

## Quarter BPS classified by Brauer algebra

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**ABSTRACT:** We analyse the one-loop dilatation operator with the help of the Brauer algebra. We find some BPS operators in  $\mathcal{N} = 4$  SYM, which are labelled by irreducible representations of the Brauer algebra. Some of them are quarter BPS operators. The result includes full non-planar corrections. Our construction and proof are based on simple algebraic arguments and are carried out for any number of fields.

**KEYWORDS:** Supersymmetric gauge theory, AdS-CFT Correspondence

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## 1 Introduction and summary

The study of conformal field theories has been an important topic especially in understanding string theory. In particular  $\mathcal{N} = 4$  Super Yang-Mills theory has been extensively investigated in the context of the AdS/CFT correspondence. Concretely a large number of progress has been made in the computation of scaling dimensions of local operators. In general, to determine the scaling dimensions is not an easy task because of the operator mixing problem [1–5, 28]. In spite of this problem, if we restrict our attention to the planar theory, the operator mixing problem can be simplified because the dilatation operator may be identified with the integrable spin-chain system [6–8]. In contrast, our understanding of the full theory including non-planar corrections is not still enough to handle the problem. Operators with definite scaling dimensions can be linear combinations of single traces and multi-traces. To tackle the mixing problem including non-planar corrections, we would need to find a good way to organise gauge invariant operators of both single trace and multi-trace.

In our previous paper [9], we have proposed a basis of gauge invariant operators constructed from two kinds of  $u(N)$  matrices<sup>1</sup> with the help of the Brauer algebra. For holomorphic operators constructed from  $m$   $X$ s and  $n$   $Y$ s, our basis is

$$O_{A,ij}^\gamma(X, Y) \equiv tr_{m,n} \left( Q_{A,ij}^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n} \right), \tag{1.1}$$

where  $Q_{A,ij}^\gamma$  is given by a linear combination of elements in the Brauer algebra. In this paper we do not need an explicit form of  $Q_{A,ij}^\gamma$ . The definition of  $Q_{A,ij}^\gamma$  and  $O_{A,ij}^\gamma(X, Y)$ ,

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<sup>1</sup> The paper [9] originally studied gauge invariant operators built from  $X$  and  $X^\dagger$ , but the construction can be straightforwardly applied to holomorphic operators built from two types of complex matrices  $X$  and  $Y$ . See [18] for another use of the Brauer algebra to deal with the global indices.

and the meaning of the labels are briefly summarised in appendix A. One striking property of the basis is the operators have diagonal two-point functions at classical level [9]:

$$\langle O_{A,ij}^\gamma[x]^\dagger O_{A',i'j'}^{\gamma'}[0] \rangle_0 \propto \delta_{\gamma\gamma'} \delta_{AA'} \delta_{ii'} \delta_{jj'} \frac{1}{x^{2(m+n)}}, \tag{1.2}$$

where  $\langle \dots \rangle_0$  means the tree level correlator. The construction of the basis was motivated by the fact that the chiral primary operators with diagonal two-point functions are classified by a Young diagram, which can be identified with giant gravitons [10, 11].

Researches along this line for the other sectors have been reported in [12–18]. An idea of these works is to exploit algebras or groups which are dual to  $U(N)$  in the meaning of Schur-Weyl duality (see [16, 19] from this point of view) to organise the multi-trace structure of gauge invariant operators.

In this paper, we study the sector composed of two types of complex scalar fields  $X$  and  $Y$  in  $\mathcal{N} = 4$  SYM (the  $su(2)$  sector) and focus on the dilatation operator including non-planar corrections, by making use of the Brauer algebra. This sector is closed under renormalisation in all order perturbation. Analysing the action of the one-loop dilatation operator on the basis, we find some gauge invariant operators which are vanished by the one-loop dilatation operator after simple algebraic manipulations of the Brauer algebra. This result would indicate that the Brauer algebra can be a useful tool at quantum level.<sup>2</sup> Former studies on the quarter BPS operators are [7, 28–31]. See also the recent paper [32].

