JORDAN BIMODULES OVER THE SUPERALGEBRAS

P(n) AND Q(n)

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Abstract. We extend the Jacobson’s Coordinatization theorem to Jordan superalgebras. Using it we classify Jordan bimodules over superalgebras of types Q(n) and JP(n), n ≥ 3. Then we use the Tits-Kantor-Koecher construction and representation theory of Lie superalgebras to treat the remaining case Q(2).

Introduction

Throughout the paper all algebras are considered over a ground field F of characteristic ≠ 2.

Let G = ⟨1, e_i, i ≥ 1|e_i e_j + e_j e_i = 0⟩ denote the Grassmann (or exterior) algebra. Then G = G_0 + G_1 is a Z/2Z-graded algebra, where G_0, G_1 are linear spans of all tensors of even and odd length, respectively.

Let V be a variety of algebras defined by homogeneous identities (see [1], [20]). A superalgebra A = A_0 + A_1 is said to be a V-superalgebra if its Grassmann envelope G(A) = A_0 ⊗ G_0 + A_1 ⊗ G_1 lies in V.

C.T.C. Wall [19] proved that every associative simple finite-dimensional superalgebra over an algebraically closed field F is isomorphic to one of the superalgebras:

I) A = M_m+n(F), A_0 = \left\{\begin{array}{c} \star \\ 0 \end{array}\right., A_1 = \left\{\begin{array}{cc} 0 & \star \\ \star & 0 \end{array}\right.\right\}
and

II) A = Q(n) = \left\{\begin{array}{c} a \\ b \end{array}: a,b \in M_n(F)\right\}

are associative superalgebras.

Given a homogeneous element a ∈ A_0 ∪ A_1, let |a| denote its parity (0 or 1).

From the definition above it follows that a Jordan superalgebra is a Z/2Z-graded algebra J = J_0 + J_1 satisfying the graded identities

xy = (-1)^{|x||y|}yx

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and

\[(xy)z^t + (-1)^{|y||z|+|y||t|+|z||t|}(xt)z^t + (-1)^{|x||y||z|+|x||t||z|+|t||z|}(yt)z^t\]

\[= (xy)(zt) + (-1)^{|y||z|}(xz)(yt) + (-1)^{|t||(|y||z|+|z|)|}(xt)(yz).\]

If \(A\) is an associative (super)algebra, then the new operation \(a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)\) defines a structure of a Jordan (super)algebra on \(A\). We will denote this Jordan (super)algebra as \(A^+\).

Similarly, the operation \([a, b] = ab - (-1)^{|a||b|}ba\) defines a Lie superalgebra \(A^-\).

A graded linear map \(\star : A \to A\) of an associative superalgebra is called a superinvolution if \((a\star)^\ast = a, (ab\star)^\ast = (-1)^{|a||b|}b^\ast a^\ast\). Then the set of symmetric elements \(H(A, \star)\) is a (Jordan) subsuperalgebra of \(A^+\). Similarly the set of skewsymmetric elements \(Skew(A, \star)\) is a Lie subsuperalgebra of \(A^-\).

Let \(I_n, I_m\) be the identity matrices, \(t\) the transposition and \(U = -U^t = -U^{-1} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}\). Then the mapping \(\star : M_{n+2m}(F) \to M_{n+2m}(F)\) defined as

\[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\ast = \begin{pmatrix} I_n & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & U^{-1} \end{pmatrix}\]

is a superinvolution.

The Jordan (resp. Lie) superalgebra of symmetric (resp. skewsymmetric) elements is called the Jordan (resp. Lie) orthosymplectic superalgebra and denoted \(Josp_{n, 2m}(F) = H(M_{n+2m}(F), \star)\) (resp. \(OSP_{n, 2m}(F) = Skew(M_{n+2m}(F), \star)\)).

The associative superalgebra \(M_{n+n}(F)\) has another superinvolution:

\[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}\]

The Jordan (resp. Lie) superalgebra of symmetric (resp. skewsymmetric) elements is denoted by \(JP_n(F)\) (resp. \(P_n(F)\)).

V. Kac [3] (see also I. Kantor [4]) classified simple finite dimensional Jordan superalgebras over an algebraically closed field \(F\) of zero characteristic. Simple finite dimensional Jordan superalgebras over fields of positive characteristics \(\neq 2\) were classified in [15] and [9].

If \(J\) is a Jordan (super)algebra, a Jordan bimodule \(V\) over \(J\) is a \(\mathbb{Z}/2\mathbb{Z}\)-graded vector space with operations \(V \times J \to V, J \times V \to V\) such that the split null extension \(V + J\) is a Jordan (super)algebra (see [1]). Recall that the split null extension is the direct sum of vector spaces \(V + J\) with the operation that extends the multiplication of \(J\) and the action of \(J\) on \(V\) while the product of two arbitrary elements in \(V\) is zero.

Given an arbitrary set \(X\), there is a unique free \(J\)-bimodule \(V(X)\) over the set of free generators \(X\). If \(V'\) is a \(J\)-bimodule, then an arbitrary map \(X \to V'\) uniquely extends to a homomorphism of bimodules \(V(X) \to V'\).

Let \(X\) be a set consisting of one element. For an element \(a \in J\) let \(R_{V(X)}(a)\) denote the multiplication operator \(R_{V(X)}(a) : V(X) \to V(X), v \mapsto va\).

The subalgebra \(U(J)\) of the algebra of all linear transformations of \(V(X)\) generated by the operators \(R_{V(X)}(a), a \in J\), is called the multiplicative enveloping algebra of \(J\).

Every Jordan bimodule over \(J\) is a right module over \(U(J)\) and vice versa.
In [1], N. Jacobson developed the representation theory of semisimple finite dimensional Jordan algebras. He proved that:

i) if $J$ is a finite dimensional Jordan algebra, then $\dim F U(J) < \infty$,

ii) if $J$ is a finite dimensional semisimple Jordan algebra, then $U(J)$ is semisimple as well. In particular, all bimodules over $J$ are completely reducible.

iii) Moreover, he determined all irreducible bimodules over simple finite dimensional Jordan algebras.

The representation theory for various types of simple Jordan superalgebras was developed in [8], [17], [13], [10], [11], [12] and [13]. For the current status of the project, see the survey [13].

In this paper we classify unital bimodules over Jordan superalgebras of the remaining type $Q(n)^{(+)}, n \geq 2$ and extend the results of [12] for $JP(n), n \geq 3$ to arbitrary characteristics $\neq 2$.

First, we adapt the arguments from [1] to obtain a Coordinatization theorem for Jordan superalgebras of capacity $\geq 3$. The latter condition is satisfied for the superalgebras $JP(n), Q(n)^{(+)}, n \geq 3$. Then we determine irreducible involutive alternative bimodules over the coordinate superalgebras of $JP(n), Q(n)^{(+)}, n \geq 3$. This yields the classification of unital irreducible bimodules over $JP(n), Q(n)^{(+)}, n \geq 3$. Recall that in [12] it was shown that the multiplicative enveloping algebra $U = U(J), J = JP(n), Q(n)^{(+)}, n \geq 3$, is a finite dimensional semisimple algebra; hence all Jordan bimodules over $J$ are completely reducible. The classification of irreducible finite dimensional Jordan bimodules over $JP(n)$ (including the case $n = 2$) is obtained in [12] by different methods, though only over fields of characteristic zero.

In order to tackle the case $J = Q(2)^{(+)}$ we had to change the point of view and to resort to the study of root-graded modules over Lie superalgebras (as in [12]). This imposes stronger assumptions on the characteristic of the ground field: char $F > 3$ or $= 0$.

