power flow models that simultaneously permit the inclusion of three phase unbalanced lines, loads, transformers and distributed generators in a unique reference frame, that is a rectangular reference frame, without need for reference transformations. The authors would like to emphasize that what it is proposed in this paper is a model, or formulation, to be used in power flow analysis, so different methods (algorithms) could be applied to solve the problem. The proposed formulation is based on the application of KCL and KVL like the BFS methods. In this particular case, the authors chose a direct approach [12], [13] to simultaneously solve the whole system of equations by means of the trust-region dogleg algorithm [27]. However, other algorithms could be used to solve the proposed formulation.

The paper is structured as follows. Sections II and III will respectively describe the complex vector model in $\alpha\beta 0$ frame of an *RL* element and a three-phase power transformer. In Section IV, these models will be incorporated into the whole grid by using a compact matrix formulation. In Section V the models are validated by means of the IEEE 4 Node Test Feeder benchmark [28]. Several cases based on the IEEE 123 Node Test Feeder [28] are also solved to evaluate the model performance in large distribution systems. Ultimately, in Section VI the conclusions are presented.

II. COMPLEX VECTOR MODEL OF A THREE-PHASE UNBALANCED *RL* ELEMENT IN $\alpha\beta 0$

The voltage drop $\Delta \mathbf{v}_{abc}$ in the abc reference frame for a series *RL* element, depending on the branch current $\mathbf{i}_{abc}$, when voltages and currents are time dependant, is:

$$\Delta \mathbf{v}_{abc} = \mathbf{R}_{abc}\ \mathbf{i}_{abc} + \mathbf{L}_{abc}\ \frac{d\,\mathbf{i}_{abc}}{dt} \tag{1}$$

Where $\mathbf{R}_{abc}$ and $\mathbf{L}_{abc}$ are the resistance and the inductance matrices considering coupling effects. These matrices are obtained through Carlson's equations [29] and Kron's reduction [30].

The $\alpha\beta 0$ to abc transformation for a generic three complex component vector **x** is defined as:

$$\mathbf{x}_{abc} = \mathbf{A}\,\mathbf{x}_{\alpha\beta 0} \tag{2}$$

Where **A** is the invertible matrix:

$$\mathbf{A} = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The choice of this matrix was made to be easily inverted. As it can be checked $\mathbf{A}^{-1} = \mathbf{A}^T$. By replacing (2) into (1) the voltage drop in $\alpha\beta 0$ is obtained:

$$\Delta \mathbf{v}_{\alpha\beta 0} = \mathbf{R}_{\alpha\beta 0}\ \mathbf{i}_{\alpha\beta 0} + \mathbf{L}_{\alpha\beta 0}\ \frac{d\,\mathbf{i}_{\alpha\beta 0}}{dt} \tag{3}$$

Where $\mathbf{R}_{\alpha\beta 0}$ and $\mathbf{L}_{\alpha\beta 0}$ are the transformed resistance and inductance matrices:

$$\mathbf{R}_{\alpha\beta 0} = \mathbf{A}^{-1}\,\mathbf{R}_{abc}\ \mathbf{A} \tag{4}$$
$$\mathbf{L}_{\alpha\beta 0} = \mathbf{A}^{-1}\,\mathbf{L}_{abc}\ \mathbf{A} \tag{5}$$

Equation (3) represents the general dynamic model for an RL element in $\alpha\beta 0$ reference frame. In the steady state analysis, the differential equation (3) can be replaced by an algebraic equation by means of phasor theory. Thus, each sine wave $x(t)$, with time invariant amplitude and angular frequency ω, and its time derivative $\frac{dx(t)}{dt}$ are represented by two phasors $\overline{\mathbf{X}}$ and $\dot{\overline{\mathbf{X}}}$ respectively as shown in equations (6) and (7):

$$\overline{\mathbf{X}} = X_r + j\,X_i \tag{6}$$
$$\dot{\overline{\mathbf{X}}} = -\omega X_i + j\,\omega X_r \tag{7}$$

Where subscripts r and i stand for real and imaginary parts respectively. With the above mentioned assumptions, equation (3) becomes:

$$\left[\Delta \mathbf{V}_{\alpha\beta 0}\right]^T = \mathbf{Z}_{\alpha\beta 0}\ \left[\mathbf{I}_{\alpha\beta 0}\right]^T \tag{8}$$

Where:

$$\left[\Delta \mathbf{V}_{\alpha\beta 0}\right] = \left[\Delta V_{\alpha_r}\ \Delta V_{\beta_r}\ \Delta V_{0_r}\ \Delta V_{\alpha_i}\ \Delta V_{\beta_i}\ \Delta V_{0_i}\right] \tag{9}$$
$$\left[\mathbf{I}_{\alpha\beta 0}\right] = \left[I_{\alpha_r}\ I_{\beta_r}\ I_{0_r}\ I_{\alpha_i}\ I_{\beta_i}\ I_{0_i}\right] \tag{10}$$
$$\mathbf{Z}_{\alpha\beta 0} = \left(\begin{array}{c|c} \mathbf{R}_{\alpha\beta 0} & -\omega\,\mathbf{L}_{\alpha\beta 0} \\ \hline \omega\,\mathbf{L}_{\alpha\beta 0} & \mathbf{R}_{\alpha\beta 0} \end{array}\right) \tag{11}$$

To implement shunt capacitors, that means *RC* type elements, the same procedure could be employed to obtain the dual equation of (8). An example of *RC* type elements in a rectangular reference frame can be seen in [31].

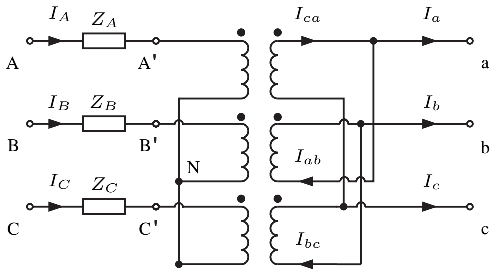

Fig. 1: Wye-delta step-up.

III. Complex vector model of a three-phase transformer in $\alpha\beta0$ reference frame

To describe the $\alpha\beta0$ transformer model, the step-up YΔ ungrounded three-phase transformer of the IEEE 4 Node Test Feeder benchmark [28] will be employed. The primary and secondary windings are connected as it is shown in Fig. 1. The chosen currents are also depicted. The transformer impedance is concentrated in the primary side of the model, but other considerations may be made without many modifications. As it can be seen in the figure, the real transformer (primary nodes A, B and C) is partitioned in a line impedance and an ideal transformer (primary nodes A', B' and C').