Here are our results. We will show that the following operators

$$O^\gamma(X, Y) \equiv tr_{m,n}(P^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n}) \tag{1.3}$$

satisfy

$$\hat{D}_2 O^\gamma(X, Y) = 0, \tag{1.4}$$

where  $\hat{D}_2$  is the one-loop dilatation operator. This result is valid for any  $m$  and  $n$ .  $P^\gamma$  is the projector associated with the irreducible representation  $\gamma$  of the Brauer algebra, which is given by

$$P^\gamma = \sum_{A,i} Q_{A,ii}^\gamma. \tag{1.5}$$

The irreducible representation of the Brauer algebra may be specified by the following set

$$\gamma = (k, \gamma_+, \gamma_-), \tag{1.6}$$

where  $k$  is an integer satisfying  $0 \leq k \leq \min(m, n)$ ,  $\gamma_+$  is a Young diagram with  $m - k$  boxes and  $\gamma_-$  is a Young diagram with  $n - k$  boxes. When  $N$  is finite, we have the constraint  $c_1(\gamma_+) + c_1(\gamma_-) \leq N$ , where  $c_1$  denotes the length of the first column of the Young diagram. It follows from (1.2) that two-point functions are orthogonal at one-loop:

$$\langle O^\gamma[x]^\dagger O^{\gamma'}[0] \rangle_1 \propto \delta_{\gamma\gamma'} \frac{1}{x^{2(m+n)}}. \tag{1.7}$$

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<sup>2</sup> See [20–23] for studies of quantum corrections on the other bases where the symmetric group  $S_{m+n}$  plays a role. See also [24–27] for attempts to study non-planar corrections.

Our way of finding eigenstates of the dilatation operator is based on a simple algebraic argument. In particular what we need to prove our claim is only the property that  $P^\gamma$  are projection operators associated with irreducible representations of the Brauer algebra. We have not disclosed the operator mixing completely, but the results in this paper may be interpreted as a message that the classification of operators in terms of Brauer algebras or symmetric groups could be a promising way to deal with full non-planar corrections.

The structure of this paper is as follows. In section 2, we derive the action of the one-loop dilatation operator in terms of the Brauer algebra. Section 3 will be given to the proof of the claim that  $O^\gamma(X, Y)$  may be quarter BPS operators at one-loop. The operators are classified by an integer  $k$ . We will study the sectors which are labelled by  $k = 0$  and  $k = m = n$  in section 4. We shall provide a brief explanation on the role of the Brauer algebra and the mathematical meaning of the operators  $P^\gamma$  and  $Q_{A,ij}^\gamma$  in appendix A. The mixing of the basis under the dilatation operator will be discussed in appendix B.

## 2 One-loop dilatation operator on the Brauer basis

In this section, we study the action of the one-loop dilatation generator on the basis composed by the Brauer algebra.

In perturbation theory, the dilatation generator can be expanded in power series of the coupling constant as

$$\hat{D} = \sum_{l=0} \left( \frac{g_{\text{YM}}^2}{16\pi^2} \right)^l \hat{D}_{2l}, \tag{2.1}$$

where  $\hat{D}_{2l}$  is the  $l$ -loop dilatation generator. For the  $su(2)$  sector, the concrete form has been known [7, 33, 34] (see [35] for a review):

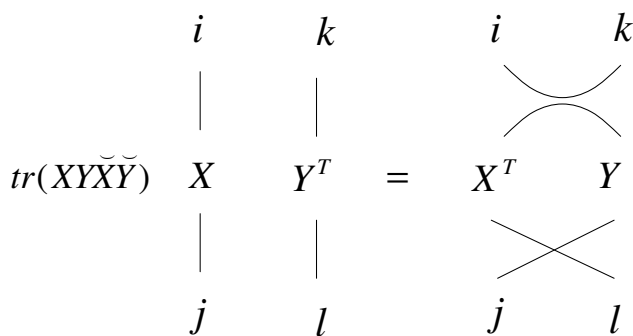
$$\begin{aligned} \hat{D}_0 &= \text{tr}(X\check{X} + Y\check{Y}), \\ \hat{D}_2 &= -2\text{tr}([X, Y][\check{X}, \check{Y}]) \equiv -2\hat{H}, \end{aligned} \tag{2.2}$$

where  $\check{X}$  is the derivative, and when  $U(N)$  is the gauge group, which is the case of this paper, it acts as  $(\check{X})_{ij}X_{kl} = \delta_{il}\delta_{jk}$ .