We prove that $U(Q(n)^{(+)})$ is finite dimensional for all $n \geq 2$. If char $F > 3$ or $= 0$, then the only unital irreducible Jordan bimodules over $Q(2)^{(+)})$ are the 4 nonisomorphic matrix bimodules over the same involutive alternative bimodules as in the case $n \geq 3$. The algebra $U(Q(2)^{(+)})$ is semisimple; that is, all unital Jordan bimodules over $Q(2)^{(+)})$ are completely reducible.

1. The Coordinatization theorem

Let $J$ be a Jordan (super)algebra with an identity element 1. Let $e_1, \ldots, e_n \in J_0$ be pairwise orthogonal idempotents such that $\sum_{i=1}^{n} e_i = 1$. Then

$$J = \sum_{i \leq j} J_{ij},$$

where $J_{ij} = \{x \in J| xe_i = x\}, J_{ij} = \{x \in J| xe_i = xe_j = \frac{1}{2} x\}.$

It is easy to see [1] that $J_{ij} \subseteq J_{ii}, J_{ij}J_{ii} \subseteq J_{ij}, J_{ij}^2 \subseteq J_{ii} + J_{jj}, J_{ij}J_{jk} \subseteq J_{ik}, J_{ii}J_{jj} = J_{ij}J_{kk} = (0)$ for distinct $i, j, k$.

The idempotents $e_i, e_j, i \neq j$ are said to be strongly connected if there exists an element $a_{ij} \in J_{ij}$ such that $a_{ij}^2 = a_{ij} = e_i + e_j$. In this case denote $U_{(ij)} = U(a_{ij} + \sum_{k \neq i,j} e_k)$.

The following theorem is one of the cornerstones in the structure theory of Jordan algebras.
Theorem 1.1 ([1]). Let $J$ be a Jordan algebra with $1$, which is a sum of $n \geq 3$ strongly connected orthogonal idempotents, $1 = \sum_{i=1}^{n} e_i$, $a_{ij} \in J_{ij}$, $a_{ij}^2 = e_i + e_j$, $1 \leq i \neq j \leq n$.

(1) Consider the Peirce space $D = J_{12}$ with the multiplication $x \star y = 2x U_{(23)j}y U_{(13)}$. Then $D$ is an alternative algebra with the identity element $a_{12}$ and the involution $x \rightarrow \bar{x} = x U_{(12)}$. If $n \geq 4$, then $D$ is associative. The symmetric elements $\{x \in D | x = \bar{x}\}$ lie in the associative center of $D$.

(2) $J$ is isomorphic to the Jordan matrix algebra $H_n(D)$.

Our aim is to extend this theorem to Jordan superalgebras. Let $J = J_0 + J_1$ be a unital Jordan superalgebra, $1 = \sum_{i=1}^{n} e_i$, $n \geq 3$, the idempotents $e_1, \ldots, e_n$ are pairwise orthogonal and strongly connected in $J_0$: $a_{ij} \in (J_0)_{ij}$, $a_{ij}^2 = e_i + e_j$, $1 \leq i \neq j \leq n$. As above, consider the automorphisms $U_{ij} = U(a_{ij} + \sum_{k \neq i,j} e_k)$ of the superalgebra $J$. On the Peirce space $J_{12}$ define the multiplication

$$x \star y = 2x U_{(23)j}y U_{(13)}.$$  

It is easy to see that the Grassmann envelope of the superalgebra $D = (J_{12}, \star)$ is isomorphic to the Peirce subspace $G(J)_{12}$ with the operation $\star$. Part (1) of Jacobson’s theorem above implies that $D$ is an alternative superalgebra, where $x \rightarrow \bar{x} = x U_{(12)}$, $x \in D$ is a superinvolution. The symmetric elements lie in the associative center of $D$; if $n \geq 4$, then $D$ is associative.

In order to prove that $J \simeq H_n(D)$, let’s recall the isomorphism from part (2) of the Coordinatization theorem. Suppose that $J$ is a Jordan algebra. Following [1] we will define $1$-$1$ linear maps $\varphi_{ij}$ from the alternative algebra $D$ to all Peirce spaces $J_{ij}$, $1 \leq i \leq j \leq n$. Let $1 \leq i < j \leq n$. If $i = 1, j = 2$, then $\varphi_{12} = Id_D$. If $i = 1, j > 2$, then $\varphi_{1j} = U_{(2j)}$. If $i = 2$, then $\varphi_{2j} = U_{(1j)}$. Let $\varphi_{11} = 2R(a_{12})R(e_1)$, $\varphi_{ii} = \varphi_{11}U_{(11)}$ for $i > 1$.

Define the linear mapping $\varphi : H_n(D) \rightarrow J$ via $(x_{ij})_{n \times m} \rightarrow \sum_{i=1}^{n} \varphi_{ii}(x_{ii}) + \sum_{i<j} \varphi_{ij}(x_{ij})$. In [1] it is proved that $\varphi$ is an algebra isomorphism.

Now let’s come back to the Jordan superalgebra $J$ and define the linear mapping $\varphi : H_n(D) \rightarrow J$ as above. Applying Jacobson’s theorem to the Grassmann envelopes we see that $\varphi \otimes Id : H_n(G(D)) \rightarrow G(J)$ is an algebra isomorphism. This implies that $\varphi$ is an isomorphism as well.

A superinvolution $\sigma : A \rightarrow A$ in an alternative superalgebra is said to be nuclear if all symmetric elements lie in the associative center of $A$.

Let $V$ be a bimodule over $A$. A linear mapping $\tau : V \rightarrow V$ is a superinvolution of the bimodule $V$ if $\sigma + \tau$ is a superinvolution of the split extension $A + V$.

Let $A$ be an alternative superalgebra with a nuclear superinvolution (if $n = 3$) or an associative superalgebra with a superinvolution (if $n \geq 4$). Then the superalgebra of Hermitian matrices $H_n(A)$ is a Jordan superalgebra with $n$ strongly connected orthogonal idempotents.

Just as was done in [1], the Coordinatization theorem implies that the category of unital Jordan bimodules over $H_n(A)$ is equivalent to the category of alternative $A$-bimodules with a nuclear involution (if $n = 3$) or to the category of involutive associative bimodules (if $n \geq 4$).

2. Alternative bimodules

Let $A = (Fe + Fu) \oplus (Ff + Fv)$; $e^2 = e$, $eu = ue = u, u^2 = e; f^2 = f$, $fv = vf = v, v^2 = -f$. The algebra $A$ is $\mathbb{Z}/2\mathbb{Z}$-graded: $A_0 = Fe + Ff$, $A_1 = Fu + Fv$,
and thus is an associative superalgebra. The graded mapping \( \sigma(e) = f, \sigma(f) = e, \sigma(u) = v, \sigma(v) = u \) is a superinvolution. It is easy to see that \( H_n(A) \simeq Q(n)^{(+)} \).

Let \( B = M_{1+1}(F) \) with the superinvolution

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \xi
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\xi & -\beta \\
\gamma & \alpha
\end{pmatrix}.
\]

Then \( H_n(B) \simeq JP(n) \).

If \( V \) is a supermodule over a superalgebra \( A \) with a superinvolution \( \star \), a bijective linear map (that we will denote also \( \star \)), \( \star : V \rightarrow V \) is a superinvolution of \( V \) if the natural extension \( \star \) to \( A + V \) is a superinvolution of the split null extension \( A + V \).