When working in the per unit (*pu*) system of representation, and choosing the appropriate voltage base for primary and secondary voltages, the turns ratio becomes 1:1, so the relationship between primary and secondary voltages in Fig. 1 can be expressed as:

$$\begin{bmatrix} V_{A'N} \\ V_{B'N} \\ V_{C'N} \end{bmatrix} = \mathbf{T}_R \begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix} \tag{12}$$

Where the matrix $\mathbf{T}_R$ in *pu* depends only on the transformer connection and phase shift. In the present case $\mathbf{T}_R$ is deduced from Fig. 1 and the dot convention:

$$\mathbf{T}_R = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$\mathbf{T}_R$ is a nonsingular matrix that satisfies $\mathbf{T}_R^T = \mathbf{T}_R^{-1}$. In (12), phase-to-neutral voltages are employed in the primary side of the ideal transformer and phase-to-phase voltages are employed in the secondary side. Uppercase characters denote primary voltages or currents, and lowercase characters are used for the secondary ones.

Due to the fact that there is no neutral conductor available in the primary side of the transformer, a fictitious neutral point should be chosen as a reference point to determine phase-to-neutral voltages in the real transformer (primary nodes A, B and C). To avoid this problem the authors have worked with phase-to-phase voltages both in primary and secondary sides of the real transformer.

The phase-to-phase voltages in the primary side of the ideal transformer are calculated from phase-to-neutral voltages and the matrix $\mathbf{T}_{DY}$:

$$\begin{bmatrix} V_{A'B'} \\ V_{B'C'} \\ V_{C'A'} \end{bmatrix} = \mathbf{T}_{DY} \begin{bmatrix} V_{A'N} \\ V_{B'N} \\ V_{C'N} \end{bmatrix} \tag{13}$$

$$\mathbf{T}_{DY} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

$\mathbf{T}_{DY}$ is a singular matrix. This implies that phase-to-neutral voltages can not be obtained from phase-to-phase voltages. However, when working with phase-to-phase voltages, this problem is overcome. In others cases the applied criteria is as follows: phase-to-phase voltages are used in delta or ungrounded wye connections and phase-to-ground voltages are chosen for grounded wye connections.

Replacing (12) into (13), the primary voltages in the ideal transformer can be rewritten as a function of the secondary voltages:

$$\begin{bmatrix} V_{A'B'} \\ V_{B'C'} \\ V_{C'A'} \end{bmatrix} = \mathbf{T}_{DY}\ \mathbf{T}_R \begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix} \tag{14}$$

The voltage drops in the power transformer impedances are obtained as:

$$\begin{bmatrix} V_{AA'} \\ V_{BB'} \\ V_{CC'} \end{bmatrix} = \mathbf{Z}_{trans} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \tag{15}$$

Where

$$\mathbf{Z}_{trans} = \begin{pmatrix} Z_\mathrm{A} & 0 & 0 \\ 0 & Z_\mathrm{B} & 0 \\ 0 & 0 & Z_\mathrm{C} \end{pmatrix} = Z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transformer windings are equal and balanced, thus the three impedances might be represented by Z. In the next equation, the voltage drops in the power transformer impedances are related to the phase-to-phase voltages in the primary side of the real and the ideal transformers.

$$\begin{bmatrix} V_{AB} - V_{A'B'} \\ V_{BC} - V_{B'C'} \\ V_{CA} - V_{C'A'} \end{bmatrix} = \begin{bmatrix} V_{AA'} - V_{BB'} \\ V_{BB'} - V_{CC'} \\ V_{CC'} - V_{AA'} \end{bmatrix} = \mathbf{T}_{DY} \begin{bmatrix} V_{AA'} \\ V_{BB'} \\ V_{CC'} \end{bmatrix} \tag{16}$$

Combining (14), (15) and (16) the relationship between primary voltages, secondary voltages and primary line currents can be written as:

$$\begin{bmatrix} V_{AB} \\ V_{BC} \\ V_{CA} \end{bmatrix} = \mathbf{T}_{DY}\ \mathbf{T}_R \begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix} + \mathbf{T}_{DY}\ \mathbf{Z}_{trans} \begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} \tag{17}$$

Equation (17) relates primary voltages and currents to secondary voltages in abc coordinates. It can be rewritten in a more compact way:

$$\begin{aligned} \left[V_{ph-ph}\right]_{abc}^{P} &= \mathbf{T}_{DY}\ \mathbf{T}_R \left[V_{ph-ph}\right]_{abc}^{S} + \\ &+ \mathbf{T}_{DY}\ \mathbf{Z}_{trans} \left[I_{Br}\right]_{abc}^{P} \end{aligned} \tag{18}$$

TABLE I: Matrices for all connections and phase shifts.

Connection	PhSh	$\mathbf{N}_{\text{I}}$	$\mathbf{N}_{\text{II}}$	$\mathbf{N}_{\text{III}}$	$\mathbf{N}_{\text{IV}}$	γ
$\text{Y}_\text{g}\text{Y}_\text{g}$		$\mathbf{I}_{d\,(3\times3)}$	$\begin{pmatrix} \mathbf{G}(-\varphi) & \\ & \delta(\gamma) \end{pmatrix}$	$\begin{pmatrix} -\mathbf{G}(\varphi) & \\ & -\delta(\gamma) \end{pmatrix}$	$\mathbf{I}_{d\,(3\times3)}$	φ
$\Delta\Delta$		$\sqrt{3}\begin{pmatrix} \mathbf{G}(30^\circ) & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{G}(-\varphi) & \\ & \delta(\gamma) \end{pmatrix}$	$\begin{pmatrix} -\mathbf{G}(\varphi) & \\ & -\delta(\gamma) \end{pmatrix}$	$\mathbf{I}_{d\,(3\times3)}$	φ
$\text{Y}_\text{g}\Delta$	φ	$\mathbf{I}_{d\,(3\times3)}$	$\begin{pmatrix} \mathbf{G}(-\varphi-30^\circ) & \\ & \delta(\gamma) \end{pmatrix}$	$\sqrt{3}\begin{pmatrix} -\mathbf{G}(\varphi) & \\ & 0 \end{pmatrix}$	$\mathbf{I}_{d\,(3\times3)}$	$\varphi+30^\circ$
$\text{Y}\Delta$		$\sqrt{3}\begin{pmatrix} \mathbf{G}(30^\circ) & \\ & 0 \end{pmatrix}$	$\sqrt{3}\begin{pmatrix} \mathbf{G}(-\varphi) & \\ & 0 \end{pmatrix}$	$\sqrt{3}\begin{pmatrix} -\mathbf{G}(\varphi) & \\ & 0 \end{pmatrix}$	$\mathbf{I}_{d\,(3\times3)}$	$\varphi+30^\circ$
ΔY_g		$\begin{pmatrix} \mathbf{G}(30^\circ-\varphi) & \\ & \delta(\gamma) \end{pmatrix}$	$\begin{pmatrix} \mathbf{G}(30^\circ-\varphi) & \\ & \delta(\gamma) \end{pmatrix}$	$-\mathbf{I}_{d\,(3\times3)}$	$\sqrt{3}\begin{pmatrix} \mathbf{G}(-\varphi) & \\ & 0 \end{pmatrix}$	$\varphi-30^\circ$

Where $\mathbf{G}(\theta)=\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$ $\mathbf{I}_{d\,(3\times3)}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ and $\delta(\gamma)=\begin{cases}1 & \text{if } \left(\frac{\gamma}{120}+1\right) \text{ is an integer}\\ -1 & \text{any other case}\end{cases}$

Where:

$\left[V_{ph-ph}\right]^P_{abc}$ are phase to phase primary voltages in abc.