First consider the action of  $\hat{H}$  on  $X_{ij}(Y^T)_{kl}$ :

$$\begin{aligned} \hat{H}X_{ij}(Y^T)_{kl} &= ([X, Y]_{mn}(\check{X})_{no}(\check{Y})_{om} - ([X, Y]_{mn}(\check{Y})_{no}(\check{X})_{om}) X_{ij}(Y^T)_{kl} \\ &= ([X, Y]_{lj}\delta_{ik} - ([X, Y]_{ik}\delta_{jl}) \\ &= X_{lm}(Y^T)_{jm}\delta_{ik} - X_{mj}(Y^T)_{ml}\delta_{ik} - X_{im}(Y^T)_{km}\delta_{jl} + X_{mk}(Y^T)_{mi}\delta_{jl}. \end{aligned} \tag{2.3}$$

The first term is depicted in figure 1. Because the derivatives act via the Leibniz rule, the



**Figure 1.** The action of  $tr(XY\tilde{X}\tilde{Y})$  on  $X \otimes Y^T$ .

action of  $\hat{H}$  on the basis (1.1) can be expressed as follows:

$$\begin{aligned}
 \hat{H}O_{A,ij}^\gamma &= \sum_{r,s} tr_{m,n}(\sigma_{r,s}Q_{A,ij}^\gamma C_{r,s} \mathbf{T}_r X^{\otimes m} \otimes \mathbf{T}_s (Y^T)^{\otimes n}) \\
 &\quad - \sum_{r,s} tr_{m,n}(Q_{A,ij}^\gamma C_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
 &\quad - \sum_{r,s} tr_{m,n}(C_{r,s} Q_{A,ij}^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
 &\quad + \sum_{r,s} tr_{m,n}(C_{r,s} Q_{A,ij}^\gamma \sigma_{r,s} \mathbf{T}_r X^{\otimes m} \otimes \mathbf{T}_s (Y^T)^{\otimes n}), \tag{2.4}
 \end{aligned}$$

where we have introduced the operation  $\mathbf{T}_i$  to take the transpose on the  $i$ -th operator as

$$\begin{aligned}
 \mathbf{T}_r X^{\otimes m} &= X^{\otimes r-1} \otimes X^T \otimes X^{m-r} \\
 \mathbf{T}_s (Y^T)^{\otimes n} &= (Y^T)^{\otimes s-1} \otimes Y \otimes (Y^T)^{n-s}. \tag{2.5}
 \end{aligned}$$

$C_{r,s}$  is an element in the Brauer algebra, which connects the  $r$ -th element of  $X^{\otimes m}$  with  $s$ -th element of  $(Y^T)^{\otimes n}$  ( $1 \leq r \leq m, 1 \leq s \leq n$ ), and is expressed in the upper part of  $X^T$  and  $Y$  in figure 1.  $\sigma_{r,s}$  is a permutation acting on the  $r$ -th element of  $X^{\otimes m}$  and the  $s$ -th element of  $(Y^T)^{\otimes n}$ , which is represented as a cross in figure 1. Note that  $\sigma_{r,s}$  is not an element in the Brauer algebra.

One might be worried about the gauge invariance of the first term and forth term because  $X^T, Y$  appear instead of  $X, Y^T$ . However, the gauge invariance is kept consistent because of the existence of  $\sigma_{r,s}$ . In order to see the gauge invariance more manifestly, it will be helpful to have another expression of those terms. The first term of (2.4) may be rewritten as

$$tr_{m,n}(\sigma_{r,s}Q_{A,ij}^\gamma C_{r,s} \mathbf{T}_r X^{\otimes m} \otimes \mathbf{T}_s (Y^T)^{\otimes n}) = tr_{m,n}(Q_{A,ij}^\gamma C_{r,s} \mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}), \tag{2.6}$$

where we have introduced the operation  $\mathbf{P}_{r,s}$  to exchange the  $r$ -th  $X$  with the  $s$ -th  $Y$

$$\mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n} = X^{\otimes r-1} \otimes Y \otimes X^{m-r} \otimes (Y^T)^{\otimes s-1} \otimes X^T \otimes (Y^T)^{n-s}. \tag{2.7}$$

Thus we have<sup>3</sup>

$$\begin{aligned}
\hat{H}O_{A,ij}^\gamma &= \sum_{r,s} tr_{m,n}(Q_{A,ij}^\gamma C_{r,s} \mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
&\quad - \sum_{r,s} tr_{m,n}(Q_{A,ij}^\gamma C_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
&\quad - \sum_{r,s} tr_{m,n}(C_{r,s} Q_{A,ij}^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
&\quad + \sum_{r,s} tr_{m,n}(C_{r,s} Q_{A,ij}^\gamma \mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}). \tag{2.8}
\end{aligned}$$

In the equation, we have neither the transpose nor  $\sigma_{r,s}$ . Note that the second term is the same as the third term because of  $(\sum_{r,s} C_{r,s})Q_{A,ij}^\gamma = Q_{A,ij}^\gamma(\sum_{r,s} C_{r,s})$  (see appendix B).