Notice that if \( \star \) is a superinvolution of the supermodule \( V \), then \(-\star\) is a superinvolution as well.

Let \( V \) be an alternative bimodule over an alternative superalgebra \( C \) with a superinvolution \( \star : C \rightarrow C \). Consider another copy of the vector space \( V \), the 1-1 linear map \( ex : V \rightarrow V^{ex} \) and define the multiplication \( av^{ex} = (-1)^{|a||v|}(va^*)^{ex}, v^{ex}a = (-1)^{|a||v|}(a^*v)^{ex}; a \in C, v \in V \).

Then \( V^{ex} \) is an alternative bimodule over \( C \), and \( V \oplus V^{ex}, v + w^{ex} \rightarrow w + v^{ex} \) is a superinvolution in the bimodule \( V \oplus V^{ex} \).

**Lemma 2.1.** (1) An irreducible involutive bimodule over an alternative superalgebra with a superinvolution is either an irreducible bimodule or isomorphic to \( V \oplus V^{ex} \), where \( V \) is an irreducible bimodule.

(2) \( V \oplus V^{ex} \) is an irreducible involutive bimodule if and only if \( V \) is an irreducible bimodule, which does not have a superinvolution that is, \( V \not\cong V^{ex} \).

**Proof.** Part (1) is standard. Let us prove (2).

Suppose that \( \sigma : V \rightarrow V \) is a superinvolution in the bimodule \( V \). Then \( \tau : V \rightarrow V^{ex}, v \rightarrow (v^\sigma)^{ex} \) is an isomorphism of bimodules. In this case, \( \{v + v^\tau, v \in V\} \) is a proper involutive subbimodule of \( V \oplus V^{ex} \).

On the other hand, let \( V \) be an irreducible bimodule and let \( W \) be a proper involutive subbimodule of \( V \oplus V^{ex} \). Then \( W \cap V = W \cap V^{ex} = (0) \).

Let \( 0 \neq v_1 + v_2^{ex} \in W; v_1, v_2 \in V \). For an arbitrary multiplication operator \( P \) (by elements from the superalgebra), \( v_1P = 0 \) implies \( v_2^{ex}P = 0 \); otherwise \( 0 \neq (v_1 + v_2^{ex})P \in W \cap V^{ex} \). Hence \( v_1P \rightarrow v_2P \) is an isomorphism of the bimodules \( V \rightarrow V^{ex} \). The lemma is proved. \( \square \)

Let \( V = V_0 + V_1 \) be a bimodule over a superalgebra \( A \). Consider the bimodule \( V^{op} = V_1^{op} + V_0^{op} \), where the parity of the subspace \( V_i^{op} \) is different from \( i \) and the action of \( A \) is defined via \( av^{op} = (-1)^{|a||v|}(av)^{op}, v^{op}a = (va)^{op} \) for arbitrary \( a \in A, v \in V \). The bimodule \( V^{op} \) is called the opposite of the bimodule \( V \).

Let us proceed with the classification of alternative involutive unital bimodules with nuclear superinvolution over \( M_{1+1}(F) \) with

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \xi
\end{pmatrix} \rightarrow
\begin{pmatrix}
\xi & -\beta \\
\gamma & \alpha
\end{pmatrix}.
\]

N. A. Pisarenko [13] proved that every alternative unital bimodule over \( M_{1+1}(F) \) is associative and completely reducible and the only irreducible \( M_{1+1}(F) \)-bimodules are the regular bimodule \( \text{Reg}(M_{1+1}(F)) \) and its opposite.
It is not difficult to check that the regular bimodule \( \text{Reg}(M_{1+1}(F)) \) has two (up to isomorphism) superinvolutions, \( \ast \) and \( -\ast \). By Lemma 2.1 the only irreducible involutive bimodules over \( M_{1+1}(F) \) are \( \text{Reg}(M_{1+1}(F)) \) with the involution \( \ast \), \( \text{Reg}(M_{1+1}(F)) \) with the involution \( -\ast \) and their opposites. This implies the following.

**Theorem 2.2.** (1) Unital Jordan bimodules over \( JP(n) \), \( n \geq 3 \) are completely reducible.

(2) The only unital irreducible Jordan bimodules over \( JP(n) \), \( n \geq 3 \) are:

(i) the regular bimodule,

(ii) the matrix bimodule over \( \text{Reg}(M_{1+1}(F)) \) with the superinvolution \( -\ast \), which is isomorphic to the bimodule of skew-symmetric matrices in \( M_{n+n}(F) \) with respect to the superinvolution \( \sigma \) (see page 2),

(iii) the opposites of (i) and (ii).

In [2] this theorem was proved over fields of zero characteristic.

Now let us consider alternative bimodules over the involutive algebra \( A = (Fe + Fu) \oplus (Ff + Fv) \).

**Lemma 2.3.** If \( V \) is an alternative unital \( A \)-bimodule with a nuclear involution, then \( V \) is an associative bimodule.

**Proof.** Let \( V \neq (0) \) be an alternative unital \( A \)-bimodule. Let us show that the identity map cannot be a superinvolution in \( V \).

Suppose that \( Id_V \) is a superinvolution; that is, \( ax = (-1)^{|a||x|}xa^\sigma \) for arbitrary elements \( x \in V, a \in A \). Then \( eVe = (0) \). Indeed, for \( x \in eVe \) we have \( x = ex = xf = 0 \). Similarly, \( fVf = (0) \).

Consider the operator \( P : eVf \to eVf, x \to uxv \). Recall that, since the symmetric element \( u + v \) lies in the associative center of \( A + V \) it follows that \( (ux)v = u(xv) \). We have \( xP^2 = u(uxv)v = -exf = -x \). On the other hand \( (ux)v = (-1)^{|x|}(xv)v = (-1)^{|x|}x \) and therefore \( xP^2 = x \). Hence \( eVf = (0) \) and similarly \( fVf = (0) \).

Now let \( \ast : V \to V \) be a nuclear superinvolution in \( V \). Consider the subbimodule \( V' \) of \( V \) generated by all symmetric elements \( x + x^\ast, x \in V \). Then \( -Id_{V'/V} \) is a superinvolution; hence \( V'/V = (0) \).

Hence the bimodule \( V \) is generated by symmetric elements \( x + x^\ast, x \in V \), which lie in the associative center of \( A + V \). This implies that \( V \) is an associative bimodule. The lemma is proved. \( \square \)

It is well known that associative bimodules over a separable finite dimensional associative superalgebra are completely reducible.

Let us first determine irreducible unital associative bimodules \( V \) over the superalgebra \( Fe + Fu \). Consider the operator \( P : V \to V, x \to uxx; P^2 = Id_V \). Hence \( V = V(1) \oplus V(-1), V(i) = \{ x \in V | P(x) = ix \} \). Since the decomposition above is again a direct sum of subbimodules it follows that \( V = V(i), i = \pm 1 \). If \( 0 \neq x \in V_0 \), then \( x, ux \) is a base of \( V \) with a clearly defined action of \( A \). We will denote these two nonisomorphic 2-dimensional bimodules as \( V(i), i = \pm 1 \). Clearly \( V(-1) \) is isomorphic to \( V(1)^{op} \).

Now we will proceed with the classification of irreducible involutive unital associative \( A \)-bimodules.
Let $V = V_0 + V_1$ be such a bimodule. Since $V = eVe + eVf + fVe + fVf$ is a direct sum of $A$-subbimodules and $(eVe)^* = fVf$, $(fVf)^* = eVe$, $(eVf)^* = eVf$, $(fVe)^* = fVe$ it follows that $V = eVe + fVf$ or $V = eVf$ or $V = fVe$.