$\left[V_{ph-ph}\right]^S_{abc}$ are phase to phase secondary voltages in abc.

$\left[I_{Br}\right]^P_{abc}$ are line primary currents in abc.

The $\alpha\beta0$ model is then obtained by using equation (2) into (18):

$$\begin{aligned}\left[\mathbf{V}_{ph-ph}\right]^P_{\alpha\beta0} &= \mathbf{A}^{-1}\mathbf{T}_{DY}\mathbf{T}_R\mathbf{A}\left[\mathbf{V}_{ph-ph}\right]^S_{\alpha\beta0} + \\ &+ \mathbf{A}^{-1}\mathbf{T}_{DY}\mathbf{Z}_{trans}\mathbf{A}\left[\mathbf{I}_{Br}\right]^P_{\alpha\beta0} \qquad (19)\end{aligned}$$

To establish the relationship between primary and secondary line currents in the same pu system than the one used for voltages, it can be considered (see Fig. 1):

$$\begin{bmatrix}I_A\\I_B\\I_C\end{bmatrix} = -\mathbf{T}_R\begin{bmatrix}I_{ab}\\I_{bc}\\I_{ca}\end{bmatrix} \qquad (20)$$

$$\begin{bmatrix}I_a\\I_b\\I_c\end{bmatrix} = -\mathbf{T}^T_{DY}\begin{bmatrix}I_{ab}\\I_{bc}\\I_{ca}\end{bmatrix} \qquad (21)$$

From the above equations it can be stated that:

$$\begin{bmatrix}I_a\\I_b\\I_c\end{bmatrix} = (\mathbf{T}_R\mathbf{T}_{DY})^T\begin{bmatrix}I_A\\I_B\\I_C\end{bmatrix} \qquad (22)$$

The currents model of the power transformer in $\alpha\beta0$ reference frame is summarized in:

$$[\mathbf{I}_{Br}]^S_{\alpha\beta0} = \mathbf{A}^{-1}(\mathbf{T}_R\mathbf{T}_{DY})^T\mathbf{A}\,[\mathbf{I}_{Br}]^P_{\alpha\beta0} \qquad (23)$$

This formulation can be extended to any other transformer connection. Thus, a generalized equation similar to (19) can be expressed as follows:

$$\left[\mathbf{V}_{ph-ph}\right]^P_{\alpha\beta0} = \mathbf{N}_{\text{II}}\left[\mathbf{V}_{ph-ph}\right]^S_{\alpha\beta0} + Z\,\mathbf{N}_{\text{I}}\left[\mathbf{I}_{Br}\right]^{PS}_{\alpha\beta0} \qquad (24)$$

Where $\mathbf{N}_{\text{I}}$ and $\mathbf{N}_{\text{II}}$ are the rotation matrices shown in Table I. The line currents $\left[\mathbf{I}_{Br}\right]^{PS}_{\alpha\beta0}$ are primary currents in all connections except in ΔY_g connection, in which it is easier to consider secondary currents with no need for modifications in (24).

Due to the fact that $\left[\mathbf{I}_{Br}\right]^{PS}_{\alpha\beta0}$ are only defined as secondary currents in ΔY_g connection, equation (23) is generalized by means of two expressions:

$$-\left[\mathbf{I}_{Br}\right]^P_{\alpha\beta0} + \mathbf{N}_{\text{IV}}\left[\mathbf{I}_{Br}\right]^{PS}_{\alpha\beta0} = 0 \qquad (25)$$

$$\left[\mathbf{I}_{Br}\right]^S_{\alpha\beta0} + \mathbf{N}_{\text{III}}\left[\mathbf{I}_{Br}\right]^{PS}_{\alpha\beta0} = 0 \qquad (26)$$

Where $\mathbf{N}_{\text{III}}$ and $\mathbf{N}_{\text{IV}}$ are the rotation matrices shown in Table I.

An inspection of Table I reveals that to obtain the rotation matrices $\mathbf{N}_{\text{I}}$, $\mathbf{N}_{\text{II}}$, $\mathbf{N}_{\text{III}}$ and $\mathbf{N}_{\text{IV}}$, it is not necessary to use $\mathbf{T}_R$ or $\mathbf{T}_{DY}$ matrices. Only the phase shift and the connection type are required to define the exact model of the power transformer. Equations (25) and (26) are the same for all connections and phase shifts. Only matrices $\mathbf{N}_{\text{I}}$, $\mathbf{N}_{\text{II}}$, $\mathbf{N}_{\text{III}}$ and $\mathbf{N}_{\text{IV}}$ need to be defined. Table I gives the generalization of the model because, unlike the transformer model in [25], in the present work all the possibilities are presented.

The $\alpha\beta0$ transformer model can be easily included in the network model as it will be explained in next section.

IV. Network modeling in $\alpha\beta 0$ reference frame

To obtain a compact matrix expression, the formulation developed in [13] is going to be adapted to the $\alpha\beta 0$ reference frame. The formulation is based on the node incidence matrix $\mathbf{\Gamma}$, which is associated to the graph describing the system topology. To obtain the matrix $\mathbf{\Gamma}$, the electrical network is represented by an oriented-type graph. Firstly, the network nodes are numbered. The graph edges represent the connections between numbered network nodes. Under this assumption, and considering that the oriented edges always go from the lower node number to the higher, the matrix $\mathbf{\Gamma}$ can be formed as:

$$\mathbf{\Gamma}_{ij} = \begin{cases} 1 \text{ when the tail of the edge } i \text{ is node } j. \\ -1 \text{ when the head of the edge } i \text{ is node } j. \\ 0 \text{ otherwise.} \end{cases}$$

This section is divided into subsections A and B. In the first one, it will be described how the formulation is adapted to an electrical network without power transformers. In the second subsection, the accommodation of the power transformer model in the proposed formulation will be explained.