Because each term in (2.4) (or (2.8)) is gauge invariant, it may be expressed by a linear combination of the basis  $O_{A,ij}^\gamma$ . This can tell us how the basis mixes under the action of the dilatation operator. We will discuss this point in appendix B.

### 3 Quarter BPS operators and central projectors

In this section, we shall prove our claim that the gauge invariant operators (1.3) are in the kernel of the one-loop dilatation operator. In this proof the only essential element is  $P^\gamma$  to be central elements in the Brauer algebra. In particular they commute with the contractions:

$$C_{r,s}P^\gamma = P^\gamma C_{r,s}. \tag{3.1}$$

Hence, for  $O^\gamma = tr_{m,n}(P^\gamma X^{\otimes m} \otimes Y^{T \otimes n})$ , we have the following action of the dilatation operator

$$\begin{aligned}
\hat{H}O^\gamma &= 2 \sum_{r,s} tr_{m,n}(C_{r,s}P^\gamma \mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
&\quad - 2 \sum_{r,s} tr_{m,n}(C_{r,s}P^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n}). \tag{3.2}
\end{aligned}$$

We shall see a cancellation between the first term and the second term.

The equation (3.1) allows us to have the following equation (note that  $C_{r,s}^2 = NC_{r,s}$ )

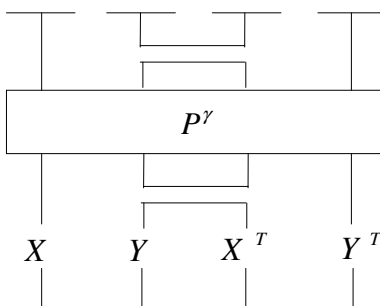
$$C_{r,s}P^\gamma = \frac{1}{N}C_{r,s}P^\gamma C_{r,s}. \tag{3.3}$$

Making use of this equation, the first term in (3.2) can be

$$\begin{aligned}
tr_{m,n}(C_{r,s}P^\gamma \mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) &= \frac{1}{N}tr_{m,n}(C_{r,s}P^\gamma C_{r,s} \mathbf{P}_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\
&= \frac{1}{N}tr(XY)tr_{m,n}(C_{r,s}P^\gamma \mathbf{I}_r X^{\otimes m} \otimes \mathbf{I}_s (Y^T)^{\otimes n}), \tag{3.4}
\end{aligned}$$

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<sup>3</sup> A similar equation was found in unpublished work of T. Brown and S. Ramgoolam (Oct. 2008).



**Figure 2.** A diagrammatic representation of the second line in (3.4) at  $m = n = 2, r = 2, s = 1$ . The top lines and the bottom lines are identified to express a trace  $tr_{2,2}$ . For more on the diagrammatic representation, see [9, 11].

where we have introduced the operation  $\mathbf{I}_i$  to replace the  $i$ -th matrix with the identity:

$$\mathbf{I}_r X^{\otimes m} = X^{\otimes r-1} \otimes 1 \otimes X^{m-r} \tag{3.5}$$

$$\mathbf{I}_s (Y^T)^{\otimes n} = (Y^T)^{\otimes s-1} \otimes 1 \otimes (Y^T)^{n-s}. \tag{3.6}$$

A diagrammatic representation of the second term in (3.4) is provided in figure 2. Similarly the second term in (3.2) may be rewritten as

$$\begin{aligned} tr_{m,n}(C_{r,s} P^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n}) &= \frac{1}{N} tr_{m,n}(C_{r,s} P^\gamma C_{r,s} X^{\otimes m} \otimes (Y^T)^{\otimes n}) \\ &= \frac{1}{N} tr(XY) tr_{m,n}(C_{r,s} P^\gamma \mathbf{I}_r X^{\otimes m} \otimes \mathbf{I}_s (Y^T)^{\otimes n}). \end{aligned} \tag{3.7}$$

Hence we conclude that

$$\hat{H}O^\gamma = 0. \tag{3.8}$$

We have shown that the operators composed by the projectors associated with irreducible representations of the Brauer algebra are in the kernel of the one-loop dilatation operator.