**Case 1.** $V = eVe + fVf$.

It is easy to see that in this case $eVe$ is an irreducible unital bimodule over $Fe + Fu$. Hence $eVe \simeq V(1)$ or $eVf \simeq V(-1)$ and $V \simeq V(1) \oplus V(1)^{ez}$ or $V \simeq V(-1) \oplus V(-1)^{ez}$. These two bimodules are the opposites.

**Case 2.** $V = eVf$.

Let us show that $V_0$ has a nonzero symmetric element. Indeed, otherwise $x^* = -x$ for an arbitrary $x \in V_0$. Then $(uxv)^* = -v^*ux^* = uvx = 0$. Since $u^2 = e$, $v^2 = -f$, this implies that $x = 0$, a contradiction. So, choose $0 \neq x \in V_0$, $x = x^*$. As we have seen above, $(uxv)^* = -uxv$ in this case; hence the elements $x, uxv$ are linearly independent. Multiplying both elements by the invertible element $u$ on the left, we conclude that the odd elements $ux, xv$ are also linearly independent. We have $(ux)^* = xv$. Hence $x, ux, uxv$ span an involutive $A$-bimodule. Hence $V = Fx + Fuxv + Fu + Fxv$.

**Case 3.** $V = fVe$.

As in the previous case we can choose $0 \neq x \in V_0$, $x = x^*$. Hence $V = Fx + Fux + Fu + Fxv$.

**Theorem 2.4.** (1) Unital Jordan bimodules over $Q(n)^{(+)}$, $n \geq 3$ are completely reducible.

(2) The only unital irreducible Jordan bimodules over $Q(n)^{(+)}$, $n \geq 3$ are the bimodules of Hermitian $n \times n$ matrices over the four irreducible involutive $A$-bimodules above. The bimodules of the cases 2, 3 are isomorphic to their opposite bimodules.

**Remark.** The four irreducible unital Jordan $Q(n)^{(+)0}$ bimodules above have a different presentation. The first two of them come from the associative $Q(n)$-bimodules $M_n(V(\pm 1))$. If $\sqrt{-1} \in F$, then the second two Jordan bimodules are the same matrix modules $M_n(V(\pm 1))$ but with a “twisted” action. The mapping $*: A \to A$, $(ae + \beta u)^* = ae + \sqrt{-1}\beta u$ is a pseudoinvolution (see [12]). It extends to a pseudoinvolution $*: Q(n) \to Q(n)$, $(a_{ij}) \to (\overline{a}_{ji})$. Define the action of $Q(n)^{(+)0}$ on $M_n(V(\pm 1))$ via $a \cdot x = \frac{1}{2}(ax + (-1)^{|a||x|}xa^*)$, $a \in Q(n)$, $x \in M_n(V(\pm 1))$.

3. **Multiplicative enveloping algebra of $Q(2)^{(+)}$**

In [12] it was shown that the multiplicative enveloping algebra $U(J)$ of a finite dimensional simple Jordan superalgebra, containing 3 orthogonal idempotents in its even part, is finite dimensional. The latter assumption is essential as $U(D_1)$ and $U(JP(2))$, for example, are infinite dimensional (see [10]). In this chapter we prove, however, that $U(Q(2)^{(+)0})$ is finite dimensional.

**Theorem 3.1.** $\dim U(Q(2)^{(+)}0) < \infty$.

**Proof.** As in the introduction, we consider the one-generator free unital module $V$ over $J = Q(2)^{(+)0}$ and denote $R(a) = R_V(a)$, the right multiplication operator. The multiplicative enveloping algebra $U$ is generated by the subspace $R(J)$. The algebra $U$ acts on any bimodule over $J$, including $J$ itself.
Denote $D(x, y) = R(x)R(y) - (-1)^{|x||y|}R(y)R(x)$.

We will need the following well-known identities (see [1], [20]).

1. $R(x)R(y)R(z) + (-1)^{|y||z|+|x||z|}R(z)R(y)R(x) + (-1)^{|y||z|}R((xz)y) = R(xy)R(z) + (-1)^{|z||y|}R(xz)R(y) + (-1)^{|z||y|+|z||z|}R(yz)R(x)$,

2. $D(x, y)$ acts on $J$ as a superderivation,

3. $D(x, y, z) = D(x, yz) + (-1)^{|z||y|}D(y, xz)$,

4. $R(x)R(y)R(z) = \frac{1}{2}((-1)^{|y||z|}R((xz)y) + R(xy)R(z) + (-1)^{|z||y|}R(xz)R(y) + (-1)^{|z||y|+|z||z|}R(yz)R(x) + R(x)D(y, z) + (-1)^{|z||y|}D(x, z)R(y) + (-1)^{|z||y|+|z||y|}R(z)D(y, x))$.

We say that an operator $R(a_1) \cdots R(a_k)$, $a_i \in J_0 \cup J_1$ is irreducible if it does not lie in $\sum_{i=1}^{k-1} R(J) \cdots R(J)$.

**Step 1** (N. Jacobson, [1]). If $a_i \in J_0$, $1 \leq i \leq k$ and $R(a_1) \cdots R(a_k)$ is irreducible, then $k \leq 8$. Indeed, by the identity (1), the element

$$R(a_1) \cdots R(a_k) + \sum_{i=1}^{k-1} R(J) \cdots R(J) \in \sum_{i=1}^{k} R(J) \cdots R(J) / \sum_{i=1}^{k-1} R(J) \cdots R(J)$$

is skew-symmetric in $a_1, a_3, a_5, \ldots$. This implies the claim.

**Step 2.** Suppose that $a_i \in J_0 \cup J_1$, and the operator $R(a_1) \cdots R(a_k)$ is irreducible. Then $\{|i| \leq i \leq k, a_i \in J_0\} = 12$.

If $a_i, a_{i+1} \in J_0$, then “push” them to the left via the Jordan identity (4). If $a_i, a_{i+1} \in J_1$ then “push” them to the right via the Jordan identity.

We will get

$$R(a_1) \cdots R(a_k) \in \sum_{i=1}^{k} R(b_1) \cdots R(b_t) \prod_{i=1}^{t} R(x_i)R(c_i)R(z_1) \cdots R(z_s) + \sum_{i=1}^{k-1} R(J) \cdots R(J)$$

and for each summand $r + 2t + s = k$; $b_1, \ldots, b_r, c_1, \ldots, c_t \in J_0; x_1, \ldots, x_t, z_1, \ldots, z_s \in J_1$ and $b_1, \ldots, b_r, x_1, \ldots, x_t, c_1, \ldots, c_t, z_1, \ldots, z_s$ is a permutation of $a_1, \ldots, a_k$.

The expression $\prod_{i=1}^{t} R(x_i)R(c_i)$ is skew-symmetric in $c_1, \ldots, c_t$ modulo $\sum_{j=1}^{2t-1} R(J) \cdots R(J)$. Hence $t \leq 4$. By Step 1, $r \leq 8$. This implies the assertion.

We will denote an even element $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in J_0$ as $a$ and an odd element $\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \in J_1$ as $b$, where $a, b \in M_2(F)$.

**Step 3.** $D(\bar{e}_{12}, \bar{e}_{12}) = 2D(\bar{e}_{11} \cdot \bar{e}_{12}, \bar{e}_{12}) = 2D(\bar{e}_{11}, \bar{e}_{12} \cdot \bar{e}_{12}) + 2D(\bar{e}_{12}, \bar{e}_{11} \cdot \bar{e}_{12}) = 0$.