A. Network description without power transformers

With the use of $\mathbf{\Gamma}$, each component in (8) can be expressed for a whole network as follows:

$$\begin{aligned} \mathbf{\Gamma}\left[\mathbf{V}_{\alpha r}\right]^T &= \mathbf{R}_{\alpha\alpha}\left[\mathbf{I}_{\alpha r}\right]^T + \mathbf{R}_{\alpha\beta}\left[\mathbf{I}_{\beta r}\right]^T + \\ &+ \mathbf{R}_{\alpha 0}\left[\mathbf{I}_{0r}\right]^T - \omega\mathbf{L}_{\alpha\alpha}\left[\mathbf{I}_{\alpha i}\right]^T - \\ &- \omega\mathbf{L}_{\alpha\beta}\left[\mathbf{I}_{\beta i}\right]^T - \omega\mathbf{L}_{\alpha 0}\left[\mathbf{I}_{0i}\right]^T \end{aligned} \tag{28}$$

$$\begin{aligned} \mathbf{\Gamma}\left[\mathbf{V}_{\alpha i}\right]^T &= \omega\mathbf{L}_{\alpha\alpha}\left[\mathbf{I}_{\alpha r}\right]^T + \omega\mathbf{L}_{\alpha\beta}\left[\mathbf{I}_{\beta r}\right]^T + \\ &+ \omega\mathbf{L}_{\alpha 0}\left[\mathbf{I}_{0r}\right]^T + \mathbf{R}_{\alpha\alpha}\left[\mathbf{I}_{\alpha i}\right]^T + \\ &+ \mathbf{R}_{\alpha\beta}\left[\mathbf{I}_{\beta i}\right]^T + \mathbf{R}_{\alpha 0}\left[\mathbf{I}_{0i}\right]^T \end{aligned} \tag{29}$$

Where $\mathbf{V}_{\alpha r}$, $\mathbf{I}_{\alpha r}$ are vectors containing the α component real parts of all node voltages and line currents respectively, and $\mathbf{V}_{\alpha i}$, $\mathbf{I}_{\alpha i}$ are vectors including the α component imaginary parts of the same voltages and currents. Analogous definitions are also applicable to $\mathbf{V}_{\beta r}$, $\mathbf{V}_{0r}$, $\mathbf{V}_{\beta i}$, $\mathbf{V}_{0i}$, $\mathbf{I}_{\beta r}$, $\mathbf{I}_{0r}$, $\mathbf{I}_{\beta i}$ and $\mathbf{I}_{0i}$. $\mathbf{R}_{**}$ and $\mathbf{L}_{**}$ are diagonal matrices extended to all network lines. They are formed with the $**$ elements of the transformed resistance and inductance matrices $\mathbf{R}_{\alpha\beta 0}$ and $\mathbf{L}_{\alpha\beta 0}$ described in (4) and (5) for each system line.

Similar expressions might be obtained for β and zero voltage components. However, by defining a new extended node incidence matrix $\mathbf{\Gamma}^*$ in which each element is replaced by itself multiplied by the identity matrix $\mathbf{I}_{d\,(6\times 6)}$, the KVL for all system lines can be readily formulated:

$$\mathbf{\Gamma}^*\left[\mathbf{V}^*\right]^T_{\alpha\beta 0} = \mathbf{Z}^*_{\alpha\beta 0}\left[\mathbf{I}^*_{Br}\right]^T_{\alpha\beta 0} \tag{30}$$

Where $[\mathbf{V}^*]_{\alpha\beta 0}$ and $[\mathbf{I}^*_{Br}]_{\alpha\beta 0}$ are the extended node voltages and branch currents vectors for the whole power system, respectively. $[\mathbf{V}^*]_{\alpha\beta 0}$ is constructed by adding 6 components with the same structure as the one defined in expression (9) for each node, and $[\mathbf{I}^*_{Br}]_{\alpha\beta 0}$ is formed by adding 6 components with the same structure as the one defined in expression (10) for each branch. $\mathbf{Z}^*_{\alpha\beta 0}$ is the extended impedance matrix for the whole system, so it is a block diagonal matrix in which each line adds a block as the one in (11).

In a similar manner, KCL for all system nodes is expressed as follows:

$$\mathbf{\Gamma}^{*T}[\mathbf{I}^*_{Br}]^T_{\alpha\beta 0} = -\mathbf{I}^*_d\left[\mathbf{I}^*_L\right]^T_{\alpha\beta 0} + \mathbf{I}^*_d\left[\mathbf{I}^*_G\right]^T_{\alpha\beta 0} \tag{31}$$

Where $\mathbf{I}^*_d$ is a block diagonal matrix. Each node will add a new block, which is the identity matrix $\mathbf{I}_{d\,(6\times 6)}$. $[\mathbf{I}^*_L]_{\alpha\beta 0}$ and $[\mathbf{I}^*_G]_{\alpha\beta 0}$ are respectively the currents demanded by the loads and the currents injected by the generators at each network node. The structure of these vectors will be equal to the structure of $[\mathbf{I}^*_{Br}]_{\alpha\beta 0}$, since each node will add the six components described in (10).

Expressions (30) and (31) can be rewritten in a really compact matrix form, including all linear KVL and KCL equations:

$$\mathbf{M}\,\mathbf{z}^T = 0 \tag{32}$$

Where $\mathbf{z}$ is the vector representing voltage and current magnitudes and it is constructed as follows:

$$\mathbf{z} = \left[\left[\mathbf{I}^*_{Br}\right]_{\alpha\beta 0} \quad \left[\mathbf{I}^*_L\right]_{\alpha\beta 0} \quad \left[\mathbf{I}^*_G\right]_{\alpha\beta 0} \quad \left[\mathbf{V}^*\right]_{\alpha\beta 0}\right] \tag{33}$$

The structure of $\mathbf{M}$ is represented in expression (34).

$$\mathbf{M} = \left(\begin{array}{c|c|c|c} \mathbf{Z}^*_{\alpha\beta 0} & & & -\mathbf{\Gamma}^* \\ \hline \mathbf{\Gamma}^{*T} & \mathbf{I}^*_d & -\mathbf{I}^*_d & \end{array}\right) \tag{34}$$

Loads and generators will add the non linear equations to the power flow problem. Each *PQ* load will add the next expressions:

$$\mathbf{P}_{abc} = \text{real}\left(\mathbf{AV}_{\alpha\beta 0} \otimes \text{conj}\left[\mathbf{AI}_{L_{\alpha\beta 0}}\right]\right) \tag{35}$$

$$\mathbf{Q}_{abc} = \text{imag}\left(\mathbf{AV}_{\alpha\beta 0} \otimes \text{conj}\left[\mathbf{AI}_{L_{\alpha\beta 0}}\right]\right) \tag{36}$$

$\mathbf{P}_{abc}$ and $\mathbf{Q}_{abc}$ are vectors including respectively the active and reactive powers demanded by the loads in abc coordinates. Since these powers can be balanced or not, they can always be calculated by means of these general expressions (35) and (36). In the present case, for the sake of simplicity, the active and reactive power are given in abc coordinates, as the problem input data. The operation $\otimes$ is defined as the element-wise product of two vectors.