Before finishing this section, let us count the number of the operators. For a fixed  $(m, n)$ , the number of the operators is

$$\sum_{k=0}^{\min(m,n)} p(m-k)p(n-k), \tag{3.9}$$

where  $p(m)$  is the number of partitions of  $m$ . When  $N$  is finite, we have to impose  $c_1(\gamma_+) + c_1(\gamma_-) \leq N$ . A list of the operators for some  $(m, n)$  is shown in table 1.

One may wonder whether the operators found here exhaust BPS operators. We may answer this question by consulting the partition function over BPS states [37–39], and so it appears that there exist more BPS operators.<sup>4</sup> It would be interesting to know how the other operators can be labelled by representations.

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<sup>4</sup> We thank S. Ramgoolam and J. Pasukonis for discussions on this point.

(1, 1)	$\gamma_+$	$\gamma_-$
$k = 0$	[1]	[1]
$k = 1$	$\emptyset$	$\emptyset$

(2, 1)	$\gamma_+$	$\gamma_-$
$k = 0$	[2]	[1]
	[1, 1]	[1]
$k = 1$	[1]	$\emptyset$

(2, 2)	$\gamma_+$	$\gamma_-$
$k = 0$	[2]	[2]
	[2]	[1, 1]
	[1, 1]	[2]
	[1, 1]	[1, 1]
$k = 1$	[1]	[1]
$k = 2$	$\emptyset$	$\emptyset$

(3, 1)	$\gamma_+$	$\gamma_-$
$k = 0$	[3]	[1]
	[2, 1]	[1]
	[1, 1, 1]	[1]
$k = 1$	[2]	$\emptyset$
	[1, 1]	$\emptyset$

(3, 2)	$\gamma_+$	$\gamma_-$
$k = 0$	[3]	[2]
	[2, 1]	[2]
	[1, 1, 1]	[2]
	[3]	[1, 1]
	[2, 1]	[1, 1]
	[1, 1, 1]	[1, 1]
$k = 1$	[2]	[1]
	[1, 1]	[1]
$k = 2$	[1]	$\emptyset$

**Table 1.** List of operators for  $(m, n) = (1, 1), (2, 1), (2, 2), (3, 1), (3, 2)$ . If  $N$  is finite, we have  $c_1(\gamma_+) + c_1(\gamma_-) \leq N$ .

In general protected operators built from  $X$  and  $Y$  are classified into quarter BPS operators and (SU(4) descendants of) the half BPS operators. (This is found by group theory. See [28, 39] for example.) Because the number of the half BPS operators built from  $(m + n) X$ 's is equivalent to the number of partitions of  $m + n$ , there can be  $p(m + n)$  half BPS operators in the basis (1.1). We have considered only the kernel condition  $\hat{H}O = 0$ , our operators are in general linear combinations of the half BPS operators and the quarter BPS operators. From the counting, at least some of quarter BPS operators are in the list of the operators we found. In this basis the global SU(4) representation is not easy to see. Hence we do not have a good way to classify our operators based on the global representation at the moment. But an interesting thing to realise may be the fact that some of quarter BPS operators are combined with the other BPS operators to be labelled by irreducible representations of the Brauer algebra, which are more relevant to the construction of the orthogonal basis. Complete classification of protected operators and non-protected operators based on the Brauer algebra is left as an important future problem.

#### 4 Characteristic sectors

Our operators are labelled by irreducible representations of the Brauer algebra, which are determined by the set  $\gamma = (k, \gamma_+, \gamma_-)$ . It is interesting to recognise that they are classified by an integer  $k$ . In this section, we will have a closer look on two interesting sectors:  $k = 0$  and  $k = m = n$ .



### 4.1 $k = 0$

The sector labelled by  $k = 0$  is characterised by the interesting equation

$$C_{r,s}P^{\gamma(k=0)} = 0, \tag{4.1}$$

for any  $(r, s)$ .<sup>5</sup> Hence we immediately conclude from (3.2) that  $\hat{H}O^{\gamma(k=0)} = 0$ .