Similarly, $D(\bar{e}_{21}, \bar{e}_{21}) = 0$.

Furthermore, $D(\bar{e}_{11}, \bar{e}_{12}) = 2D(\bar{e}_{11}, \bar{e}_{12} \cdot \bar{e}_{22}) = D(\bar{e}_{12}, \bar{e}_{22}) \in D(J_0, J_0)$.

Similarly, $D(\bar{e}_{ji}, \bar{e}_{jk}) \in D(J_0, J_0)$, where $1 \leq j \neq k \leq 2$, $1 \leq i \leq 2$.

Finally, $D(\bar{e}_{12}, \bar{e}_{21}) = 2D(\bar{e}_{11} \cdot \bar{e}_{12}, \bar{e}_{21}) = 2D(\bar{e}_{11}, \bar{e}_{12} \cdot \bar{e}_{21}) + 2D(\bar{e}_{12}, \bar{e}_{11} \cdot \bar{e}_{21}) = D(\bar{e}_{11}, \bar{e}_{11} + \bar{e}_{22}) - D(\bar{e}_{12}, \bar{e}_{21}) = D(\bar{e}_{11}, \bar{e}_{11}) - D(\bar{e}_{12}, \bar{e}_{21})$.

Similarly, $D(\bar{e}_{12}, \bar{e}_{21}) = D(\bar{e}_{22}, \bar{e}_{22}) + D(\bar{e}_{12}, \bar{e}_{21})$. 


We have proved that
\[ D(J_1, J_1) \subseteq FD(\overline{e}_{11}, \overline{e}_{11}) + D(J_0, J_0) = FD(\overline{e}_{22}, \overline{e}_{22}) + D(J_0, J_0) = FD(\overline{e}_{12}, \overline{e}_{21}) + D(J_0, J_0). \]

**Step 4.** In view of the identities (1), (2) and (3) it is sufficient to bound the length of irreducible operators of the type
\[ U = R(a_1) \cdots R(a_n) \prod_{i=1}^r R(x_i) R(b_i) R(y_1) \cdots R(y_{\nu})(\prod_{i=1}^\mu D(z_i, u_i)), \]
where \( a_1, \ldots, a_n, b_1, \ldots, b_t \in J_0; \ x_1, \ldots, x_t, y_1, \ldots, y_{\nu}, z_1, \ldots, z_{\mu}, u_1, \ldots, u_{\mu} \in J_1, \ r \leq 8, \ t \leq 4 \) and \( \nu \leq 2. \)

**Step 5.** For even elements \( a, b \) of \( J_0 \) we denote \( U(a) = 2R(a)^2 - R(a^2), \ U(a, b) = R(a)R(b) + R(b)R(a) - R(ab). \) Since \( V \) is a unital module it follows that \( ID_V = U(e_11 + e_22) = U(e_11) + U(e_22) + U(e_{11}, e_{22}). \)

We claim that \( U(e_{11}) U(J) \subseteq U(e_{11}) \sum_{i=0}^{18} \left[ \prod_{i=0}^i R(J) \right]. \)

Indeed, in the multiplication operator above \( D(z_i, u_i) \) can be moved to the left modulo shorter operators. By step 3, \( U(e_{11}) D(z_i, u_i) \in U(e_{11}) (FD(\overline{e}_{22}, \overline{e}_{22}) + D(J_0, J_0)) \subseteq U(e_{11}) D(J_0, J_0). \)

In this way we can get rid of all the derivations \( D(z_i, u_i), 1 \leq i \leq \mu. \)

Similarly, \( U(e_{22}) U(J) \subseteq U(e_{22}) \sum_{i=0}^{18} \left[ \prod_{i=0}^i R(J) \right]. \)

Finally, \( U(e_{11}, e_{22}) D(z_i, u_i) \in U(e_{11}, e_{22}) (FD(\overline{e}_{12}, \overline{e}_{21}) + D(J_0, J_0)), U(e_{11}, e_{22}), D(\overline{e}_{12}, \overline{e}_{21}) = U(e_{11}, e_{22}) D(\overline{e}_{12}, \overline{e}_{21}) (U(e_{11}) + U(e_{22})). \)

Hence
\[ U(e_{11}, e_{22}) U(J) \subseteq U(e_{11}, e_{22}) \sum_{i=0}^{18} \left[ \prod_{i=0}^i R(J) \right] + U(e_{11}, e_{22}) D(\overline{e}_{12}, \overline{e}_{21}) U(e_{11}) \sum_{i=0}^{18} \left[ \prod_{i=0}^i R(J) \right] + U(e_{11}, e_{22}) D(\overline{e}_{12}, \overline{e}_{21}) U(e_{22}) \sum_{i=0}^{18} \left[ \prod_{i=0}^i R(J) \right]. \]

We have that \( \dim \sum_{i=0}^{18} \left[ \prod_{i=0}^i R(J) \right] < 1 + 8 + \cdots + 8^{18} < 8^{19}. \) Hence \( \dim U(J) < 5.8^{19}. \) The theorem is proved.

### 4. General Facts

Let us recall some constructions relating Lie and Jordan algebras.

**Definition 4.1** ([7]). A Jordan (super)pair \( P = (P^-, P^+) \) is a pair of vector (super)spaces with a pair of trilinear operations\n\[ \{, , \} : P^- \times P^+ \times P^- \to P^-, \{, , \} : P^+ \times P^- \times P^+ \to P^+ \]
that satisfies the following identities:
\[ (P.1) \{ x^\sigma, y^-^\sigma, \{ x^\sigma, z^-^\sigma, x^\sigma \} \} = \{ x^\sigma, \{ y^-^\sigma, x^\sigma, z^-^\sigma \}, x^\sigma \}, \]
\[ (P.2) \{ x^\sigma, y^-^\sigma, x^\sigma \}, y^-^\sigma, u^\sigma \} = \{ x^\sigma, \{ y^-^\sigma, x^\sigma, y^-^\sigma \}, u^\sigma \}, \]
\[ (P.3) \{ x^\sigma, y^-^\sigma, x^\sigma \}, z^-^\sigma, \{ x^\sigma, y^-^\sigma, x^\sigma \} \} = \{ x^\sigma, \{ y^-^\sigma, x^\sigma, z^-^\sigma, x^\sigma \}, y^-^\sigma \}, x^\sigma \}, \]
for every \( x^\sigma, u^\sigma \in P^+, y^-^\sigma, z^-^\sigma \in P^-^\sigma, \sigma = \pm. \)
Let $L = L_{-1} + L_0 + L_1$ be a $\mathbb{Z}$-graded Lie (super)algebra. Then $(L_{-1}, L_1)$ is a Jordan (super)pair.

For an arbitrary Jordan (super)pair $P = (P^-, P^+)$, there exists a unique $\mathbb{Z}$-graded Lie (super)algebra $K = K_{-1} + K_0 + K_1$ such that $(K_{-1}, K_1) \simeq P$, $K_0 = [K_{-1}, K_1]$ and for every 3-graded Lie (super)algebra $L = L_{-1} + L_0 + L_1$, an arbitrary homomorphism of the Jordan pairs $P \to (L_{-1}, L_1)$ uniquely extends to a homomorphism of Lie (super)algebras $K \to L$.

We will refer to $K = K(P)$ as the Tits-Kantor-Koecher (in short TKK) construction of the pair $P$.

If $J$ is a Jordan superalgebra, then $(J^-, J^+)$ is a Jordan superpair. The Lie superalgebra $K = K(J^-, J^+)$ is called the TKK-construction of $J$.