A slack bus imposes the node voltage as shown in (37) (*pu* system).

$$\mathbf{AV}_{\alpha\beta 0} = \begin{bmatrix} e^{0j} \\ e^{\frac{-2\pi}{3}j} \\ e^{\frac{2\pi}{3}j} \end{bmatrix} \tag{37}$$

A *PQ* generator can be added in the same way as the *PQ* load just by substituting $\mathbf{I}_{L_{\alpha\beta 0}}$ by $\mathbf{I}_{G_{\alpha\beta 0}}$.

A *PV* node is modeled by replacing $\mathbf{I}_{L_{\alpha\beta 0}}$ by $\mathbf{I}_{G_{\alpha\beta 0}}$ in (35), and introducing (38) to state the voltage magnitude constraint in the *pu* system.

$$\text{abs}(\mathbf{AV}_{\alpha\beta 0}) = \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} \tag{38}$$

Where V_a, V_b and V_c are the specified voltage magnitudes that are usually given in abc coordinates.

$$\mathbf{M} = \left(\begin{array}{ccc|cccc|cccc|cccc}
\mathbf{Z}_{\alpha\beta 0}^{12} & & & & & & & & & & & -\mathbf{I}_d & \mathbf{I}_d & & \\
 & \mathbf{Z}_{\alpha\beta 0}^{23}\,\mathbf{N}_{\mathrm{I}}^{*} & & & & & & & & & & & -\mathbf{I}_d & \mathbf{N}_{\mathrm{II}}^{*} & \\
 & & \mathbf{Z}_{\alpha\beta 0}^{34} & & & & & & & & & & & -\mathbf{I}_d & \mathbf{I}_d \\
\hline
\mathbf{I}_d & & & \mathbf{I}_d & & & & -\mathbf{I}_d & & & & & & & \\
-\mathbf{I}_d & \mathbf{N}_{\mathrm{IV}}^{*} & & & \mathbf{I}_d & & & & -\mathbf{I}_d & & & & & & \\
 & \mathbf{N}_{\mathrm{III}}^{*} & \mathbf{I}_d & & & \mathbf{I}_d & & & & -\mathbf{I}_d & & & & & \\
 & & -\mathbf{I}_d & & & & \mathbf{I}_d & & & & -\mathbf{I}_d & & & &
\end{array}\right) \tag{27}$$

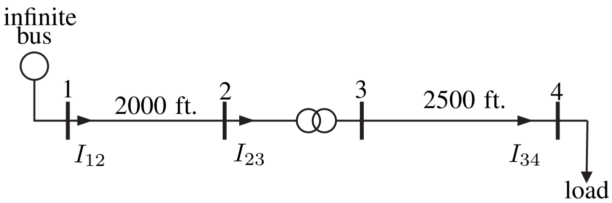

Fig. 2: IEEE 4 Node Test Feeder system.

In a similar way as the above mentioned loads and generators, other models of loads and/or generators can be included into the problem by adding the element equations. The great advantage of the proposed formulation is that the general matrix $\mathbf{M}$ (34) and the grid equation (32) depend only on the grid interconnections (nodes and lines). As it will be explained later, a change in the location of loads or generators does not modify this matrix.

B. Network description with power transformers

To describe the inclusion of the power transformer model into the grid model, the IEEE 4 Node Test Feeder benchmark in Fig. 2 will be used as an example.

The transformer addition requires some modifications in matrices $\mathbf{Z}_{\alpha\beta 0}^{*}$, $-\mathbf{\Gamma}^{*}$ and $\mathbf{\Gamma}^{*T}$ in (34). As it was previously mentioned, $\mathbf{Z}_{\alpha\beta 0}^{*}$ is a block diagonal matrix where each network line will add a block. These blocks are sorted through the line (edge) enumeration criteria established by $\mathbf{\Gamma}$. A transformer is then considered as a new diagonal block to be included in the matrix $\mathbf{Z}_{\alpha\beta 0}^{*}$. For example, in the IEEE 4 Node Test Feeder case, there are two lines and a power transformer, so the resulting $\mathbf{Z}_{\alpha\beta 0}^{*}$ matrix is formed by three blocks. Since the transformer is connected between nodes 2 and 3, it will add a block matrix $(\mathbf{Z}_{\alpha\beta 0}^{23}\,\mathbf{N}_{\mathrm{I}}^{*})$ at the second position as it can be observed in (27). $\mathbf{Z}_{\alpha\beta 0}^{23}$ is the transformer impedance matrix, with the same structure as (11) and $\mathbf{N}_{\mathrm{I}}^{*}$ is defined as follows:

$$\mathbf{N}_{\mathrm{I}}^{*} = \left(\begin{array}{c|c} \mathbf{N}_{\mathrm{I}} & \\ \hline & \mathbf{N}_{\mathrm{I}} \end{array}\right) \tag{39}$$

In the same way, the block matrices $\mathbf{N}_{\mathrm{II}}^{*}$, $\mathbf{N}_{\mathrm{III}}^{*}$ and $\mathbf{N}_{\mathrm{IV}}^{*}$ can be built. Such matrices will modify $-\mathbf{\Gamma}^{*}$ and $\mathbf{\Gamma}^{*T}$ following equations (24), (25) and (26). Due to the fact that the primary side of the transformer is connected to node 2 and the secondary to node 3, $\mathbf{N}_{\mathrm{II}}^{*}$ is embedded in $-\mathbf{\Gamma}^{*}$ at the same row block (node 2, second position) as $\mathbf{N}_{\mathrm{I}}^{*}$ in $\mathbf{Z}_{\alpha\beta 0}^{*}$ and at the third column (node 3). $\mathbf{N}_{\mathrm{IV}}^{*}$ and $\mathbf{N}_{\mathrm{III}}^{*}$ are embedded in $\mathbf{\Gamma}^{*T}$ at the same column block (node 2, second position) as $\mathbf{N}_{\mathrm{I}}^{*}$ in $\mathbf{Z}_{\alpha\beta 0}^{*}$, and at row blocks 2 and 3 respectively.