We also find out that this class does not receive quantum corrections at two-loop, using the equation (4.1). The dilatation operator at two-loop was given in [7], where each term contains  $(\check{X}\check{Y})_{ij}$  or  $(\check{Y}\check{X})_{ij}$ . The action of them on  $X \otimes Y^T$  is

$$\begin{aligned} (\check{X}\check{Y})_{pq}X_{ij}(Y^T)_{kl} &= \delta_{ik}\delta_{pj}\delta_{ql}, \\ (\check{Y}\check{X})_{pq}X_{ij}(Y^T)_{kl} &= \delta_{jl}\delta_{pk}\delta_{qi}. \end{aligned} \tag{4.2}$$

The appearance of  $\delta_{ik}$  in the first line and  $\delta_{jl}$  in the second line has the same effect as the contraction  $C$  on  $P^{\gamma(k=0)}$  when the action on  $O^{\gamma(k=0)}$  is considered. Therefore the equation (4.1) means that the  $k = 0$  operators are annihilated by the dilatation operator at two-loop.

One unique property of this sector is that the  $k = 0$  operators do not appear in the image of  $\hat{D}_2$ . The leading term of the  $k = 0$  operators is the product of a Schur polynomial of  $X$  and a Schur polynomial of  $Y$  [9]. On the other hand, the dilatation operator always combine an  $X$  and a  $Y$  in a trace because of the contraction. Therefore we can not get the leading term of the  $k = 0$  operators as an image of the dilatation operator. This indicates that no mixing happens between  $k = 0$  operators and  $k \neq 0$  operators.

The  $k = 0$  operators have the interesting form:

$$\begin{aligned} &tr_{m,n}(P^{(k=0,\gamma_+,\gamma_-)}X^{\otimes m} \otimes Y^{T\otimes n}) \\ &= Dim_{\gamma_+\gamma_-} \frac{m!n!}{d_{\gamma_+}d_{\gamma_-}} \frac{1}{N^{m+n}} tr_{m,n}(\Omega_{m+n}^{-1}p_{\gamma_+}p_{\gamma_-}X^{\otimes m} \otimes Y^{\otimes n}). \end{aligned} \tag{4.3}$$

This was obtained in [9] (see (4.7) and (4.8) in the paper). In this expression,  $\Omega_{m+n}^{-1}$  is the omega factor considered in the symmetric group  $S_{m+n}$ , and  $p_{\gamma_+}$  and  $p_{\gamma_-}$  are projection operators in  $S_m$  and  $S_n$ .

The  $k = 0$  operators for some  $(m, n)$  are shown explicitly for the  $X-X^\dagger$  sector in [9]. The  $X-Y$  system is obtained by replacing  $X^\dagger$  with  $Y$ .

Let me mention on the role of (4.1) for non-holomorphic operators constructed from  $X$ . A  $k = 0$  operator is a linear combination of single traces and multi-traces. An interesting property is that divergences arising from self-contractions cancel among those terms due to the equation (4.1) [9, 18]. One may keep in mind that the equation (4.1) plays interesting and different roles in the two systems ( $X-Y$  and  $X-X^\dagger$ ). See also [36] for physics related to the  $k = 0$  sector.

### 4.2 $k = m = n$

In this subsection we study the operators at  $k = m = n$ . This sector was investigated in [36]. We leave some details (the definition of  $C_{(k)}$ , the derivation of (4.4) and how to

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<sup>5</sup> This equation was exploited to provide concrete forms of  $k = 0$  projectors in [9].

rewrite the first line to get the last expression in (4.7) to [36] because the purpose of this subsection is not to review the calculations given in it.

When  $k = m = n$ , some labels of  $Q_{A,ij}^\gamma$  are trivial, thereby  $Q_{A,ij}^\gamma$  are labelled by a single Young diagram  $\alpha$  with  $k$  boxes (we define  $P_\alpha^{\gamma(k=m)} \equiv Q_\alpha^{\gamma(k=m)}$ ):

$$P_\alpha^{\gamma(k=m)} = \frac{d_\alpha}{k! \text{Dim} \alpha} C_{(k)} p_\alpha. \tag{4.4}$$

The central projector  $P^{\gamma(k=m)}$  can be obtained by the sum of all projectors in this sector<sup>6</sup>

$$P^{\gamma(k=m)} = \sum_{\alpha \vdash k} P_\alpha^{\gamma(k=m)} = \frac{1}{N^k} \Omega_k^{-1} C_{(k)}, \tag{4.5}$$

where we have used the formula for the inverse of the omega factor:

$$\Omega_k^{-1} = \frac{N^k}{k!} \sum_{\alpha \vdash k} \frac{d_\alpha}{\text{Dim} \alpha} p_\alpha. \tag{4.6}$$

The omega factor played an important role in the large  $N$  expansion of two-dimensional Yang-Mills [40–42].