Let $J = J_0 + J_1$ be a simple finite dimensional Jordan superalgebra. Let us consider $L = K(J)$ its TKK-construction.

If $V$ is a Jordan bimodule over $J$, then the null extension $V + J$ is a Jordan superalgebra, so we can consider its TKK Lie superalgebra $K(V + J) = (V^- + J^-) + [V^-, J^+] + [V^-, J^+] + [V^+, J^+] + V^+$. Denote $K(V) = V^- + [V^-, J^+] + [J^-, J^+] + J^+$ which is isomorphic to $K(J)$. Let $W$ be the maximal $(J)$-submodule, which is contained in $K(V)_0 = [V^-, J^+] + [J^-, J^+]$. Let $\tilde{K}(V) = K(V)/W$.

The following two lemmas were proved in [12].

**Lemma 4.2 ([12]).** Let $J$ be a unital Jordan (super)algebra and let $V_1, V_2$ be two unital Jordan $J$-bimodules. The following assertions are equivalent:

1. $V_1 \simeq V_2$,
2. $K(V_1) \simeq K(V_2)$,
3. $\tilde{K}(V_1) \simeq \tilde{K}(V_2)$.

**Lemma 4.3 ([12]).** For a unital Jordan bimodule $V$ over a unital Jordan (super)algebra $J$, the following assertions are equivalent:

1. $V$ is an irreducible $J$-bimodule,
2. $\tilde{K}(V)$ is an irreducible $K(J)$-module.

The Tits-Kantor-Koecher Lie superalgebra of $J = Q(2)^{(1)}$ is the Lie superalgebra $L = \{(a, b) \mid a, b \in M_4(F), \text{tr}(b) = 0\} = [Q(4)^-, Q(4)^-]$.

We will denote the element $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ as $a$ and the element $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$ as $\bar{b}$.

Let $H = \{\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mid \sum_{i=1}^4 \alpha_i = 0\}$ be a Cartan subalgebra of $[L_0, L_0]$. Let $\Lambda = \bigoplus_{i=1}^4 \mathbb{Z} w_i / \mathbb{Z}(w_1 + \cdots + w_4)$ be the free abelian group of rank 3. The associative algebra $M_4(F)$ is $\Lambda$-graded with $\deg(e_{ij}) = w_j - w_i + \mathbb{Z}(w_1 + \cdots + w_4)$, $1 \leq i, j \leq 4$. This gradation induces a $\Lambda$-gradation of the Lie superalgebra $L$.

Clearly, $L_0 = \{a + \bar{b} \mid a, b \text{ is diagonal and } \text{tr}(b) = 0\}$.

An arbitrary element $\lambda = \sum_{i=1}^4 \lambda_i w_i + \mathbb{Z}(w_1 + \cdots + w_4)$ induces a functional on $H$. If $h = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\sum_{i=1}^4 \alpha_i = 0$, we let $\langle \lambda, h \rangle = \sum_{i=1}^4 \lambda_i \alpha_i$. Thus, $[a, h] = \langle \lambda, h \rangle a$ for elements $a \in L_\lambda$, $h \in H$.

Let $\{V_i\}$ be the family of the four finite dimensional irreducible unitil bimodules over $J_0 = M_2(F)^{+}$. Consider the modules $\{K(V_i)_0\}$ over $K(J_0) = sl(3)$. From the description of the modules $K(V_i)$ (see [1], [12]) it follows that the $\Lambda$-gradation can
be extended to those modules, $K(V_i) = \sum_{\lambda \in \Lambda} K(V_i)_{\lambda}$ and for arbitrary elements $a \in K(V_i)$, $h \in H$ we have $ah = (a,h)a$.

Let $\Delta = \{0 \neq w_i \pm w_j + Z(w_1 + \cdots + w_4), 1 \leq i,j \leq 4\}$.

In [12] it was shown that $K(V_i) = \sum_{\lambda \in \Delta \cup \{0\}} K(V_i)_{\lambda}$.

**Lemma 4.4.** Let $\alpha, \beta \in \{0 \neq w_i \pm w_j\}$.

1. If $(\alpha,h) = (\beta,h)$ for all $h \in H$, then $\alpha - \beta \in Z(w_1 + \cdots + w_4)$. 
2. If $\langle w_i - w_j + \alpha + \beta, h \rangle = 0$ for all $h \in H$, then $w_i - w_j + \alpha + \beta + Z(w_1 + \cdots + w_4) = 0$ in $\Lambda$.

**Proof.** The assertion (1) is obvious. Let’s prove (2). We have $w_i - w_j + \alpha + \beta = \sum \mu k_\mu w_\mu$, $\sum \mu k_\mu$ is even, $\sum \mu |k_\mu| \leq 6$.

Suppose at first that at least one $k_\mu$ is equal to zero. Let $k_4 = 0$. Then $\sum \mu k_\mu \alpha_\mu = 0$ for all $\alpha_1, \alpha_2, \alpha_3 \in F$. Hence $k_1, k_2, k_3$ are divisible by $p = \text{char} F$.

If $p \geq 7$, then $k_1 = k_2 = k_3 = 0$ since $\sum \mu |k_\mu| \leq 6$.

If $p = 5$, then at most one $k_\mu, 1 \leq \mu \leq 3$ is not equal to zero and equal to $\pm 5$. This contradicts the fact that $\sum \mu k_\mu$ is even.

From now on we will assume that all $k_\mu$ are different from zero. Suppose that at least one of them is equal to $\pm 1$. Without loss of generality we can assume that $k_4 = -1$. Then $\langle \sum k_\mu (w_\mu + w_4), h \rangle = 0$. Hence $k_1, k_2, k_3 + 1$ are divisible by $p$.

If $p \geq 7$, then at most one of $k_\mu + 1$ is not equal to zero. In this case $p = 7$, $k_\mu = 6, k_\mu = -1$ for $\nu \neq \mu, 1 \leq \nu \leq 3$.

Then, $\sum \mu k_\mu = 3$, an odd number.

Hence $k_1 = k_2 = k_3 = k_4 = -1$, which means that $\sum k_\mu w_\mu + Z(w_1 + \cdots + w_4) = 0$ in $\Lambda$.

Let $p = 5$. If $k_1 = 1 = \pm 5, k_2 + 1 = \pm 5$, then $|k_1| + |k_2| > 6$.

If $k_1 = 1 = \pm 5, k_2 = k_3 = k_4 = -1$, then again $\sum \mu |k_\mu| \geq 7$.

Hence $k_1 = k_2 = k_3 = k_4 = -1$ and again $\sum \mu k_\mu w_\mu + Z(w_1 + \cdots + w_4) = 0$ in $\Lambda$.

Finally if $|k_\mu| \geq 2$ for all $\mu$, then $\sum |k_\mu| \geq 8$, a contradiction. The lemma is proved.

**Remark.** If $p = 3$, then $\alpha = \beta = w_i - w_j$ and $\alpha = w_i - w_k, \beta = 2w_1$, where $i, j, k, l$ are distinct, are counterexamples to the assertion (2).

Let $V$ be a unital Jordan bimodule over $J = Q(2)^+$. Then $V$ is a direct sum of irreducible bimodules over $J_0 = M_2(F)^+$. This defines the decomposition $K(V) = \sum_{\lambda \in \{0\} \cup \Delta} K(V)_{\lambda}$. By Lemma 4.4(1), each nonzero $K(V)_{\lambda}$ is an eigenspace with respect to the action of $H$.