The resulting matrix $\mathbf{M}$ for the study case is represented in (27), it is formed by 7 row blocks and 15 column blocks. The structure of this matrix is the same as in (34). The transformer is similar to a line but including matrices $\mathbf{N}_{\mathrm{I}}$ to $\mathbf{N}_{\mathrm{IV}}$. As it can be deducted from the matrix, for each node, load and generator currents are included (there are four $\mathbf{I}_d$ block matrices and four $-\mathbf{I}_d$ block matrices) although there are no actual loads and generators in all nodes. Contrary to what it seem this procedure will make the system to be easily modified, as it will be demonstrated in Section V.

The inclusion or elimination of a transformer or a line is quite simple because it is only needed to add or remove the corresponding row and column blocks from the system matrix. For example, to eliminate the transformer from the system, the second row block and the second column block should be removed from $\mathbf{M}$. Since a node should be eliminated too, then two more block matrices $\mathbf{I}_d$ and $-\mathbf{I}_d$ would be removed too. The resulting system would have 2 lines and 3 nodes, and matrix $\mathbf{M}$ would be formed by 5 row blocks and 11 column blocks. To change the transformer connection and/or phase shift instead of removing it, just matrices $\mathbf{N}_{\mathrm{I}}$ to $\mathbf{N}_{\mathrm{IV}}$ need to be modified following the instructions in Table I. This demonstrated the simplicity and generalization of the proposed formulation.

V. Validation

The authors have employed the IEEE 4 Node Test Feeder and the IEEE 123 Node Test Feeder [28] as guides to validate the model. As it was stated by the Distribution Test Feeder Working Group that developed this set of standards, the purpose of publishing the data was to make available a common set of data that could be used by program developers and users to verify the correctness of their solutions, so the authors have considered these standards as the most appropriate to validate and evaluate the proposed model.

A. IEEE 4 Node Test Feeder

The model was tested by means of the IEEE 4 Node Test Feeder. This is the most adequate feeder to represent transformers of various configurations, full three phase lines and unbalanced loads [28]. The system is represented in Fig. 2. All power transformer connections were analyzed for step-up and step-down configurations under balanced and unbalanced

TABLE II: Test results: step-up unbalanced loading.

Connection	Y_gY_g	$Y_g\Delta$	$Y\Delta$	ΔY_g	$\Delta\Delta$
Voltage Node 2					
V_1	$7161\angle -0.1^\circ$	$7121\angle -0.4^\circ$	$12348\angle 29.7^\circ$	$12364\angle 29.8^\circ$	$12362\angle 29.8^\circ$
V_2	$7120\angle -120.3^\circ$	$7146\angle -120.3^\circ$	$12393\angle -90.3^\circ$	$12391\angle -90.5^\circ$	$12392\angle -90.4^\circ$
V_3	$7128\angle 119.3^\circ$	$7149\angle 119.5^\circ$	$12354\angle 149.5^\circ$	$12333\angle 149.6^\circ$	$12334\angle 149.5^\circ$
Voltage Node 3					
V_1	$13839\angle -2.1^\circ$	$23703\angle 57.2^\circ$	$23703\angle 57.2^\circ$	$13792\angle 27.7^\circ$	$23675\angle 27.2^\circ$
V_2	$13663\angle -123.3^\circ$	$24040\angle -63.6^\circ$	$24040\angle -63.6^\circ$	$13733\angle -93.5^\circ$	$24060\angle -93.6^\circ$
V_3	$13655\angle 115.1^\circ$	$23576\angle 176.1^\circ$	$23576\angle 176.1^\circ$	$13641\angle 145.4^\circ$	$23573\angle 146^\circ$
Voltage Node 4					
V_1	$13815\angle -2.2^\circ$	$23637\angle 57.1^\circ$	$23637\angle 57.1^\circ$	$13768\angle 27.7^\circ$	$23610\angle 27.2^\circ$
V_2	$13614\angle -123.4^\circ$	$23995\angle -63.8^\circ$	$23995\angle -63.8^\circ$	$13684\angle -93.6^\circ$	$24015\angle -93.7^\circ$
V_3	$13615\angle 114.9^\circ$	$23495\angle 175.9^\circ$	$23495\angle 175.9^\circ$	$13600\angle 145.2^\circ$	$23492\angle 145.9^\circ$
Current I_{12}					
I_a	$216.8\angle -34^\circ$	$332.6\angle -28.1^\circ$	$333.5\angle -28.2^\circ$	$309.3\angle -35.2^\circ$	$312.3\angle -34.8^\circ$
I_b	$293.3\angle -149.2^\circ$	$269.5\angle -155.6^\circ$	$269.6\angle -155.4^\circ$	$249.5\angle -146.5^\circ$	$248.1\angle -147.2^\circ$
I_c	$366.7\angle 96.7^\circ$	$275.5\angle 100.3^\circ$	$274.3\angle 100.2^\circ$	$319.2\angle 98.1^\circ$	$319.5\angle 98.7^\circ$
Current I_{34}					
I_a	$108.6\angle -34^\circ$	$156.4\angle -4.8^\circ$	$156.4\angle -4.8^\circ$	$108.9\angle -4.1^\circ$	$156.4\angle -34.8^\circ$
I_b	$146.9\angle -149.2^\circ$	$124.2\angle -117.2^\circ$	$124.2\angle -117.2^\circ$	$146.2\angle -119.4^\circ$	$124.2\angle -147.2^\circ$
I_c	$183.6\angle 96.7^\circ$	$158.4\angle 128.7^\circ$	$158.4\angle 128.7^\circ$	$183.8\angle 127.0^\circ$	$158.5\angle 98.7^\circ$

Where V_1, V_2 and V_3 are phase to phase voltages in delta and ungrounded wye connections and phase to ground voltages in grounded wye connections.

load conditions. As an example, in Table II the results obtained for the power transformer step-up connections with unbalanced load are presented. It must be pointed out that for three wire configurations the voltages in Table II (V_1, V_2 and V_3) are phase-to-phase voltages and phase to neutral voltages are considered in four wire configurations.

It is established by the Distribution Test Feeder Working Group in [28] that a good match would have an error less than 0.05%. In view of the results in Table II, and comparing with the results presented in the website, it is demonstrated that the model has been properly validated. In the next subsection the evaluation of the formulation in a more complex and larger system is conducted.