Gauge invariant operators relevant to the projectors can be written down as

$$\begin{aligned} O^{\gamma(k=m)}(X, Y) &= \text{tr}_{k,k}(P^\gamma X^{\otimes k} \otimes (Y^T)^{\otimes k}) \\ &= \frac{1}{N^k} \text{tr}_k(\Omega_k^{-1} S^{\otimes k}), \end{aligned} \tag{4.7}$$

where we have defined  $S = XY$ . This sector may be characterised by the fact that these operators are invariant under  $X \rightarrow gX$ ,  $Y \rightarrow Yg^{-1}$  or  $X \rightarrow Xg$ ,  $Y \rightarrow g^{-1}Y$ . These transformations are introduced in [16] to measure labels of orthogonal sets. More remarks are in [36].

For concreteness, we present explicit forms:

$$O^{\gamma(k=2)} = \frac{1}{N(N^2 - 1)} (N(\text{tr} S)^2 - \text{tr}(S^2)) \tag{4.8}$$

for  $m = n = 2$ , and

$$O^{\gamma(k=3)} = \frac{1}{N(N^2 - 1)(N^2 - 4)} (6(N^2 - 2)(\text{tr} S)^3 - 18N(\text{tr} S)\text{tr}(S^2) + 24\text{tr}(S^3)) \tag{4.9}$$

for  $m = n = 3$ .

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<sup>6</sup> In this sector, there is only one central projector because  $\gamma_+ = \gamma_- = \emptyset$ .

## A On the role of the Brauer algebra

In this section, we shall make a brief description on the role of the Brauer algebra in constructing a set of gauge invariant operators, with emphasis on group theoretic structure. A more useful review of Brauer algebras may be found in [9], and references therein.

An  $N \times N$  matrix  $X$  can be viewed as an endomorphism acting on an  $N$ -dimensional vector space  $V$ , i.e.  $X : V \rightarrow V$ . The tensor product  $X^{\otimes n} = X \otimes \dots \otimes X$  acts on  $V^{\otimes n}$ . The symmetric group  $S_n$  can be introduced as a tool to organise both single trace and multi-trace. We define the action of the symmetric group  $S_n$  as the permutations of  $n$  vector spaces. The tensor product space can be decomposed into irreducible representations as

$$V^{\otimes n} = \bigoplus_R V_R^{U(N)} \otimes V_R^{S_n}. \tag{A.1}$$

This is a consequence of the fact that the  $U(N)$  action and the symmetric group action commute each other on the space  $V^{\otimes n}$  [Schur-Weyl duality]. The sum is taken for all irreducible representations with  $n$  boxes satisfying  $c_1(R) \leq N$ , where  $c_1(R)$  is the length of the first column of  $R$ . The projection operator  $p_R$  associated with an irreducible representation  $R$  can be introduced as an element in the group algebra of the symmetric group  $S_n$ . The operators defined by  $tr_n(p_R X^{\otimes n})$  were shown to form a complete set of gauge invariant operators in the chiral primary sector [10]. The trace  $tr_n$  is taken in  $V^{\otimes n}$ .

Let us next consider the space  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$  by including the complex conjugated space  $\bar{V}$ . Elements of the symmetric group  $S_m \times S_n$  can act on the space as linear maps. The contraction  $C$  can also be introduced as a map from  $V \times \bar{V}$  to itself. (It acts as  $Cv_i \bar{v}_j = \delta_{ij} \sum_k v_k \bar{v}_k$  for components.) An algebra formed by the group algebra of the symmetric group  $S_m \times S_n$  and the contractions is the Brauer algebra. We shall denote it by  $B_N(m, n)$ . Note that the Brauer algebra is sensitive to  $N$  while the symmetric group is not. Schur-Weyl duality relevant to this case is