From Lemma 4.4(2) it follows that $K(V)_{\lambda}L_{\alpha} \subseteq K(V)_{\lambda + \alpha}$ for any $\alpha \in \{w_i - w_j, 1 \leq i,j \leq 4\}$. Indeed, each nonzero vector from $K(V)_{\lambda}L_{\alpha}$ is an eigenvector with respect to the action of $h$, which belongs to the eigenfunctional $h \rightarrow (\lambda + \alpha, h)$. Hence there exists $\beta \in \{0\} \cup \Delta$ such that $K(V)_{\beta} \neq 0$ and $\langle \lambda + \alpha, h \rangle = (\beta, h)$ for all $h \in H$. By Lemma 4.3(1), $\lambda + \alpha = \beta$.

We have proved 4.4.

**Lemma 4.5.** The decomposition $K(V) = \sum_{\lambda \in \{0\} \cup \Delta} K(V)_{\lambda}$ makes $K(V)$ a $\Lambda$-graded $L$-module.

Consider a functional $f : \bigoplus_{i=1}^4 Z w_i \rightarrow Z$ such that $f(w_1 + \cdots + w_4) = 0$ and all $\pm f(w_i)$ are distinct. For example, $f(w_1) = 4, f(w_2) = -3, f(w_3) = 1, f(w_4) = -2$. 

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Let $\Delta_+ = \{ \gamma \in \Delta | f(\gamma) > 0 \}$, $\Delta_- = \{ \gamma \in \Delta | f(\gamma) < 0 \}$, $L_+ = \sum_{\gamma \in \Delta_+} L_{\gamma}$, $L_- = \sum_{\gamma \in \Delta_-} L_{\gamma}$, $L = L_- + L_0 + L_+$. Let $M$ be an irreducible module over $L_0$. From $[(L_0)_{\bar{1}}, (L_0)_{\bar{1}}] = (L_0)_{\bar{0}}$ it follows that $M_{\bar{0}} \neq (0)$.

**Lemma 4.6.** For an arbitrary $\lambda \in \Delta$ there exists at most one irreducible $\Lambda$-graded module $V$ over the Lie superalgebra $L$, such that $V = V_0 + \sum_{\alpha \in \Delta} V_{\alpha}$, $V_{\bar{0}} \neq (0)$, $V_{\bar{1}}L_+ = (0)$ and the $L_0$-module $V_{\lambda}$ is isomorphic to $M$.

**Proof.** Choose a nonzero element $x \in M_{\bar{0}}$ and consider the right ideal $I = \{ a \in U(L_0) | xa = 0 \}$ of $U(L_0)$, $M \cong U(L_0)/I$.

The $\Lambda$-gradation on $L$ extends to the $\Lambda$-gradation on $U(L)$.

Consider the free one-generated $U(L)$-module $W = wU(L)$. Assigning the degree $\lambda$ to $w$ we make $W$ a $\Lambda$-graded module. Let $W'$ be the submodule of $W$ generated by $wI$, $wL_+$ and $\sum_{\alpha \in \Delta} w_\alpha$. Let $W = W/W'$. Since the $L_0$-module $W_{\lambda}$ is a homomorphic image of $M$ it follows that either $W_{\lambda} = (0)$, in which case the module of the lemma does not exist, or $W_{\lambda} \simeq M$. In the latter case, $W$ has a unique proper submodule, which implies the lemma.  

We say that a $\Lambda$-graded $L$-module $V$ is $\Delta$-graded if $V = \sum_{\alpha \in (0) \cup \Delta} V_{\alpha}$ and $V$ is generated by $V = \sum_{\alpha \in \Delta} V_{\alpha}$.

If $\lambda \in \Delta$, $V_{\lambda} \neq (0)$, $V_{\lambda}L_+ = (0)$ and $V_{\lambda}$ generates $V$, then we say that $\lambda$ is the highest weight of the $\Delta$-graded module $V$.

**Lemma 4.7.** Only $2w_1, w_1 - w_2, -2w_2$ can be highest weights of a $\Delta$-graded $L$-module.

**Proof.** Let $V$ be a $\Delta$-graded $L$-module. Suppose that $V_{2w_1} = V_{w_1 - w_2} = V_{-2w_2} = (0)$. Since $V_{L^{03}w_i - w_j} = (0)$, $1 \leq i \neq j \leq 4$ and $\text{char} F \geq 5$, it follows that the Weyl group acts on $V$ permuting weight spaces. This implies that $V_{2w_i} = V_{w_i - w_j} = V_{-2w_i} = 0$ for all $1 \leq i \neq j \leq 4$.

We have $V_{w_1 + w_2} e_{12} \subseteq V_{w_2} = (0)$, $V_{w_1 + w_2} e_{21} \subseteq V_{2w_1} = (0)$.

Hence $V_{w_1 + w_2} [e_{12}, e_{21}] = V_{w_1 + w_2} (e_{11} + e_{22}) = (0)$. On the other hand, $V_{w_1 + w_2} (e_{11} - e_{22}) = V_{w_1 + w_2} (e_{11} + e_{22}) = (0)$.

We also have $V_{w_1 + w_2} e_{34} \subseteq V_{w_1 + w_2 - w_4} = V_{-2w_2} = (0)$, $V_{w_1 + w_2} e_{43} \subseteq V_{-2w_4} = (0)$; hence $V_{w_1 + w_2} (e_{33} + e_{44}) = (0)$. On the other hand, $V_{w_1 + w_2} (e_{33} - e_{44}) = (0)$, which implies $V_{w_1 + w_2} e_{ij} = 0$, $1 \leq i \leq 4$. However, for an arbitrary element $v \in V_{w_1 + w_2}$ we have $v(e_{11} - e_{33}) = v$. Hence $V_{w_1 + w_2} = V_{w_1 + w_j} = (0)$, $1 \leq i \neq j \leq 4$.

Similarly, $V_{w_1 - w_2} = (0)$, $1 \leq i \neq j \leq 4$. This contradicts the assumption that $V$ is generated by $\sum_{\alpha \in \Delta} V_{\alpha}$. The lemma is proved.

Denote

$$ z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L_0 $$

a central element. Clearly, $L_0 = H + Fz + \bar{H}$.

Let $V$ be a $\Delta$-graded $L$-module of the highest weight $2w_1$. Let $2 \leq i \neq j \leq 4$. Then $V_{2w_1} (e_{ii} - e_{jj}) = (0)$, $V_{2w_1} e_{ij} = V_{2w_1} e_{ji} = (0)$; hence $V_{2w_1} (e_{ii} + e_{jj}) = (0)$.  


This implies $V_{2w_i}e_{ii} = (0)$. On the other hand, for an arbitrary element $v \in V_{2w_i}$ we have $v(e_{11} - e_{22}) = 2v$. Hence $ve_{ii} = 2\delta_{i1}v$, $1 \leq i \leq 4$.

The element $z$ acts on $V$ as the multiplication by $2$. Again, if $2 \leq i \neq j \leq 4$, then $v_{2w_i}e_{ij} = v_{2w_i}e_{ji} = (0)$; hence $v_{2w_i}(e_{ii} - e_{jj}) = (0)$.

Denote $x = e_{11} - e_{22}$. Then $x^2 = \frac{1}{2}(e_{11} + e_{22})$, $vx^2 = v$ for $v \in L_{2w_1}$. Thus, the even and the odd parts of $V_{2w_1}$ can be identified, $V_{2w_1} = (V_{2w_1})_0 + (V_{2w_1})_o x$.