B. IEEE 123 Node Test Feeder

To evaluate the performance of the proposed model in large power systems, extensive tests have been carried out in a system based in the IEEE 123 Node Test Feeder (see Fig. 3). The following considerations were taken into account:

- The voltage regulators and shunt capacitors were not included.
- 6 power transformer were placed in a way that they divide the network in 7 different zones, as it is shown in Fig. 3. The load flow was performed using *pu* system on a phase-to-phase voltage basis of 115 kV for zone 1 and 4160 V for other zones. Depending on the zone type (three wire or four wire) phase-to-neutral voltage basis might be needed instead of phase-to-phase. The base power is 5 MVA. Several study cases were solved changing the power transformer connections and phase shifts. The resistance and reactance of the power transformers are respectively considered as 1% and 6% in all cases. In Table III three different combinations of power transformer connections and phase shifts are presented.
- A case without transformers, named base case, was also considered. In such case, each transformer was replaced by a 200 feet line.
- Each zone has 3 or 4 wires depending on the power transformer connection feeding the zone, so a given zone can be a three or a four wire zone depending on the study case. When a line is four wire type the configuration 1 described in the IEEE 123 Node Test Feeder is considered while configuration 12 is used for 3 wires zones.
- Regarding the loads, two different scenarios were contemplated. The former uses spot *PQ* balanced loads of 20 kW and 10 kVAr per phase placed in the same nodes as the standard case (IEEE 123 Node Test Feeder, input data extracted from [28]). The latter uses the same unbalanced

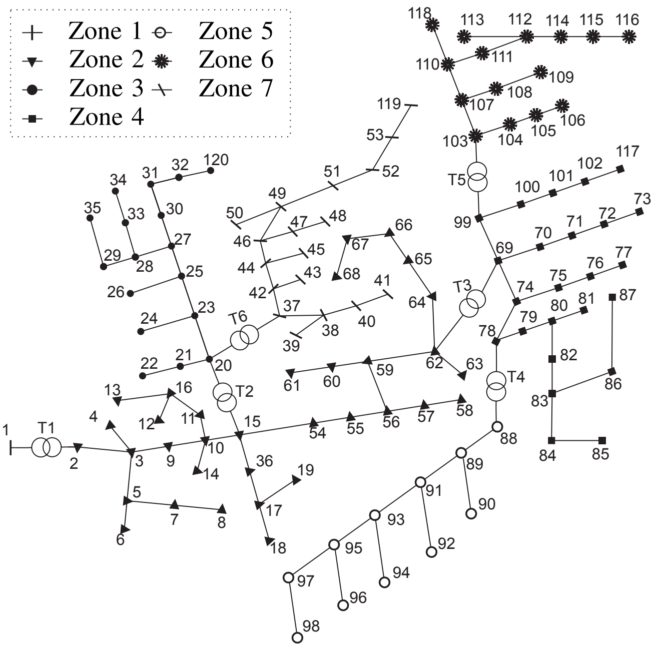

Fig. 3: Modified IEEE 123 Node Test Feeder.

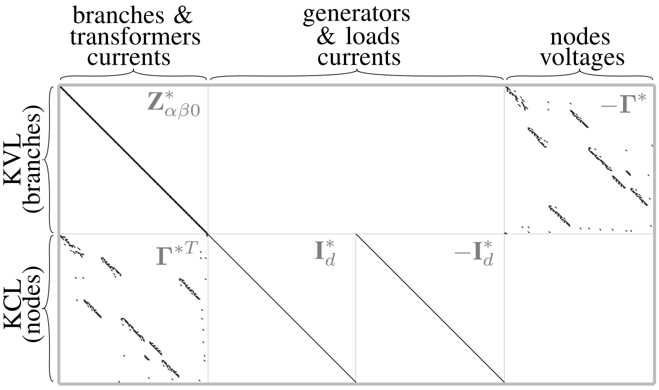

Fig. 4: Matrix M. Case 3 IEEE 123 Node Test Feeder. (See equation (34)).

load configuration described in the standard.

Eight study cases were examined: six combining the above described load scenarios with the three different power transformer configurations, and two combining the base case with the two load scenarios. The formulation was solved by means of the optimization toolbox of MATLAB and the algorithm called trust-region-dogleg, which is specifically designed for non linear systems. The algorithm is an iterative method for nonlinear minimization. Although an initial guess is needed, the trust-region methods improve robustness when starting far from the solution [27]. In all cases the initial guess vector was the null vector. All study cases are reflected in Table IV, in which the number of iterations required to solve the system are shown.

Fig. 4 presents the matrix $\mathbf{M}$ obtained for the study case 3, where blanks stand for zeros. The structure of this matrix is the same as that in (34). The rows at the top correspond to the KVL equations applied to all system branches and transformers. In this case there are 113 lines and 6 transformers, so the number of KVL equations is 714 ($(113 + 6) \times 6$). Due to the fact that these are the voltage drops at the lines (including the real transformers) there are only elements different from zero pointing at the branches and transformers currents (columns, matrix $\mathbf{Z}^*_{\alpha\beta 0}$) and node voltages (matrix $-\mathbf{\Gamma}^*$).

The rows at the bottom are the KCL equations, so the unknowns involved are the branch currents and the transformer currents (matrix $\mathbf{\Gamma}^{*T}$), the generator currents (matrix $\mathbf{I}^*_d$) and the load currents (matrix $-\mathbf{I}^*_d$).

In this case, for the sake of simplicity, the authors have included generator currents and load currents in all system nodes, so the dimension of matrices $\mathbf{I}^*_d$ and $-\mathbf{I}^*_d$ are the same for all the study cases. When building the nonlinear equations in the nodes with no actual generators it is stated that the vector of generator currents equals to zero ($\mathbf{I}_{G_{\alpha\beta 0}}$ =[0]). The same procedure is applied to the load currents at nodes with no actual connected loads ($\mathbf{I}_{L_{\alpha\beta 0}}$ =[0]).

This matrix is the same for the eight study cases. To change from a transformer configuration to a different one, only matrices $\mathbf{N}^*_{\mathrm{I}}$, $\mathbf{N}^*_{\mathrm{II}}$, $\mathbf{N}^*_{\mathrm{III}}$ and $\mathbf{N}^*_{\mathrm{IV}}$ need to be defined accordingly to Table I. The obtained node incidence matrix is always the same, so in case of changes in branch parameters, just parameters in matrix $\mathbf{Z}^*_{\alpha\beta 0}$ should be updated. In this way, it has been demonstrate the simplicity and the generalization of the proposed model.

It has to be remarked that the authors propose a model for power flow studies. In this work, the trust-region dogleg algorithm [27] was employed to solve the system of equations but other solving methods could be applied to the same formulation.

TABLE III: Transformers configurations.

	Conf 1		Conf 2		Conf 3	
	Conn.	PhSh	Conn.	PhSh	Conn.	PhSh
T1	YgYg	0	ΔY_g	150	$\Delta\Delta$	180
T2	YgYg	0	$Y_g\Delta$	-30	ΔY_g	-30
T3	YgYg	0	$Y_g\Delta$	150	$Y\Delta$	150
T4	YgYg	0	$\Delta\Delta$	0	ΔY_g	150
T5	YgYg	0	$Y\Delta$	150	$\Delta\Delta$	0
T6	YgYg	0	$\Delta\Delta$	180	Y_gY_g	0

TABLE IV: Tests in the IEEE 123 Node Test Feeder.