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_N(m, n)}. \tag{A.2}$$

This gives the decomposition of the tensor product space in terms of irreducible representations of  $U(N)$  and  $B_N(m, n)$ . We have the projection operator  $P^{\gamma}$  associated with an irreducible representation  $\gamma$  which is a linear combination of elements in the Brauer algebra. Note that it is in the centre of the Brauer algebra, i.e.

$$bP^{\gamma} = P^{\gamma}b, \tag{A.3}$$

where  $b$  is any element in  $B_N(m, n)$ . The space  $V_{\gamma}^{B_N(m, n)}$  can be further decomposed into the group algebra of  $S_m \times S_n$ , which we denote by  $\mathbb{C}(S_m \times S_n)$ , as

$$V_{\gamma}^{B_N(m, n)} = \bigoplus_A V_{\gamma \rightarrow A} \otimes V_A^{\mathbb{C}(S_m \times S_n)}, \tag{A.4}$$

where  $A$  runs over irreducible representations of the symmetric group  $S_m \times S_n$ , and  $V_{\gamma \rightarrow A}$  represents the space of the multiplicity relevant to this decomposition. We introduce the

operator  $Q_{A,ij}^\gamma$  which acts on the space  $V_A^{S_m \times S_n}$  inside  $V_\gamma^{B_N(m,n)}$ . The indices  $i, j$  run over the space of the multiplicity, and besides, they behave like matrix indices as

$$Q_{A,ij}^\gamma Q_{A',i'j'}^{\gamma'} = \delta^{\gamma\gamma'} \delta_{AA'} \delta_{j'i'} Q_{A,ij}^\gamma. \tag{A.5}$$

The relation between  $P^\gamma$  and  $Q_{A,ij}^\gamma$  is

$$P^\gamma = \sum_{A,i} Q_{A,ii}^\gamma. \tag{A.6}$$

The readers who are interested in the explicit form of  $Q_{A,ij}^\gamma$  would be recommended to see [16]. Acting with  $Q_{A,ij}^\gamma$  on  $X^{\otimes m} \otimes Y^{T \otimes n}$  and taking a trace in  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ , we get  $tr_{m,n} \left( Q_{A,ij}^\gamma X^{\otimes m} \otimes Y^{T \otimes n} \right)$ , which is the basis proposed in [9].

## B Mixing of the basis

In order to realise how operators mix under the action of the dilatation operator, it will be helpful to rewrite each term of (2.4) in terms of the basis. In this section, we perform it for the third term.

Introduce the sum of contractions,

$$C \equiv \sum_{r,s} C_{r,s} = \frac{1}{(m-1)!(n-1)!} \sum_{h \in S_m \times S_n} h C_{1,1} h^{-1}. \tag{B.1}$$

We can easily show that it commutes with any element in the symmetric group  $S_m \times S_n$ :

$$C\tau = \tau C, \quad \tau \in S_m \times S_n. \tag{B.2}$$

Such an element can be expressed by a linear combination of  $Q_{A,ij}^\gamma$ , and the formula was given in appendix B.2 of [16]:

$$C = \sum_{\gamma,A,ij} \frac{1}{d_A} \chi_{A,ij}^\gamma(C) Q_{A,ij}^\gamma, \tag{B.3}$$

where  $d_A$  is the dimension of the symmetric group  $S_m \times S_n$  associated with the representation  $A$ , and  $\chi_{A,ij}^\gamma$  is the restricted character. Using this, we get

$$\begin{aligned} C Q_{A,ij}^\gamma &= \sum_{\gamma',A',i'j'} \frac{1}{d_{A'}} \chi_{A',i'j'}^{\gamma'}(C) Q_{A',i'j'}^{\gamma'} Q_{A,ij}^\gamma \\ &= \sum_{i'} \frac{1}{d_A} \chi_{A,i'i}^\gamma(C) Q_{A,i'j}^\gamma, \end{aligned} \tag{B.4}$$

where (A.5) has been used. Hence we can rewrite the third term in (2.4) as

$$\sum_{r,s} tr_{m,n} (C_{r,s} Q_{A,ij}^\gamma X^{\otimes m} \otimes (Y^T)^{\otimes n}) = \frac{1}{d_A} \sum_{i'} \chi_{A,i'i}^\gamma(C) O_{A,i'j}^\gamma. \tag{B.5}$$

As far as this term is concerned, the mixing is strictly restricted because only the multiplicity index is relevant for the mixing.

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