If $\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (L_0)_1$, $\sum_{i=1}^4 \alpha_i = 0$, then $\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 x + \alpha_3 e_{33} - e_{22} + \alpha_4 e_{44} - e_{22}$.

If $v_1, \ldots, v_r$ is a base of $(V_{2w_1})_0$, then the $L_0$-module $V_{2w_1}$ is a direct sum of $r$ isomorphic irreducible $2$-dimensional $L_0$-modules, $V_{2w_1} = \bigoplus_{i=1}^r (Fv_i + Fv_i x)$.

Now suppose that $V$ is a $\Delta$-graded $L$-module such that $V_{2w_1} = (0)$, $1 \leq i \leq 4$, but $V_{w_1-w_2} \neq (0)$.

Then for an arbitrary element $v \in V_{w_1-w_2}$ we have $v(e_{11} - e_{22}) = 2v$. Arguing as above we get $ve_{12} = ve_{21} = 0$, which implies $v(e_{11} + e_{22}) = 0$. Hence $ve_{11} = v$, $ve_{22} = -v$.

For $3 \leq i \neq j \leq 4$ we have $v(e_{ii} - e_{jj}) = v(e_{ii} + e_{jj}) = 0$; hence $ve_{ii} = 0$.

In this case $Vz = (0)$.

From $V_{w_1-w_2}[e_{34}, e_{43}] = (0)$ we deduce that $V_{w_1-w_2}e_{34} - e_{44} = (0)$.

Denote $x = e_{11} - e_{33}$, $y = e_{22} - e_{44}$. Then, for an arbitrary element $v \in V_{w_1-w_2}$ we have $vx^2 = \frac{1}{2}v$, $vy^2 = -\frac{1}{2}v$, $v(xy + yx) = 0$.

Consider the operator $\varphi : V_{w_1-w_2} \to V_{w_1-w_2}$, $\varphi(v) = (vx)y$. Then $\varphi^2(v) = \frac{1}{4}v$. The decomposition $V_{w_1-w_2} = V_{w_1-w_2}[\frac{1}{2}] \oplus V_{w_1-w_2}[-\frac{1}{2}]$, where $V_{w_1-w_2}(i) = \{v \in V_{w_1-w_2} | \varphi(v) = iv \}$ is a direct sum of $L_0$-modules.

Each summand $V_{w_1-w_2}(i)$ is a direct sum of isomorphic copies of the irreducible $2$-dimensional $L_0$-modules $Fv + Fvx$, the element $v$ is even, $vy = ivx$, $(vx)y = iv = iv, i = \pm \frac{1}{2}$.

If $V_{2w_i} = V_{w_1-w_2} = (0)$, $1 \leq i \neq j \leq 4$, then arguing as above we can show that $z$ acts on $V$ as multiplication by $-2$ and $V_{-2w_2}$ is a direct sum of isomorphic copies of a uniquely determined irreducible $2$-dimensional module over $L_0$.

Recall that for all highest weights $\gamma$ the irreducible components of the bimodule $V_{\gamma}$ are isomorphic to their opposites.

Now we are ready to classify irreducible unital Jordan bimodules over $J = Q(2)^+$. Let $V$ be such a bimodule. Then $K(V)$ is an irreducible $\Delta$-graded module over the Lie superalgebra $L$. Let $\lambda \in \Lambda$ be the highest weight of $K(V)$.

The $L_0$-module $\overline{K(V)}_\lambda$ is irreducible. If $\lambda = 2w_1$ or $-2w_2$, then the $L_0$-module $\overline{K(V)}_\lambda$ is uniquely determined. If $\lambda = w_1 - w_2$, then there are two possibilities for the $L_0$-module $\overline{K(V)}_\lambda$. By Lemma 4.7 there are at most 4 possibilities for the module $K(V)$; hence, by Lemma 4.2, there are at most four nonisomorphic bimodules over $J$, all of them isomorphic to their opposites. The $4$ Hermitian $2 \times 2$ matrices over the $4$ nonisomorphic irreducible involutive associative bimodules over the algebra $A = (Fe + Fu) \oplus (Ff + Fv)$ provide these bimodules. We proved the following theorem:

**Theorem 4.8.** Let $\text{char}F > 3$. Then an arbitrary irreducible unital bimodule over $Q(2)^+$ is isomorphic to the bimodule of Hermitian $2 \times 2$ matrices over one of the $4$ irreducible involutive associative bimodules over the algebra $A$.

Now our aim is to establish that all unital Jordan bimodules over $Q(2)^+$ are completely reducible.
Lemma 4.9. (1) Every homomorphism of unital Jordan $J$-bimodules $V_1 \to V_2$ gives rise to a homomorphism of $L$-modules $\bar{K}(V_1) \to \bar{K}(V_2)$.

(2) If $V_1 \to V_2$ is an embedding, then $\bar{K}(V_1) \to \bar{K}(V_2)$ is an embedding.

Proof. By the universal property of $K(V_1)$ a homomorphism $V_1 \to V_2$ gives rise to a homomorphism $\varphi : K(V_1) \to \bar{K}(V_2)$. Let $W$ be the largest submodule of $K(V_1)$ lying in $[V_1^-, J^+] + [V_1^+, J^-]$. The image of $W$ lies in $[V_2^-, J^+] + [V_2^+, J^-]$ and therefore is zero. This proves (1).

If $V_1 \to V_2$ is an embedding, then the kernel of $\bar{K}(V_1) \to \bar{K}(V_2)$ has zero intersection with $V_1^-$ and with $V_1^+$; hence it is zero. The lemma is proved.

Theorem 4.10. Every unital Jordan $J$-bimodule is completely reducible.

Proof. Let $V_1, V_2$ be irreducible unital Jordan $J$-bimodules and let $(0) \to V_1 \to V \to V_2 \to (0)$ be a short exact sequence. It gives rise to $(0) \to \bar{K}(V_1) \to \bar{K}(V) \to \bar{K}(V_2) \to (0)$.

We do not claim that this sequence is exact, but its restrictions $(0) \to V_1^\pm \to V^\pm \to V_2^\pm \to (0)$ are exact.

Suppose at first that the irreducible modules $\bar{K}(V_1), \bar{K}(V_2)$ have different highest weights. Then $\bar{K}(V_1)(z-\alpha) = \bar{K}(V_2)(z - \beta) = 0, \alpha \neq \beta$. Hence $V^\pm(z-\alpha)(z - \beta) = (0)$. Now $V = \text{Ker}(z-\alpha) \oplus \text{Ker}(z-\beta)$ is a direct sum of Jordan bimodules.

Now let $\bar{K}(V_1), \bar{K}(V_2)$ have the same highest weight $\gamma$ (which does not imply that they are isomorphic if $\gamma = w_1 - w_2$). We have shown above that for each of the highest weights $\gamma = 2w_1, w_1 - w_2, -2w_2$, the action of $L_0$ on $\bar{K}(V)$ is completely reducible.

Hence $\bar{K}(V)_\gamma = \bar{K}(V_1)_\gamma \oplus M$. Let $W$ be the submodule of $\bar{K}(V)$ generated by $M$. It is easy to see that $W \cap \bar{K}(V)_\gamma = M$. Hence $W \cap \bar{K}(V_1) = (0)$.

Since every nonzero submodule of $\bar{K}(V)$ has a nonzero intersection with $V^-$ it follows that $W \cap V^- \neq (0)$. Now $\{v \in V \mid v^- \in W\}$ is a nonzero $J$-subbimodule of $V$ which has zero intersection with $V_1$.

This proves that $V \simeq V_1 \oplus V_2$. The theorem is proved.

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