Case	1	2	3	4	5	6	7	8
Trafo Conf.	base	base	1	2	3	1	2	3
Load Scennario	Bal.	Unb.	Bal.	Unb.	Bal.	Unb.	Bal.	Unb.
Iterations	11	11	13	13	14	14	15	15

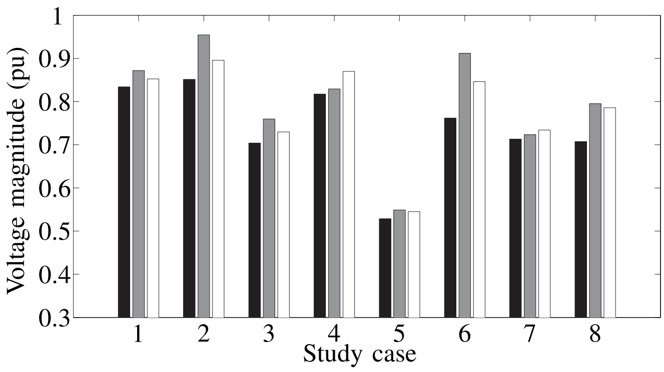

Fig. 5: Voltage magnitude (pu) per phase at node 116 (Zone 6) for all study cases. Black: phase A, grey: phase B, white: phase C.

In Fig. 5 the voltage magnitudes per phase at node 116, far from the slack node, are presented for all study cases. As it was expected the voltage unbalances are mostly displayed in cases with unbalanced loading (cases 2, 4, 6 and 8). However unbalanced voltages are also present in balanced loading scenarios. This is derived from the non transposed unbalanced distributed lines. The different transformer configurations have an important effect on the unbalanced too. The worst case in terms of voltage drop is case 5. Although this case has the same balanced loading level as cases 1, 3, 5 and 7, the transformer configurations have an influence on the voltage drops in the whole system.

Fig. 6 shows the voltage profile for the whole grid in study case 8, that is an unbalanced loading case. The figure is divided into six subplots corresponding to zones from 2 to 6. Zone 1 has been removed because it is only formed by the slack node 1 so it does not give any additional information. As it was expected, the zones near the slack bus present less voltage unbalances than the distant zones.

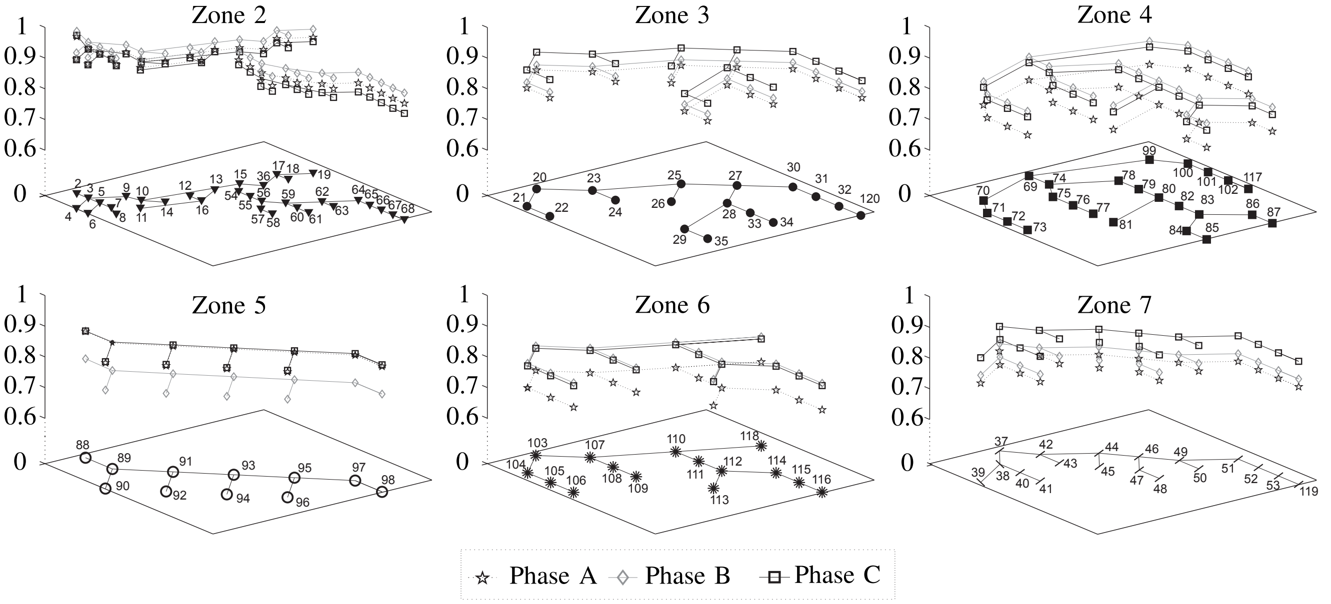

Fig. 6: Voltage per phase in *pu*, case of study 8. In $z = 0$ plane the node numbers are displayed.

VI. Conclusion

In the present work it has been demonstrated how a simple model of an AC network for three-phase unbalanced power flow studies, with embedded transformers, can be obtained conjugating the use of a complex vector based model in $\alpha\beta 0$ stationary reference frame and the node incidence matrix based formulation. The orthogonal frame simplifies the accommodation of different kind of devices in the network, as power transformers or distributed generators, and the node incidence matrix overcomes all the admittance matrix drawbacks. A power transformer model was also proposed and tested. With the suggested formulation, an exact power transformer model can be implemented just by using the connection type and the phase shift. Models of *PQ* loads, *PQ*, *PV* and slack generators were described.

The proposed formulation separately organizes the linear and the non-linear equations. All the linear equations representing the KVL and the KCL for the whole network are expressed in a compact matrix form and will be independent of load or generator models. For this reason, the authors can state that the model allows the inclusion of other complex models of loads or generators just by adding the corresponding non-linear equations, without modifying the compact matrix formulation that represents the core of the power flow problem.

The proposed model has been validated by comparing the obtained results for the IEEE 4 Node Test Feeder with those presented in [28], and evaluated in a larger distribution system, the IEEE 123 Node Test Feeder. The authors obtained appropriated results in all cases. To solve the proposed formulation the trust-region dogleg algorithm was used. However, other algorithms could be applied.

In future works, the use of an orthogonal reference frame will allow the modeling of power converters controls and constraints for distributed generators, lines and loads, in the same reference frame as the one used for the power flow analysis.

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